

Exercise walk-through

2.6

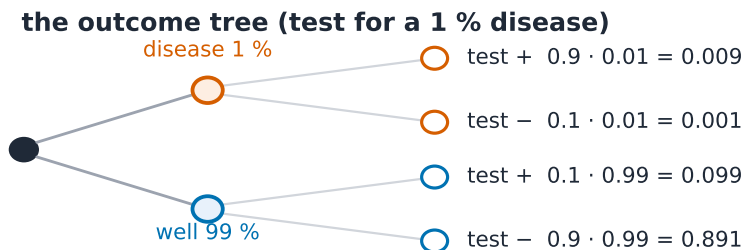
Discrete probability

University Computer Science

Discrete probability: counting with weights

- 1 **Sample** Two dice: 36 pairs; $P(\text{sum } 7) = 6/36 = 1/6$ - counting from 2.4 does the work.
- 2 **Condition** $P(A|B) = P(A \cap B)/P(B)$ - the B-branch renormalised; independence multiplies $P(A)P(B)$.
- 3 **Bayes** 1 % disease, $Se = Sp = 0.9$: $P(\text{dis}|+) = 0.009/(0.009+0.099) = 1/12 \approx 8.3\%$ - rare stays rare.

An outcome tree with conditional branches



Bayes: $P(\text{dis} \mid +) = 0.009 / (0.009 + 0.099)$
 $= 0.009 / 0.108 = 1/12 \approx 8.3 \%$

the + branch keeps 0.108 of the world;
 the diseased slice of it is 0.009

the rules

$P(E) = |E| / |\Omega|$
 (equal outcomes — counting!)

$P(A \mid B) = P(A \cap B) / P(B)$
 — the B-branch, renormalised

Bayes: $P(A|B) = P(B|A)P(A) / P(B)$
 slices rescaled between worlds

independence: $P(A \cap B) = P(A)P(B)$

two dice: $P(\text{sum } 7) = 6/36 = 1/6$

The tree shows what conditioning removes: the branch, renormalised.

Check yourself

- **Power set** of a 3-element set...? (8 members.)
- $P(7,2)$ and $C(7,2)$...? (42 and 21.)
- Edges in a 12-vertex tree...? (11.)
- $P(\text{sum } 7)$ with two dice...? ($1/6$.)

Exercises 2.1·2.3

2.1 One fair die: $P(\text{even})$?

[3]

2.2 Two coins: $P(\text{two heads} \mid \text{at least one head})$?

[3]

2.3 Independent events with $P(A) = 0.5$, $P(B) = 0.4 - P(A \cap B)$?

[3]

Exercises 2.1-2.3

2.1 One fair die: $P(\text{even})$? [3]

2.1

- $3/6 = 1/2$ ✓.

2.2 Two coins: $P(\text{two heads} \mid \text{at least one head})$? [3]

2.2

- Outcomes with at least one head: HH, HT, TH – $1/3$ ✓.

2.3 Independent events with $P(A) = 0.5$, $P(B) = 0.4 - P(A \cap B)$? [3]

2.3

- $0.5 \times 0.4 = 0.2$ ✓.

Exercise 3.1

3.1 Two fair dice, $n = 36$ pairs.

Compute $P(\text{sum} = 5) = 4/36$, $P(\text{sum} = 7) = 6/36$ and $P(\text{sum} = 5 \text{ or } 7) = 10/36$ (disjoint – add) ✓. Then $P(\text{at least one } 6)$ by the complement: $1 - (5/6)^2 = 11/36$ ✓. Verify the identity $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ on $A = \text{first die } 6$, $B = \text{second die } 6$: $1/6 + 1/6 - 1/36 = 11/36$ ✓. [8]

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3.1

- Sum 5: (1,4),(2,3),(3,2),(4,1) – 4/36; sum 7: 6/36 ✓.
- Disjoint events add: 10/36 ✓.
- At least one 6 by complement: $1 - 25/36 = 11/36$ ✓.
- Check: $1/6 + 1/6 - 1/36 = 6/36 + 6/36 - 1/36 = 11/36$ ✓ – inclusion-exclusion agrees.

Exercise 3.2

3.2 A box: 3 red, 2 blue. Two draws without replacement.

Build the outcome tree with products on the branches; compute $P(\text{two red}) = \frac{3}{5} \cdot \frac{2}{4} = \frac{3}{10}$, $P(\text{one of each}) = \frac{3}{10} + \frac{3}{10} = \frac{3}{5} \checkmark$. Then Bayes: given the second draw is red, $P(\text{first red} \mid \text{second red}) = \frac{3/10}{3/10+3/10} = \frac{1}{2} \checkmark$ – state why it differs from the unconditional $3/5$.

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[7]

3.2

- Branches: RR $\frac{3}{5} \cdot \frac{2}{4} = \frac{3}{10}$; RB and BR $\frac{3}{10}$ each; BB $\frac{1}{10} \checkmark$.
- One of each: $\frac{3}{10} + \frac{3}{10} = \frac{3}{5} \checkmark$.
- Bayes: $P(R_1 \mid R_2) = \frac{P(R_1 \cap R_2)}{P(R_2)} = \frac{3/10}{3/10+3/10} = \frac{1}{2} \checkmark$.
- It differs from $3/5$ because a red second draw is evidence – the condition reshapes the sample space $\checkmark$.