

Exercise walk-through

2.5

Graphs and trees

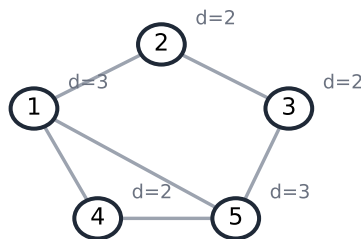
University Computer Science

Graphs and trees: one graph, three representations

- 1 **Represent Adjacency matrix** n^2 space with $O(1)$ edge test; **adjacency list** $O(|V|+|E|)$ - density decides.
- 2 **Handshake** Every edge adds 2: $\sum \deg = 2|E|$ - drawn graph $3+2+2+2+3 = 12 = 2 \cdot 6$; odd-degree count always even.
- 3 **Trees** Connected + acyclic $\rightarrow |E| = |V| - 1$; 12-vertex tree, 5 leaves $\rightarrow$ internal degree sum $22 - 5 = 17$.

Small explicit graphs with adjacency structures

the picture (5 vertices, 6 edges)



$$\sum \text{deg} = 3+2+2+2+3 = 12 = 2 \cdot 6 \quad \checkmark$$

adjacency matrix

0	1	0	1	1
1	0	1	0	0
0	1	0	0	1
1	0	0	0	1
1	0	1	1	0

n^2 space, $O(1)$ edge test

adjacency list

1: 2 4 5

2: 1 3

3: 2 5

4: 1 5

5: 1 3 4

$O(|V|+|E|)$ — for sparse graphs

**tree: connected + acyclic $\rightarrow |E| = |V| - 1$; a 12-vertex tree with 5 leaves:
7 internal vertices carry degree sum $22 - 5 = 17$**

BFS: queue, level order, unweighted shortest · DFS: stack, components · model by symmetry, represent by density

Three representations of one graph - and the degree sum checked by handshake.

Exercises 2.1-2.3

2.1 Edges in a 9-vertex tree?

[3]

2.2 A graph has degrees 3, 3, 2, 2 – how many edges?

[3]

2.3 Can exactly one vertex have odd degree? Why not?

[3]

Exercises 2.1-2.3

2.1 Edges in a 9-vertex tree?

[3]

2.1

- $9 - 1 = 8$ ✓.

2.2 A graph has degrees 3, 3, 2, 2 – how many edges?

[3]

2.2

- Sum 10, so $|E| = 5$ ✓.

2.3 Can exactly one vertex have odd degree? Why not?

[3]

2.3

- No – the degree sum must be even, so odd degrees come in pairs ✓.

Exercise 3.1

3.1 A graph has 7 vertices with degrees 4, 4, 3, 3, 3, 2, 2.

Check the handshake total (21 – odd, so this degree list is IMPOSSIBLE) ✓. Correct the list by changing one entry so the sum is even and equals $2|E|$ with $|E| = 11$ (e.g. change a 2 to a 3, total 22) ✓. State the corollary used (odd-degree count even) and draw-the-check: any proposed drawing must satisfy it ✓. [8]

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3.1

- Total $4 + 4 + 3 + 3 + 3 + 2 + 2 = 21$ – odd, violating handshake ✓.
- Change one 2 to a 3: total $22 = 2 \times 11$ edges ✓.
- Corollary: the count of odd-degree vertices is even – the original list had three odd entries (impossible) ✓.
- Any drawing must satisfy handshake; the arithmetic screens drawings before pencil moves ✓.

Exercise 3.2

3.2 A 10-vertex tree has 4 leaves.

Compute $|E| = 9$ and the total degree sum 18 ✓. The 6 internal vertices carry degree sum $18 - 4 = 14$; state the minimum degree of an internal vertex (2, since one neighbour must lead onward and one back) and derive the maximum possible number of degree-2 internals (if x are degree 2 and the rest degree 3: $2x + 3(6 - x) = 14$ gives $x = 4$) ✓. [7]

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3.2

- $|E| = 10 - 1 = 9$; degree sum $= 2 \times 9 = 18$ ✓.
- Internal sum: $18 - 4 = 14$ across 6 vertices ✓.
- Minimum internal degree 2 (path through) ✓.
- Let x degree-2 and $6 - x$ degree-3: $2x + 18 - 3x = 14 \Rightarrow x = 4$ ✓.