

Exercise walk-through

2.3

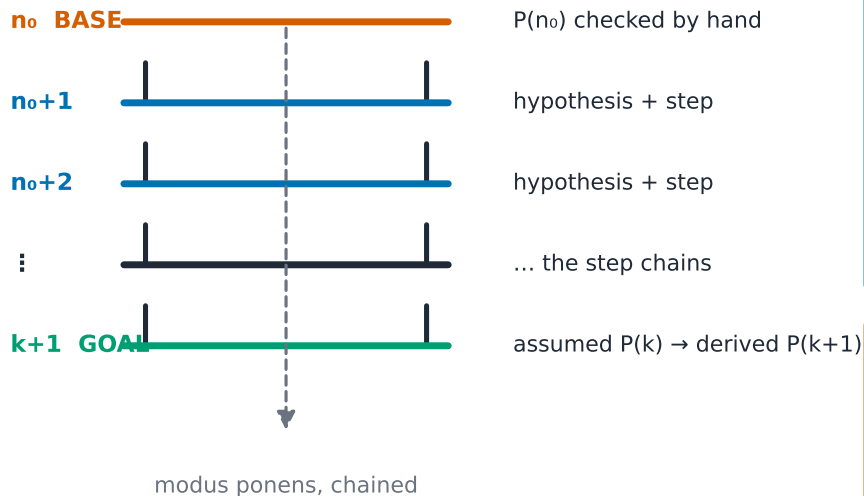
Induction and recursion

University Computer Science

Induction and recursion: the ladder

- 1 **Ladder** Base $P(n_0)$ by hand; step: assume $P(k)$, derive $P(k+1)$ - induction is chained modus ponens.
- 2 **Sum** $\sum_1^n i = n(n+1)/2$: base $1 = 1 \cdot 2/2$; step $k(k+1)/2 + (k+1) = (k+1)(k+2)/2$.
- 3 **Faults** No base (ladder on air); hypothesis unused; step silently needs $k \geq 2$ - the horses proof fails at $k+1 = 2$.

The induction ladder: base, hypothesis, step



$\Sigma i = n(n+1)/2$, proved

base: $1 = 1 \cdot 2/2 \checkmark$
 step: assume $\Sigma_{i=1}^k i = k(k+1)/2$
 then $\Sigma_{i=1}^{k+1} i = k(k+1)/2 + (k+1)$
 $= (k+1)(k+2)/2 \checkmark$

the hypothesis is USED —
 the algebra is SHOWN

the three faults

no base — ladder on air
 hypothesis never invoked
 step silently needs $k \geq 2$
 (the all-horses classic)

Base plants the ladder; the step climbs one rung.

Exercises 2.1-2.3

2.1 Name the two mandatory parts of an induction proof. [3]

2.2 In the step, what may you assume about k ? [3]

2.3 Why does the all-horses proof fail? [3]

Exercises 2.1-2.3

2.1 Name the two mandatory parts of an induction proof. [3]

2.1

- Base case and inductive step (hypothesis used) ✓.
-

2.2 In the step, what may you assume about k ? [3]

2.2

- $P(k)$ for an arbitrary $k \geq n_0$ – no more ✓.
-

2.3 Why does the all-horses proof fail? [3]

2.3

- The step needs the two size- k sets to overlap – fails when $k + 1 = 2$ ✓.

Exercise 3.1

3.1 Prove by induction: for all $n \geq 1$, $2 + 4 + \cdots + 2n = n(n + 1)$.

Write the base ($n = 1$: $2 = 1 \cdot 2$) ✓, the induction hypothesis, the step algebra $k(k + 1) + 2(k + 1) = (k + 1)(k + 2)$ with every line shown ✓, and the closing statement. Then prove the geometric sum $\sum_{i=0}^n 2^i = 2^{n+1} - 1$ with the same template ✓. [8]

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3.1

- Base: $2 = 1 \cdot 2$ ✓.
- Hypothesis stated; then $k(k + 1) + 2(k + 1) = (k + 1)(k + 2)$ ✓.
- Closing statement: holds for $k + 1$, hence all $n \geq 1$ ✓.
- Geometric: base $1 = 2^1 - 1$; step $(2^{k+1} - 1) + 2^{k+1} = 2 \cdot 2^{k+1} - 1 = 2^{k+2} - 1$ ✓.

Exercise 3.2

3.2 Two faulty proofs to diagnose.

(A) The sum-of-odds identity $\sum_{i=1}^n (2i - 1) = n^2$ presented with a step but the base written for $n = 0$ only – identify the fault and repair it ✓. (B) A claim on all $n \geq 0$ whose step goes from k to $k + 2$ only – name the missing half of the step and give the general rule (a step must cover EVERY increment) ✓. [7]

Exercise 3.2

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3.2

- (A) Fault: base for $n = 0$ proves the empty case only – the claim starts at $n = 1$; repair: verify $1 = 1^2$ ✓.
- (B) Fault: the step skips $k \rightarrow k + 1$; two chains start from separate bases or the step must cover every increment ✓.
- General rule: every increment from base to goal must be spanned by the step (or a second base added) ✓.