

Exercise walk-through

10.1

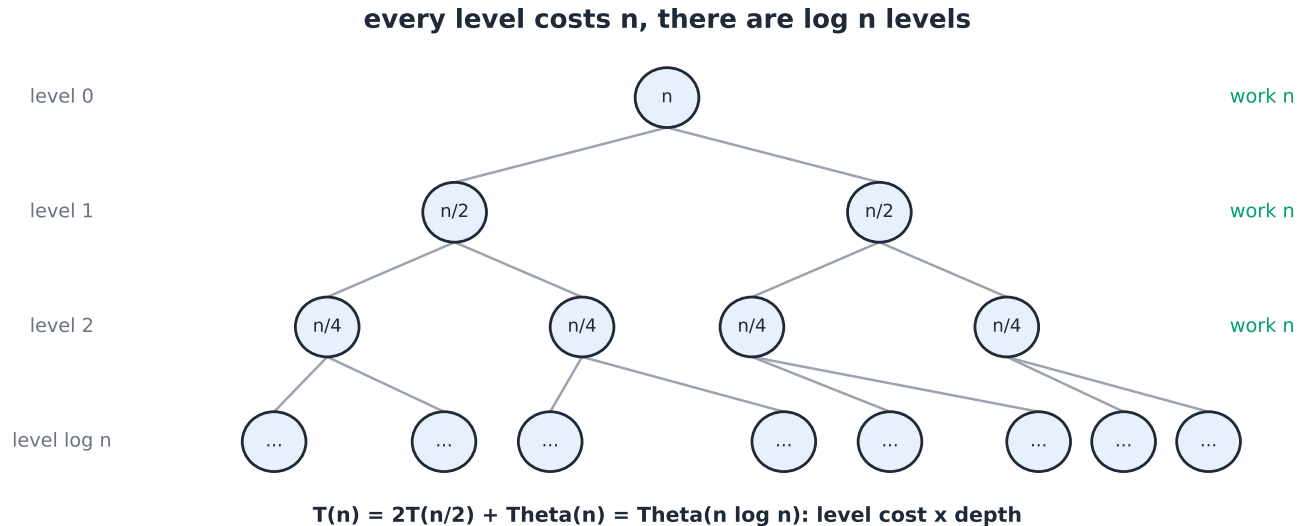
Recurrences and analysis tools

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Recurrences: the tree is the analysis

- 1 **Tree** $T(n) = 2T(n/2) + n$: level i has 2^i nodes of $n/2^i$ - every level costs n , depth $\log n$, product $n \log n$.
- 2 **Master** Leaves win: $n^{\log_b a}$; balanced: $f(n) \log n$; root wins: $f(n) - 4T(n/2) + n$ is n^2 , $T(n/2) + 1$ is $\log n$.
- 3 **Measure Merge sort** on the 64-element course array: 310 comparisons, under the 321 ceiling $n \log n - n + 1$.

Recursion tree



Every level costs the same n ; the depth is $\log n$.

Exercises 2.1·2.3

2.1 $T(n) = 3T(n/3) + n$: which master case, and what is the answer? [3]

2.2 $T(n) = T(n/4) + 1$: solve it. [3]

2.3 In the tree for $2T(n/2) + n$, what does level i cost? [3]

Exercises 2.1-2.3

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2.1

- Balanced: $n^{\log_3 3} = n = f(n)$ – case 2 ✓. $T(n) = \Theta(n \log n)$.
-

2.2 $T(n) = T(n/4) + 1$: solve it. [3]

2.2

- $\log_4 n$ levels of constant work: $\Theta(\log n)$ ✓.
-

2.3 In the tree for $2T(n/2) + n$, what does level i cost? [3]

2.3

- 2^i nodes of $n/2^i$: the level product is n at every level ✓.

Exercise 3.1

3.1 Solve each by the master method, naming the case:

$T(n) = 9T(n/3) + n$; then $T(n) = T(2n/3) + 1$; then $T(n) = 3T(n/4) + n \log n$.

For each, state the leaf total $n^{\log_b a}$ (or note the polynomial comparison), the dominating side, and the final class. Justify the middle case's comparison explicitly. [8]

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3.1

- $9T(n/3) + n$: leaves $n^{\log_3 9} = n^2$ dominate the linear root – $T(n) = \Theta(n^2)$ ✓.
- $T(2n/3) + 1$: leaf total is $n^0 = 1$ ($a = 1$); the root work 1 matches – case 2 with a shrinking ratio: $T(n) = \Theta(\log n)$ (base $3/2$) ✓.
- $3T(n/4) + n \log n$: leaves $n^{\log_4 3} \approx n^{0.79}$; the root $n \log n$ dominates (polylog factor included) – $T(n) = \Theta(n \log n)$ ✓.

Exercise 3.2

3.2 Prove by substitution that $T(n) = 2T(n/2) + n$ satisfies $T(n) \leq c n \log_2 n$ for some $c \geq 1$.

Write the induction hypothesis. Substitute it into the recurrence and simplify to show the residual is ≤ 0 exactly when $c \geq 1$. State what the measured count on the 64-element course array (310 against the 321 ceiling) adds to the proof. [7]

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3.2

- Hypothesis: $T(m) \leq cm \log_2 m$ for all $m < n$ ✓.
- Substitute: $T(n) \leq 2c \frac{n}{2} \log_2 \frac{n}{2} + n = cn \log_2 n - cn + n$ ✓.
- Residual $n(1 - c) \leq 0$ exactly when $c \geq 1$ – the induction closes ✓.
- The measurement (310 under 321) does not prove the bound – it confirms the constant is realistic on real data: theory bounds, measurement witnesses ✓.