UNIVERSITY COMPUTER SCIENCE • EXERCISE SHEET

2.3 Induction and recursion

Name: _____ Class: _____ Date: _____ **Total: 24 marks**

KEY WORDS induction hypothesis • base case

Objective

Write complete inductions; catch the faulty ones.

1 Key knowledge and worked examples

■ The template

Base $P(n_0)$ by hand + step: assume $P(k)$, derive $P(k+1)$ – the hypothesis must be USED ✓. Induction is chained modus ponens.

■ Worked example: the sum

$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$: base $1 = \frac{1\cdot 2}{2}$; step $\frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+2)}{2}$ ✓ – hypothesis invoked, algebra shown.

2 Practice

2.1 Name the two mandatory parts of an induction proof. [3]

2.2 In the step, what may you assume about k? [3]

2.3 Why does the all-horses proof fail? [3]

3 Exam-style questions

3.1 Prove by induction: for all $n \geq 1$, $2 + 4 + \cdots + 2n = n(n+1)$.

Write the base ($n = 1$: $2 = 1 \cdot 2$) ✓, the induction hypothesis, the step algebra $k(k+1) + 2(k+1) = (k+1)(k+2)$ with every line shown ✓, and the closing statement. Then prove the geometric sum $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$ with the same template ✓. [8]

3.2 Two faulty proofs to diagnose.

(A) The sum-of-odds identity $\sum_{i=1}^{n}(2i-1) = n^2$ presented with a step but the base written for $n = 0$ only – identify the fault and repair it ✓. (B) A claim on all $n \geq 0$ whose step goes from k to $k+2$ only – name the missing half of the step and give the general rule (a step must cover EVERY increment) ✓. [7]

Solutions

2.1

- Base case and inductive step (hypothesis used) ✓.

2.2

- $P(k)$ for an arbitrary $k \geq n_0$ – no more ✓.

2.3

- The step needs the two size-k sets to overlap – fails when $k + 1 = 2$ ✓.

3.1

- Base: $2 = 1 \cdot 2$ ✓.
- Hypothesis stated; then $k(k + 1) + 2(k + 1) = (k + 1)(k + 2)$ ✓.
- Closing statement: holds for $k + 1$, hence all $n \geq 1$ ✓.
- Geometric: base $1 = 2^1 - 1$; step $(2^{k+1} - 1) + 2^{k+1} = 2 \cdot 2^{k+1} - 1 = 2^{k+2} - 1$ ✓.

3.2

- (A) Fault: base for $n = 0$ proves the empty case only – the claim starts at $n = 1$; repair: verify $1 = 1^2$ ✓.
- (B) Fault: the step skips $k \to k + 1$; two chains start from separate bases or the step must cover every increment ✓.
- General rule: every increment from base to goal must be spanned by the step (or a second base added) ✓.

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