10.6 Reductions and NP-completeness

Name: _____ Class: _____ Date: _____ **Total: 24 marks**

KEY WORDS reduction • NP-completeness NP

Objective

Build reductions and read their direction correctly.

1 Key knowledge and worked examples

■ **The definition**

$A \leq_p B$: a polynomial translator from A-instances to B-instances – a B-solver solves A ✓. NP-complete = in NP + every NP problem reduces to it ✓.

■ **Worked example: the chain**

SAT → 3-SAT (clause splitting) → CLIQUE (triangle gadget) → VERTEX-COVER (complement) ✓.

2 Practice

2.1 Which direction carries hardness: $A \to B$ means – is hard? [3]

2.2 What two conditions define NP-completeness? [3]

2.3 Translate: G = triangle ABC + isolated D, $k = 3$. What VERTEX-COVER instance results? [3]

3 Exam-style questions

3.1 Reduce CLIQUE to VERTEX-COVER in full.

State the translator (graph, parameter) and its running time. Prove both directions: a k-clique exists in G iff a cover of size $n - k$ exists in $\bar{G}$. Run it on G = triangle ABC + isolated D with $k = 3$: list $\bar{G}$'s edges and the cover. State what a polynomial vertex-cover solver would then imply. [8]

3.2 A colleague proves their problem WARS hard by writing a SAT encoding of WARS and running a SAT solver.

Name the direction they actually proved (WARS $\leq_p$ SAT) and what it implies about a SAT solver. State what they must do instead to prove WARS NP-hard. State the two checks a full NP-completeness claim needs, and what remains unproven if the membership check is skipped. [7]

Solutions

2.1

- B (the target): if A is hard, B must be at least as hard ✓.

2.2

- The problem is in NP (polynomial verification); every NP problem reduces to it ✓.

2.3

- $(\bar{G}, n-k) = (\bar{G}, 1)$: complement edges A–D, B–D, C–D; cover {D} ✓.

3.1

- Translator: $(G, k) \mapsto (\bar{G}, n-k)$ – build the complement in $O(V^2)$ ✓.
- Forward: a k-clique's nodes are mutually non-adjacent in $\bar{G}$, so every $\bar{G}$-edge has an endpoint outside it – the other $n-k$ nodes form a cover ✓.
- Backward: a cover S of size $n-k$ leaves $n-|S| = k$ nodes touching no $\bar{G}$-edge – an independent set in $\bar{G}$, i.e. a clique in G ✓.
- Run: $\bar{G}$'s edges are exactly A–D, B–D, C–D; {D} covers them all at size $1 = n-k$ ✓.
- Implication: a polynomial vertex-cover solver would solve CLIQUE, hence SAT, hence every NP problem – P = NP ✓.

3.2

- They proved WARS $\leq_p$ SAT: a SAT solver solves WARS instances – it says nothing about WARS being hard ✓.
- Instead: reduce a known NP-complete problem (SAT, 3-SAT, CLIQUE, …) TO WARS with a polynomial translator ✓.
- NP-completeness needs: membership in NP (a verifiable certificate in polynomial time) plus NP-hardness by such a reduction ✓.
- Skipping membership: WARS might be harder than NP (e.g. outside NP) – then calling it NP-complete is simply unproven ✓.

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