UNIVERSITY COMPUTER SCIENCE • EXERCISE SHEET

10.1 Recurrences and analysis tools

Name: _____ Class: _____ Date: _____ **Total: 24 marks**

KEY WORDS recurrence • recursion tree

Objective

Unroll recurrences, apply the master method, and close by substitution.

1 Key knowledge and worked examples

■ **The tree**

$T(n) = 2T(n/2) + \Theta(n)$: every level costs $\Theta(n)$, depth $\log_2 n$, so $T(n) = \Theta(n \log n)$ ✓.

■ **Worked example: the three master cases**

$2T(n/2)+n \to n \log n$ (balanced); $T(n/2)+1 \to \log n$ (leaves trivial); $4T(n/2)+n \to n^2$ (leaves n^2 dominate) ✓.

2 Practice

2.1 $T(n) = 3T(n/3) + n$: which master case, and what is the answer? [3]

2.2 $T(n) = T(n/4) + 1$: solve it. [3]

2.3 In the tree for $2T(n/2) + n$, what does level i cost? [3]

3 Exam-style questions

3.1 Solve each by the master method, naming the case:

$T(n) = 9T(n/3) + n$; then $T(n) = T(2n/3) + 1$; then $T(n) = 3T(n/4) + n \log n$.

For each, state the leaf total $n^{\log_b a}$ (or note the polynomial comparison), the dominating side, and the final class. Justify the middle case's comparison explicitly. [8]

3.2 Prove by substitution that $T(n) = 2T(n/2) + n$ satisfies $T(n) \le c\, n \log_2 n$ for some $c \ge 1$.

Write the induction hypothesis. Substitute it into the recurrence and simplify to show the residual is ≤ 0 exactly when $c \ge 1$. State what the measured count on the 64-element course array (310 against the 321 ceiling) adds to the proof. [7]

Solutions

2.1

- Balanced: $n^{\log_3 3} = n = f(n)$ – case 2 ✓. $T(n) = \Theta(n \log n)$.

2.2

- $\log_4 n$ levels of constant work: $\Theta(\log n)$ ✓.

2.3

- 2^i nodes of $n/2^i$: the level product is n at every level ✓.

3.1

- $9T(n/3) + n$: leaves $n^{\log_3 9} = n^2$ dominate the linear root – $T(n) = \Theta(n^2)$ ✓.
- $T(2n/3) + 1$: leaf total is $n^0 = 1$ (a = 1); the root work 1 matches – case 2 with a shrinking ratio: $T(n) = \Theta(\log n)$ (base 3/2) ✓.
- $3T(n/4) + n \log n$: leaves $n^{\log_4 3} \approx n^{0.79}$; the root $n \log n$ dominates (polylog factor included) – $T(n) = \Theta(n \log n)$ ✓.

3.2

- Hypothesis: $T(m) \leq c\, m \log_2 m$ for all $m < n$ ✓.
- Substitute: $T(n) \leq 2c\frac{n}{2} \log_2 \frac{n}{2} + n = cn \log_2 n - cn + n$ ✓.
- Residual $n(1 - c) \leq 0$ exactly when $c \geq 1$ – the induction closes ✓.
- The measurement (310 under 321) does not prove the bound – it confirms the constant is realistic on real data: theory bounds, measurement witnesses ✓.