Linear parameters, solution counts and inequalities

Original skill preparation; versioned source scope

Linear Equations: Parameters, Solution Counts and Inequalities

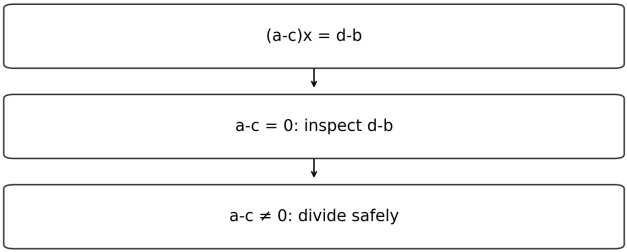
- Decide whether a linear equation with a **parameter** 参数 has no solution, one solution or infinitely many.
- Recognise an **identity** 恒等式, an equation true for every value.
- Interpret the **slope** 斜率 and **intercept** 截距 of a model with units.
- Classify a system of two linear equations, and solve an **inequality** 不等式 in one or two variables.

Counting the solutions of one equation

- If $a - c \neq 0$, divide: one solution, $x = \dfrac{d-b}{a-c}$.
- If $a - c = 0$ and $d - b = 0$, the equation reads $0 = 0$: an **identity**, with infinitely many solutions.
- If $a - c = 0$ but $d - b \neq 0$, it reads $0 =$ (a non-zero number): no solution.
- Never divide by an expression containing the parameter until you know it is not zero.

$$ax + b = cx + d \;\Rightarrow\; (a-c)x = d - b$$

Figure: Counting the solutions of one equation



- Worked steps: (a-c)x = d-b → a-c = 0: inspect d-b → a-c ≠ 0: divide safely
- Match each part of the diagram to the text.

Linear models with units

- In $y = mx + b$, the **slope** m is the change in y per one unit of x, so its units are "units of y per unit of x".
- The **intercept** b is y when $x = 0$: often a fixed fee or a starting amount.
- An intercept can exist algebraically but mean nothing if $x = 0$ is outside the situation.
- Given two data points, find the slope first, then substitute one point to find b.

Systems of two linear equations

- If one equation is a multiple of the other (both sides), they are the same line: infinitely many solutions.
- If the x and y coefficients are in the same ratio but the constants are not, the lines are parallel: no solution.
- Otherwise the lines cross once. Solve by elimination or substitution, and check the pair in both original equations.

Inequalities

- Solve as for an equation, except that multiplying or dividing by a negative number reverses the sign: $-2x < 6$ gives $x > -3$.
- A strict inequality ($<$, $>$) excludes the boundary; $\leq$ and $\geq$ include it.
- In two variables, graph the boundary line (dashed if strict, solid if not), then test a point such as $(0, 0)$ to choose the side.
- In a word problem, the solution must also be sensible: whole tickets, non-negative hours.

Worked example: a system with a parameter

- Doubling the first equation gives $4x + 6y = 14$.
- If $k = 6$ and $c = 14$, the equations are the same line: infinitely many solutions.
- If $k = 6$ but $c \neq 14$, the lines are parallel: no solution.
- If $k \neq 6$, the slopes differ: exactly one solution, whatever c is.

Worked example: a budget inequality

- With a adults and c children: $8a + 5c \leq 100$, with a, c whole numbers at least 0.
- Check $(10, 4)$: $80 + 20 = 100 \leq 100$, so it is allowed; the boundary is included.
- Check $(10, 5)$: $80 + 25 = 105 > 100$, not allowed.
- With 6 adults, $5c \leq 52$, so $c \leq 10.4$: at most 10 children.

Common errors

- Dividing by $k-3$ without first checking the case $k=3$.
- Forgetting to reverse the inequality sign when dividing by a negative.
- Calling a system "no solution" when the constants also match (it is the same line).

Common errors (2)

- Giving the **intercept** as a meaning when $x = 0$ is impossible in the context.
- Shading a half-plane from the **slope** instead of testing a point.

Key terms

- **parameter**: A letter that stands for a fixed but unknown number in an equation.
- **identity**: An equation that is true for every allowed value of the variable.
- **slope**: The change in y for each one-unit increase in x.
- **intercept**: The value of y when $x = 0$.

Check yourself

- For which k does $kx - 4 = 3x + 2$ have no solution?
- Solve $5 - 2x > 11$.
- Does $(1, 2)$ satisfy $y < 3x - 1$?