Course Companion

IGCSE Mathematics

Every topic handout in one volume

全部专题讲义合集

exercises.lol

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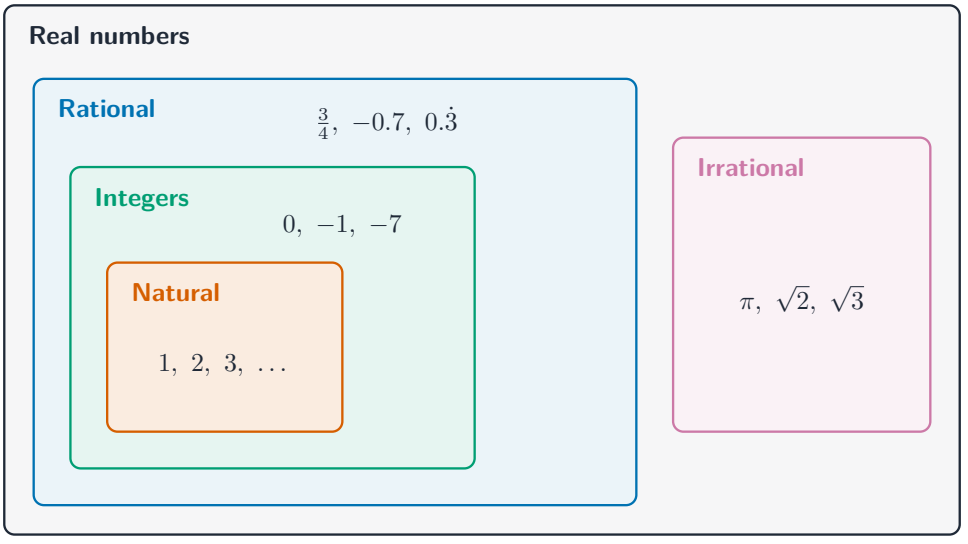
Number

IGCSE Mathematics

This handout covers Topic 1, Number. Cambridge Maths has two levels: Core and Extended. Parts marked **(Extended)** are only tested on the Extended papers; everything else is for both levels.

Types of number

You must know the words for the different kinds of number. Examiners give marks for using them correctly.



Each set sits inside the next: every natural number is an integer, every integer is rational, every rational is real. Irrational numbers are real but not rational.

Counting numbers and integers

- **natural numbers** 自然数 – the counting numbers $1, 2, 3, 4, \ldots$
- **integers** 整数 – whole numbers, positive, negative or zero: $\ldots, -2, -1, 0, 1, 2, \ldots$

Factors and multiples

- A **factor** 因数 of a number divides into it exactly, leaving no **remainder** 余数. The factors of 18 are $1, 2, 3, 6, 9, 18$.
- A **multiple** 倍数 of a number is that number times an integer. Multiples of 6 are $6, 12, 18, 24, \ldots$
- A **common factor** 公因数 of two numbers is a factor of both.
- A **common multiple** 公倍数 of two numbers is a multiple of both.

Prime, square and cube numbers

- A **prime number** 质数 has exactly two factors: 1 and itself. The first primes are $2, 3, 5, 7, 11, 13, \ldots$ Note that 1 is **not** prime.
- A **square number** 平方数 is a whole number times itself: $1, 4, 9, 16, 25, \ldots$
- A **cube number** 立方数 uses a whole number three times: $1, 8, 27, 64, \ldots$

Rational, irrational and reciprocal

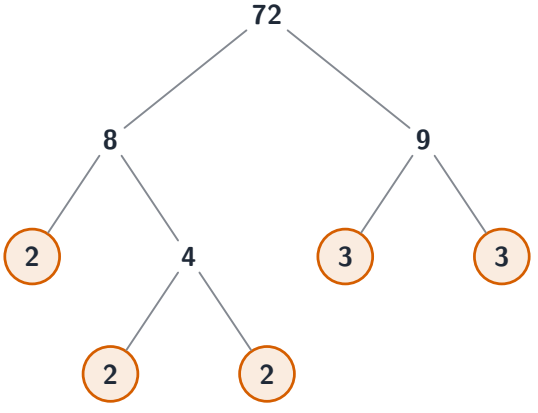
- A **rational number** 有理数 can be written as a **fraction** 分数 $\frac{a}{b}$ of two integers, with $b \neq 0$. Examples: $\frac{3}{4}$, 5, 0.7.
- An **irrational number** 无理数 cannot be written this way. Examples: π and $\sqrt{2}$.
- The **reciprocal** 倒数 of a non-zero number is 1 divided by that number. The reciprocal of 4 is $\frac{1}{4}$; the reciprocal of 0.25 is 4; the reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$.

Prime factors, HCF and LCM

Every integer above 1 is prime, or can be written as a **product** 乘积 of prime numbers. To write a number as a product of its **prime factors** 质因数, keep dividing by the smallest prime that fits.

Worked example. Write 72 as a product of its prime factors.

$$72 = 2 \times 36 = 2 \times 2 \times 18 = 2 \times 2 \times 2 \times 9 = 2 \times 2 \times 2 \times 3 \times 3 = 2^3 \times 3^2.$$



$$72 = 2^3 \times 3^2$$

Keep splitting until every branch ends on a prime (circled); collecting them gives $72 = 2^3 \times 3^2$.

The **highest common factor (HCF)** 最大公因数 of two numbers is the largest factor they share. The **lowest common multiple (LCM)** 最小公倍数 is the smallest multiple they share. Prime factors give a quick method.

Worked example. Find the HCF and LCM of 72 and 120.

First write each as a product of primes:

$$72 = 2^3 \times 3^2, \qquad 120 = 2^3 \times 3 \times 5.$$

- HCF: take the **lowest** power of each prime that appears in *both*: $2^3 \times 3 = 24$.
- LCM: take the **highest** power of *every* prime that appears: $2^3 \times 3^2 \times 5 = 360$.

Sets and Venn diagrams

A **set** 集合 is a collection of objects. Each object in the set is an **element** 元素 of the set. You should know this notation:

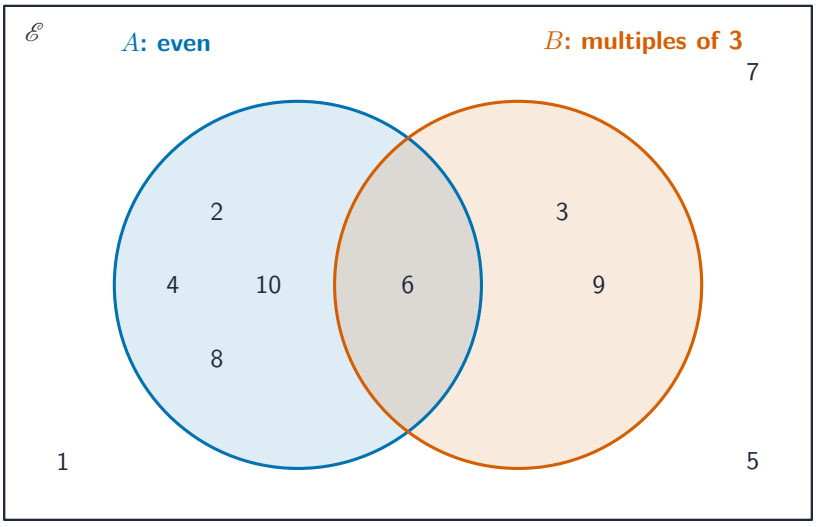
Symbol	Meaning
$n(A)$	the number of elements in set A
$x \in A$	x is an element of A
$x \notin A$	x is not an element of A
$\mathscr{E}$	the **universal set** 全集 – everything being talked about
A'	the **complement** 补集 of A – everything not in A
$\varnothing$	the **empty set** 空集 – a set with no elements
$A \subseteq B$	A is a **subset** 子集 of B – every element of A is also in B
$A \cup B$	the **union** 并集 – elements in A or B or both
$A \cap B$	the **intersection** 交集 – elements in both A and B

A **Venn diagram** 维恩图 draws each set as a circle inside a rectangle (the universal set). Core uses two sets; Extended may use three.

Worked example. $\mathscr{E} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{\text{even numbers}\}$, $B = \{\text{multiples of } 3\}$.

- $A = \{2, 4, 6, 8, 10\}$ and $B = \{3, 6, 9\}$.
- $A \cap B = \{6\}$ – the only number in both.
- $A \cup B = \{2, 3, 4, 6, 8, 9, 10\}$ – the numbers in either set.
- $n(A \cup B) = 7$.

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$A \cap B = \{6\}$ is the only number in both circles; $A \cup B$ is everything inside either circle.

You may also see a set written as a rule, e.g. $C = \{x : 1 \leqslant x \leqslant 5\}$ means "all values x such that $1 \leqslant x \leqslant 5$".

Powers and roots

- A **power** 幂 (also called an **index** 指数, plural *indices*) shows how many times to multiply a number by itself: $2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32$.
- A **square root** 平方根 of a number gives that number when squared: $\sqrt{169} = 13$ because $13^2 = 169$.
- A **cube root** 立方根 works the same way for cubes: $\sqrt[3]{8} = 2$ because $2^3 = 8$.

You should be able to recall the squares from 1^2 to 15^2 (and their roots), and the cubes of $1, 2, 3, 4, 5$ and 10.

> **Worked example.** Work out $5^2 \times \sqrt[3]{8}$.

$$5^2 \times \sqrt[3]{8} = 25 \times 2 = 50.$$

The laws of indices

When you multiply or divide powers of the **same base** 底数, use these rules:

$$a^m \times a^n = a^{m+n}, \qquad a^m \div a^n = a^{m-n}, \qquad (a^m)^n = a^{mn}.$$

Some special powers:

$$a^0 = 1, \qquad a^{-n} = \frac{1}{a^n}, \qquad a^{\frac{1}{n}} = \sqrt[n]{a}, \qquad a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m.$$

(Fractional powers like $a^{\frac{m}{n}}$ are **Extended**.)

Worked examples.

- $2^{-3} \times 2^4 = 2^{-3+4} = 2^1 = 2$.
- $(2^3)^2 = 2^6 = 64$.
- $2^3 \div 2^4 = 2^{3-4} = 2^{-1} = \dfrac{1}{2}$.
- $7^{-2} = \dfrac{1}{7^2} = \dfrac{1}{49}$.
- $81^{\frac{1}{2}} = \sqrt{81} = 9$.
- $8^{-\frac{2}{3}} = \dfrac{1}{8^{\frac{2}{3}}} = \dfrac{1}{\left(\sqrt[3]{8}\right)^2} = \dfrac{1}{2^2} = \dfrac{1}{4}$.

Common mistakes. Multiplying the indices when you multiply powers ($2^3 \times 2^4$ is 2^7, not 2^{12}), thinking a negative power makes a negative number (7^{-2} is positive), and writing $a^0 = 0$ (it is 1).

multiply the powers

$a^m \times a^n = a^{m+n}$

add:

$2^{-3} \times 2^4 = 2^1 = 2$

divide the powers

$a^m \div a^n = a^{m-n}$

subtract:

$2^3 \div 2^4 = 2^{-1} = \frac{1}{2}$

power of a power

$(a^m)^n = a^{mn}$

multiply:

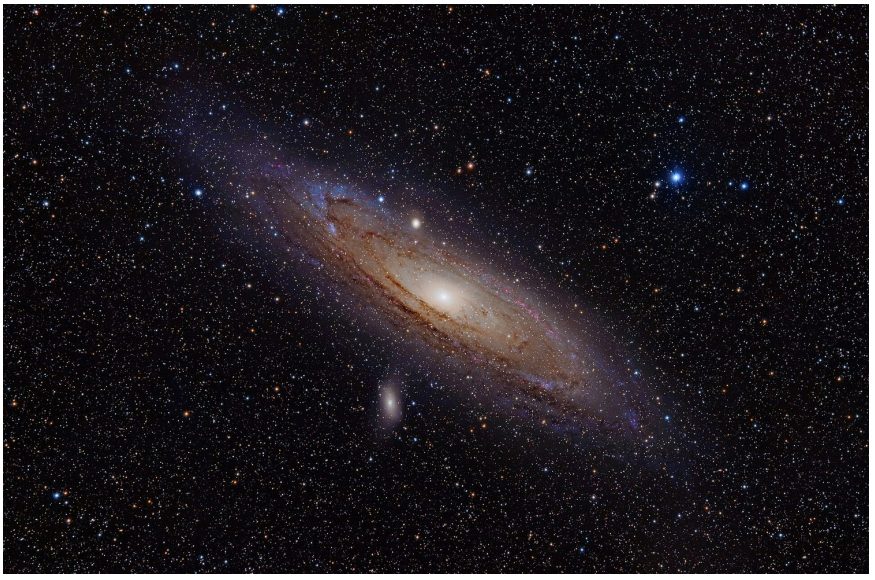
$(2^3)^2 = 2^6 = 64$

The three index laws: multiply adds the powers, divide subtracts them, a power of a power multiplies them

Standard form

Standard form 科学记数法 writes a number as $A \times 10^n$, where $1 \leqslant A < 10$ and n is an integer. It is used for very large or very small numbers.

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A galaxy: huge distances are written compactly in standard form.

To convert, count how many places the decimal point moves:

- $4\,500\,000 = 4.5 \times 10^{6}$ – the point moves 6 places left, so the power is positive.
- $0.00072 = 7.2 \times 10^{-4}$ – the point moves 4 places right, so the power is negative.

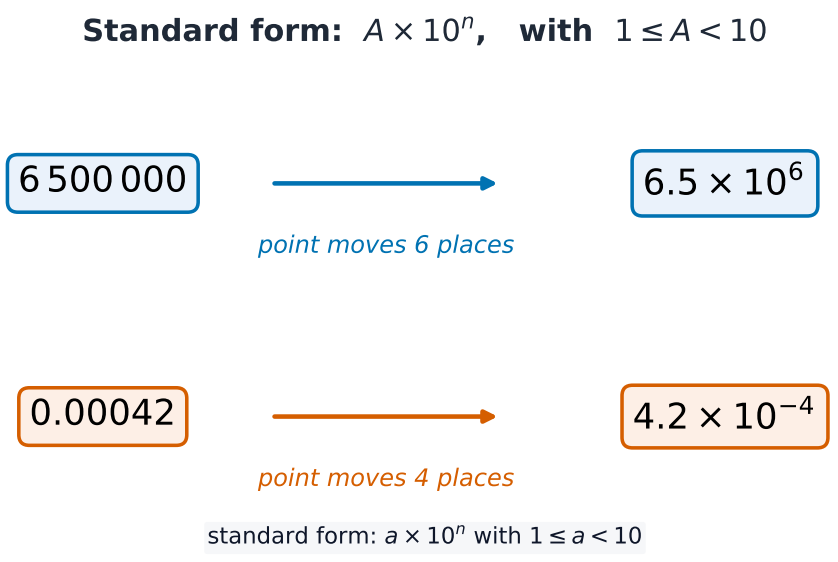
Worked example. Work out $(3 \times 10^{5}) \times (2 \times 10^{-2})$.

Multiply the front numbers and add the powers:

$$3 \times 2 = 6, \qquad 10^{5} \times 10^{-2} = 10^{3}, \qquad \text{so the answer is } 6 \times 10^{3}.$$

Common mistake. Leaving an answer such as 15×10^{4}. The front number must be between 1 and 10, so write 1.5×10^{5}.

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Standard form: a number written as a × 10^n with 1 ≤ a < 10

Surds (Extended)

A **surd** 根式 is a root that is irrational, such as $\sqrt{5}$. Leave it in exact form instead of rounding. Two useful rules:

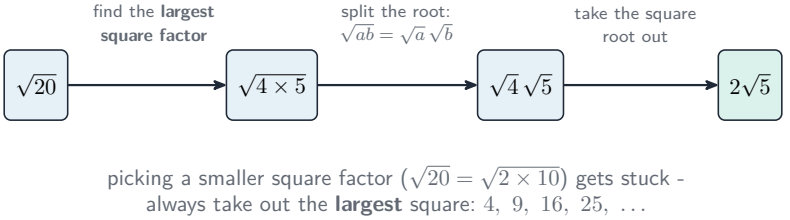
$$\sqrt{a} \times \sqrt{b} = \sqrt{ab}, \qquad \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}.$$

Simplify a surd by taking out the largest square factor.

Worked example. Simplify $\sqrt{20}$ and $\sqrt{200} - \sqrt{32}$.

$$\sqrt{20} = \sqrt{4 \times 5} = \sqrt{4}\,\sqrt{5} = 2\sqrt{5}.$$

$$\sqrt{200} - \sqrt{32} = \sqrt{100 \times 2} - \sqrt{16 \times 2} = 10\sqrt{2} - 4\sqrt{2} = 6\sqrt{2}.$$



Simplifying a surd: take out the largest square factor

To **rationalise the denominator** 分母有理化 means to remove a surd from the bottom of a fraction (the **denominator** 分母). Multiply the top and bottom by a value that clears the surd.

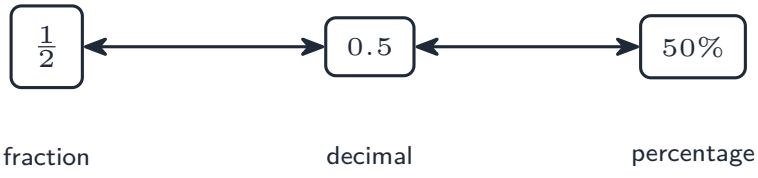
Worked example. Rationalise $\dfrac{10}{\sqrt{5}}$ and $\dfrac{1}{-1+\sqrt{3}}$.

$$\frac{10}{\sqrt{5}} = \frac{10}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} = \frac{10\sqrt{5}}{5} = 2\sqrt{5}.$$

$$\frac{1}{-1+\sqrt{3}} = \frac{1}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1} = \frac{\sqrt{3}+1}{3-1} = \frac{1+\sqrt{3}}{2}.$$

Fractions, decimals and percentages

A fraction has a **numerator** 分子 (the top) and a **denominator** (the bottom).



One value three ways: a fraction, a decimal and a percentage

- **proper fraction** 真分数: numerator smaller than denominator, e.g. $\frac{3}{4}$.
- **improper fraction** 假分数: numerator the same or larger, e.g. $\frac{7}{4}$.
- **mixed number** 带分数: a whole number plus a fraction, e.g. $1\frac{3}{4}$.

Change between improper and mixed: $\frac{7}{4} = 1\frac{3}{4}$ because $7 \div 4 = 1$ remainder 3.

A **decimal** 小数 uses place value after a point. A **percentage** 百分比 means "out of 100", so $37\% = \frac{37}{100} = 0.37$.

Converting between forms

To change	Method	Example
fraction → decimal	divide top by bottom	$\frac{3}{8} = 3 \div 8 = 0.375$
decimal → percentage	multiply by 100	$0.07 = 7\%$
percentage → fraction	put over 100, then simplify	$7\% = \frac{7}{100}$
percentage → decimal	divide by 100	$34\% = 0.34$

Write a fraction in its **simplest form** 最简形式 by dividing the top and bottom by their HCF: $\frac{18}{24} = \frac{3}{4}$ (both divided by 6).

■ Recurring decimals (Extended)

A **recurring decimal** 循环小数 repeats the same digits forever. Dots mark the repeating part: $0.1\dot{7} = 0.1777\ldots$ and $0.\dot{1}2\dot{3} = 0.123123\ldots$

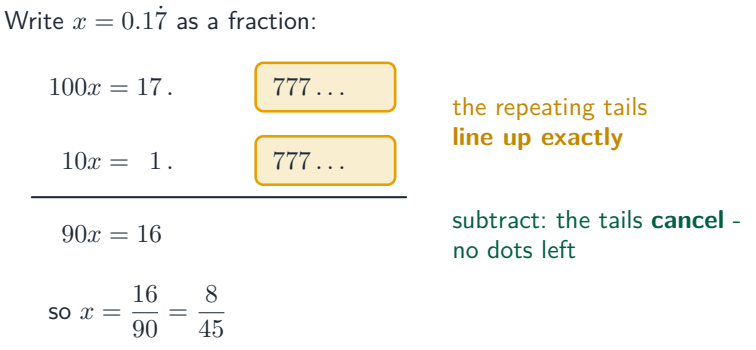
To turn a recurring decimal into a fraction, multiply so the repeating parts line up, then subtract.

Worked example. Write $0.1\dot{7}$ as a fraction.

Let $x = 0.1777\ldots$ Only the 7 repeats, so use $10x$ and $100x$:

$$100x = 17.777\ldots, \qquad 10x = 1.777\ldots$$

$$100x - 10x = 16, \qquad 90x = 16, \qquad x = \frac{16}{90} = \frac{8}{45}.$$



Line up the repeating tails, subtract, and the tails cancel

The four operations

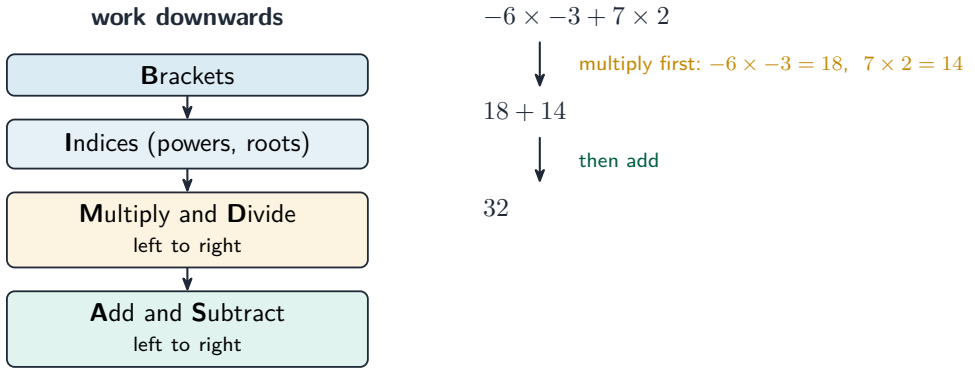
■ Order of operations

Work in this order – the **order of operations** 运算顺序: brackets first, then indices (powers and roots), then multiply and divide (left to right), then add and subtract (left to right).

Worked example. Work out $-6 \times -3 + 7 \times 2$.

Do the multiplications first: $-6 \times -3 = 18$ and $7 \times 2 = 14$. Then add: $18 + 14 = 32$.

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The order of operations, applied to the worked example

■ Negative numbers

- Adding a negative: $5 + (-3) = 5 - 3 = 2$.
- Subtracting a negative: $5 - (-3) = 5 + 3 = 8$.
- Multiplying or dividing: same signs give a positive; different signs give a negative. So $-6 \times -3 = 18$ but $-12 \div 4 = -3$.

A change in temperature from $-5\,°\text{C}$ to $3\,°\text{C}$ is a rise of $8\,°\text{C}$.

■ Calculating with fractions

- **Multiply:** multiply the tops, multiply the bottoms: $\frac{2}{3} \times \frac{4}{5} = \frac{8}{15}$.
- **Divide:** multiply by the reciprocal of the second fraction: $\frac{2}{3} \div \frac{4}{5} = \frac{2}{3} \times \frac{5}{4} = \frac{10}{12} = \frac{5}{6}$.
- **Add or subtract:** use a common denominator (the LCM of the bottoms).

> **Worked example.** Work out $1\frac{7}{15} - \frac{4}{5}$, giving the answer in its simplest form.

Change the mixed number to an improper fraction, then use denominator 15:

$$1\frac{7}{15} = \frac{22}{15}, \qquad \frac{4}{5} = \frac{12}{15}, \qquad \frac{22}{15} - \frac{12}{15} = \frac{10}{15} = \frac{2}{3}.$$

Ordering

Use these symbols to compare numbers by **magnitude** 大小 (size):

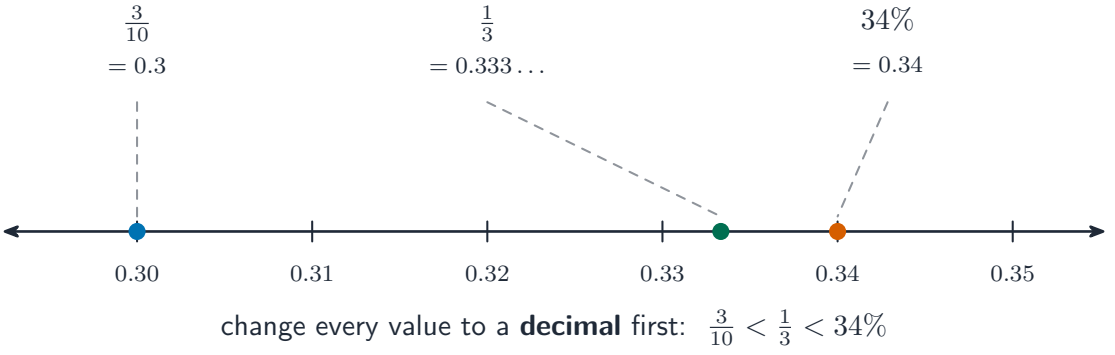
Symbol	Meaning
$=$	is equal to
$\neq$	is not equal to
$>$	is greater than
$<$	is less than
$\geqslant$	is greater than or equal to
$\leqslant$	is less than or equal to

To put a mixed list in order, change every value to a decimal first.

Worked example. Put 34%, $\frac{1}{3}$ and $\frac{3}{10}$ in order, smallest first.

As decimals: $34\% = 0.34$, $\frac{1}{3} = 0.333\ldots$, $\frac{3}{10} = 0.3$. So the order is

$$\frac{3}{10} < \frac{1}{3} < 34\%.$$



Convert every value to a decimal, then place them on a number line

Percentages

Find a percentage of an amount. 15% of \$80 $= 0.15 \times 80 =$ \$12.

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Shops use percentages for discounts, sales tax and profit margins.

Write one amount as a percentage of another. A score of 18 out of 25 is $\frac{18}{25} \times 100\% = 72\%$.

Percentage increase or decrease uses

$$\text{percentage change} = \frac{\text{change}}{\text{original amount}} \times 100\%.$$

> **Worked example.** A price rises from \$40 to \$50. Find the percentage increase.

The change is \$10, so $\frac{10}{40} \times 100\% = 25\%$ increase.

Common mistake. Dividing by the new amount ($\frac{10}{50} = 20\%$). A percentage change is always worked out from the **original** amount.

A quick way is a **multiplier** 乘数. To increase by 15%, multiply by 1.15; to decrease by 15%, multiply by 0.85.

Profit and loss. A shop buys an item at the **cost price** 成本价 and sells it at the **selling price** 售价. The **profit** 利润 is selling price − cost price (a **loss** 亏损 if it is negative). Percentage profit is always worked out from the **cost price**:

$$\text{percentage profit} = \frac{\text{profit}}{\text{cost price}} \times 100\%.$$

> **Worked example.** A trader buys 40 shirts for \$300 in total. She sells 30 of them at \$12 each and the rest at \$6 each in a sale. Find her percentage profit.

Income: $30 \times 12 + 10 \times 6 = 360 + 60 = \$420$. Profit: $420 - 300 = \$120$. Percentage profit: $\dfrac{120}{300} \times 100\% = 40\%$.

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Two changes one after the other. Multiply the multipliers; do not add the percentages. A price rises by 20% and then falls by 20%: $1.2 \times 0.8 = 0.96$, so the final price is 4% **less** than at the start, not the same.

■ Simple and compound interest

Interest 利息 is money paid for borrowing or for saving. The **principal** 本金 is the starting amount.

- **Simple interest** 单利 pays the same amount each year, worked out on the principal only:

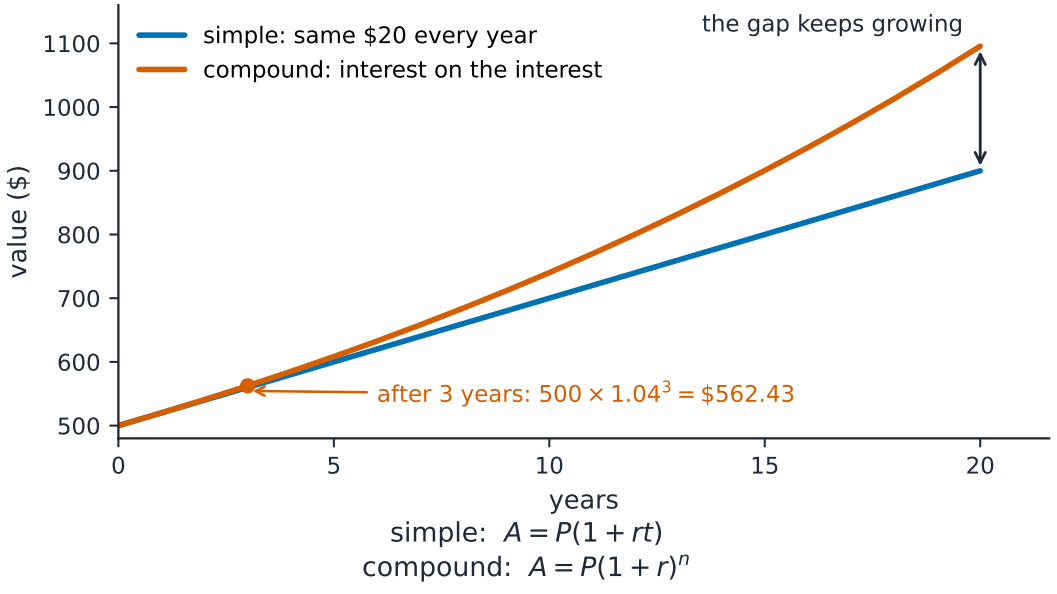
$$I = \frac{P \times r \times t}{100},$$

  where P is the principal, r is the rate per year (as a percentage) and t is the number of years.
- **Compound interest** 复利 adds the interest on each year, so the next year earns interest on a larger total:

$$\text{final value} = P\left(1 + \frac{r}{100}\right)^{t}.$$

Worked example. Find the value of \$500 saved at 4% compound interest for 3 years.

$$500 \times 1.04^3 = 500 \times 1.124864 = \$562.43 \text{ (to the nearest cent).}$$



Simple interest grows in a straight line; compound interest grows faster every year

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Reverse percentages (Extended)

A **reverse percentage** 逆百分比 problem gives the amount **after** a change and asks for the original. To solve it, divide by the multiplier – do not just take the percentage off.

> **Worked example.** A coat costs \$60 after a 20% increase. Find the original price.

\$60 is 120% of the original, so the original price is $60 \div 1.2 = \$50$.

Common mistake. Taking 20% off the new price: $60 \times 0.8 = \$48$ is wrong, because the 20% was a percentage of the **original** price, not of \$60. Check: $50 \times 1.2 = 60$.

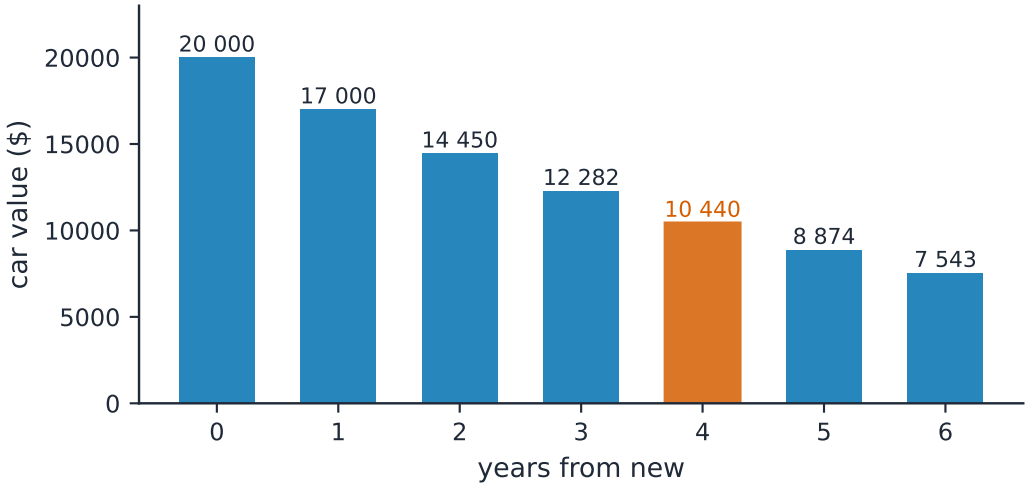
Exponential growth and decay (Extended)

When a quantity changes by the same percentage in each time period, it shows **exponential growth** 指数增长 (it gets bigger) or **exponential decay** 指数衰减 (it gets smaller). Use the compound formula. **Depreciation** 折旧, where something like a car loses value each year, is decay.

> **Worked example.** A car worth \$20 000 loses 15% of its value each year. Find its value after 4 years.

The multiplier is 0.85, so

$$20\,000 \times 0.85^4 = 20\,000 \times 0.522\ldots = \$10\,440 \text{ (to the nearest dollar).}$$



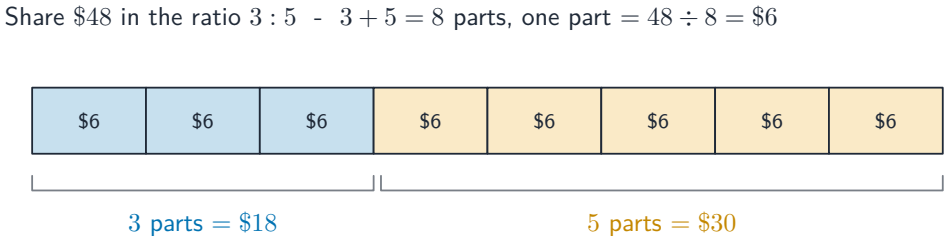
Exponential decay: the car loses 15 percent of its current value every year

Ratio and proportion

A **ratio** 比 compares quantities, written like $a : b$. Simplify it like a fraction by dividing by the HCF: $20 : 30 : 40 = 2 : 3 : 4$.

Dividing in a ratio. Share \$48 in the ratio $3 : 5$.

The total number of parts is $3 + 5 = 8$. One part is $48 \div 8 = \$6$. So the shares are $3 \times 6 = \$18$ and $5 \times 6 = \$30$.



A bar model for sharing 48 dollars in the ratio 3 to 5

> **Worked example (one share known).** Amir and Bella share some sweets in the ratio $2 : 7$. Bella gets 35 more sweets than Amir. How many sweets are there altogether?

The difference is $7 - 2 = 5$ parts, so 5 parts = 35 and 1 part = 7. The total is $2 + 7 = 9$ parts $= 9 \times 7 = 63$ sweets.

Proportion 比例 means two ratios are equal. Use it for recipes, map **scales** 比例尺 and finding best value.

> **Worked example (map scale).** A map has a scale of $1 : 50\,000$. Two villages are 6.4 cm apart on the map. Find the real distance in kilometres.

Real distance $= 6.4 \times 50\,000 = 320\,000$ cm. Divide by 100 for metres (3200 m), then by 1000 for kilometres: 3.2 km.

> **Worked example (best value).** Rice is sold in a 2 kg bag for \$3.40 or a 5 kg bag for \$8.25. Which is better value?

Find the cost of 1 kg of each: $3.40 \div 2 = \$1.70$ and $8.25 \div 5 = \$1.65$. The 5 kg bag is better value, because each kilogram costs less. Always compare the **same** amount, and say which is better in words.

> **Worked example.** 3 pens cost \$1.80. Find the cost of 7 pens.

One pen costs $1.80 \div 3 = \$0.60$. So 7 pens cost $7 \times 0.60 = \$4.20$.

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Rates and average speed

A **rate** 比率 compares two quantities measured in different units, such as price per kilogram, or distance per hour.

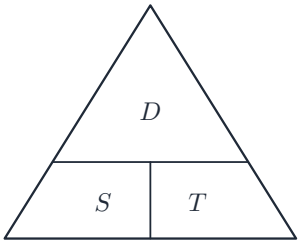
Average speed 平均速度 uses

$$\text{average speed} = \frac{\text{total distance}}{\text{total time}}.$$

> **Worked example.** A cyclist travels 45 km in 3 hours 45 minutes. Find the average speed.

First change the time to hours: 3 h 45 min = 3.75 h. Then

$$\text{average speed} = \frac{45}{3.75} = 12 \text{ km/h}.$$



cover the one you want

$S = \dfrac{D}{T}$ (cover S: D over T)

$D = S \times T$ (cover D: S next to T)

$T = \dfrac{D}{S}$ (cover T: D over S)

cyclist: 45 km in 3 h 45 min = 3.75 h $\Rightarrow$ $S = \dfrac{45}{3.75} = 12$ km/h

The distance-speed-time triangle: cover the one you want

Other rates work the same way. **Density** 密度 is found from **mass** 质量 and **volume** 体积:

$$\text{density} = \frac{\text{mass}}{\text{volume}}.$$

Other examples are flow rate, fuel consumption and **population density** 人口密度. If a rate needs a special formula (such as **pressure** 压强), the question will give it to you.

Rounding and estimation

Rounding

- **decimal places (d.p.)** 小数位: digits counted after the point. 3.14159 to 2 d.p. is 3.14.

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- **significant figures (s.f.)** 有效数字: digits counted from the first non-zero digit. 5764 to 1 s.f. is 6000; 0.004067 to 2 s.f. is 0.0041.

Rule: look at the next digit. If it is 5 or more, round up; if it is less, round down.

■ Estimation

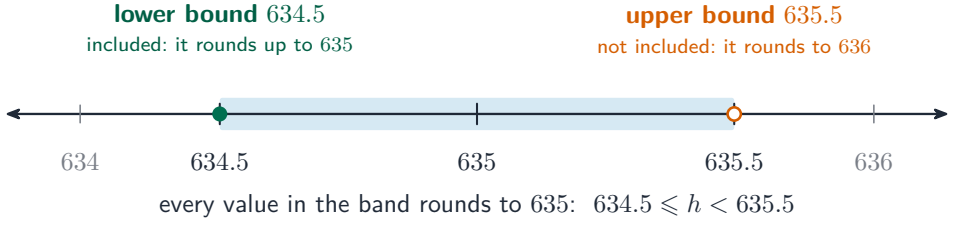
To **estimate** 估算 an answer, round every number to 1 s.f., then calculate.

> **Worked example.** Estimate $\dfrac{41.3}{9.79 \times 0.765}$.
>
> $$\frac{41.3}{9.79 \times 0.765} \approx \frac{40}{10 \times 0.8} = \frac{40}{8} = 5.$$

Limits of accuracy

A rounded value could really be anything that rounds to it. The **lower bound** 下界 and **upper bound** 上界 are the limiting values. An upper bound need not itself be allowed. For a value rounded to the nearest unit, the bounds lie half a unit on each side.

> **Worked example.** A height h is 635 m, correct to the nearest metre. Give the bounds.
>
> $$634.5 \leqslant h < 635.5.$$



Everything in the band rounds to 635: the lower bound is included, the upper bound is not

So the lower bound is 634.5 m and the upper bound is 635.5 m.

■ Bounds in calculations (Extended)

Combine the bounds to get the bound you want.

> **Worked example.** A rectangle is 8 cm by 5 cm, each side to the nearest cm. Find the upper bound for its **area** 面积.

Use the upper bounds of both sides: $8.5 \times 5.5 = 46.75$ cm^2. (The smallest area uses the lower bounds: $7.5 \times 4.5 = 33.75$ cm^2.)

For a divided quantity such as speed $= \dfrac{\text{distance}}{\text{time}}$, the **largest** speed comes from the **largest** distance divided by the **smallest** time.

Common mistakes. Writing the upper bound as 635.4 or 635.49 (it is 635.5, even though 635.5 itself would round up), and using the upper bound of the time to get the largest speed. Dividing by a **smaller** number gives a **bigger** answer.

Time

- 60 seconds = 1 minute, 60 minutes = 1 hour, 24 hours = 1 day, and 1 year = 365 days.
- The 24-hour clock writes a time as four digits: 3.15 p.m. is 15 15.

> **Worked example.** A film starts at 19 35 and lasts 70 minutes. Find the time it finishes.

70 min = 1 h 10 min. Adding 1 hour gives 20 35; adding 10 minutes gives 20 45.

For timetables and **time zone** 时区 problems, add or subtract the time difference between the places.

> **Worked example (time zones).** A flight leaves Singapore at 23 50 local time and lands in London 13 hours 25 minutes later. London time is 8 hours **behind** Singapore. Find the local time in London when it lands.

Landing time in Singapore time: 23 50 + 13 h 25 min = 13 15 the next day. London is 8 hours behind, so the local time in London is 13 15 − 8 h = 05 15 (the next day). Change the time zone **once**, at the end, so you do not mix the two clocks.

Money

Work with money as ordinary decimals, but give answers to 2 d.p. (so \$4.8 is written \$4.80).

Currency conversion. Use the **exchange rate** 汇率 as a multiplier.

> **Worked example.** The exchange rate is \$1 = €0.92. Convert \$150 to euros, and convert €138 back to dollars.

$$150 \times 0.92 = €138, \qquad 138 \div 0.92 = \$150.$$

Using a calculator

- Do not round part-way through a calculation. Keep the full value and round only the final answer.
- Enter time as a decimal of an hour: 2 hours 30 minutes is 2.5 hours, not 2.30.
- Read the display in context: in money, 4.8 means \$4.80; in time, 3.25 hours means 3 hours 15 minutes.

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Exam tips

- Follow **BIDMAS** (brackets, indices, division/multiplication, addition/subtraction) in order, and remember a negative times a negative is positive.
- In **standard form** the number in front is between 1 and 10; a small number (like 0.0004) has a *negative* power of 10.
- A percentage *change* is worked out on the **original** amount. For a reverse percentage, divide by the multiplier (e.g. $\div 1.2$ undoes a 20% rise).
- Do not round part-way through – keep the full value and round only at the end, to the accuracy the question asks for (decimal places or significant figures).
- For **limits of accuracy**, a value rounded to the nearest whole number can be up to 0.5 either side (so 8 means $7.5 \leq x < 8.5$).

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Algebra and graphs

IGCSE Mathematics

This handout covers Topic 2, Algebra and graphs. Parts marked **(Extended)** are only tested on the Extended papers; everything else is for both levels.

Working with algebra

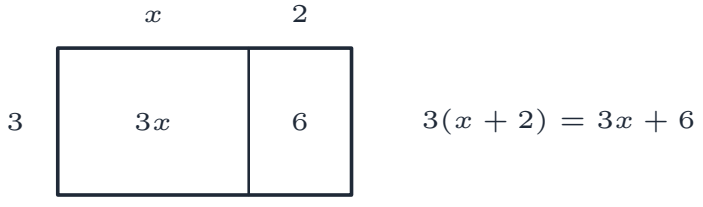
In **algebra** 代数 we use letters to stand for numbers. A letter whose value can change is a **variable** 变量. To **substitute** 代入 means to put a number in place of a letter.

Worked example. Find the value of $3x^2 - 2y$ when $x = 4$ and $y = 5$.

$$3 \times 4^2 - 2 \times 5 = 3 \times 16 - 10 = 48 - 10 = 38.$$

Simplifying and expanding

A **term** 项 is a single part of an **expression** 表达式, such as $5a$ or $-9b$. **Like terms** 同类项 have the same letters raised to the same powers; you may add or subtract them. The number in front of the letter is the **coefficient** 系数.



Expanding a bracket with an area model

$$3a + 2a + 4b - b$$

$$\downarrow \text{ collect like terms}$$

$$5a + 3b$$

Collecting like terms: add terms with the same letter

Worked example. Simplify $2a^2 + 3ab - 1 + 5a^2 - 9ab + 4$.

Collect like terms: $2a^2 + 5a^2 = 7a^2$, $3ab - 9ab = -6ab$, $-1 + 4 = 3$. So the answer is

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$$7a^2 - 6ab + 3.$$

To **expand** 展开 means to multiply out **brackets** 括号. Multiply every term inside by the term outside; for two brackets, multiply every term in the first by every term in the second.

Worked examples.

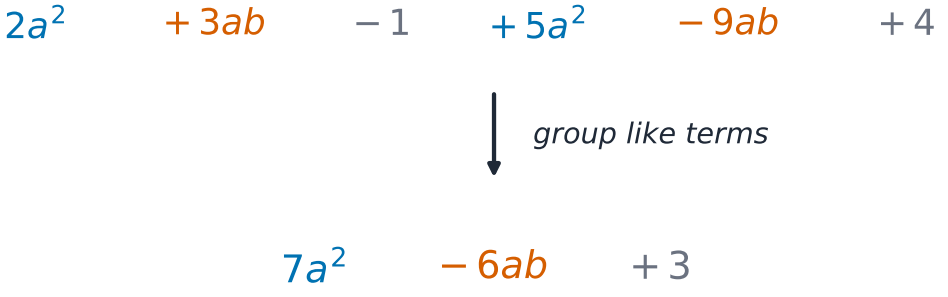
$$3x(2x - 4y) = 6x^2 - 12xy.$$

$$(2x + 1)(x - 4) = 2x^2 - 8x + x - 4 = 2x^2 - 7x - 4.$$

Common mistakes. Multiplying only the first term inside a bracket ($3(x + 4) = 3x + 12$, not $3x + 4$), losing a minus sign ($-2(x - 5) = -2x + 10$), and writing $(x + 3)^2 = x^2 + 9$. Squaring a bracket gives three terms: $(x + 3)^2 = x^2 + 6x + 9$.

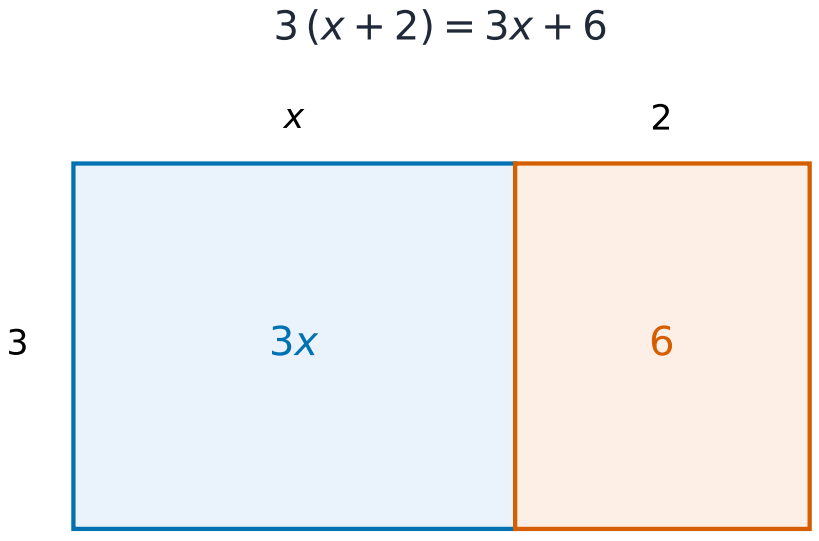
For three brackets (**Extended**), expand two first, then multiply by the third:

$$(x - 2)(x + 3)(2x + 1) = (x^2 + x - 6)(2x + 1) = 2x^3 + 3x^2 - 11x - 6.$$



Like terms share the same letters and powers – add coefficients only

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Expand by distributing: a(b+c)=ab+ac (area model)

Factorising

To **factorise** 因式分解 is the opposite of expanding: write the expression as a product of brackets. Always take out the **common factor** 公因式 first.

> **Worked example.** $9x^2 + 15xy = 3x(3x + 5y)$, because $3x$ divides both terms.

The following patterns are **Extended**.

Grouping (four terms): take a common factor from each pair.

$$xy + 2x + 3y + 6 = x(y + 2) + 3(y + 2) = (x + 3)(y + 2).$$

Difference of two squares 平方差: $a^2 - b^2 = (a + b)(a - b)$.

$$9x^2 - 16 = (3x + 4)(3x - 4).$$

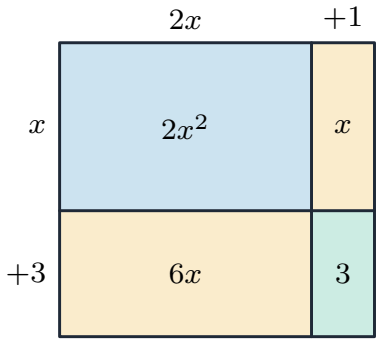
Perfect square 完全平方: $a^2 + 2ab + b^2 = (a + b)^2$.

$$x^2 + 6x + 9 = (x + 3)^2.$$

Quadratic 二次 expressions $ax^2 + bx + c$: find two numbers that multiply to $a \times c$ and add to b, then split the middle term.

> **Worked example.** Factorise $2x^2 + 7x + 3$.

Here $a \times c = 6$ and $b = 7$. The numbers 1 and 6 work. Split and group:



$a \times c = 2 \times 3 = 6$

$1 \times 6 = 6$ and $1 + 6 = 7$

split $7x$ into $x + 6x$

$(x + 3)(2x + 1)$

Splitting the middle term: 1 and 6 multiply to 6 and add to 7, so 7x becomes x + 6x and the rectangle has sides (x + 3) and (2x + 1)

$$2x^2 + x + 6x + 3 = x(2x + 1) + 3(2x + 1) = (x + 3)(2x + 1).$$

For $ax^3 + bx^2 + cx$, take out the common x first: $2x^3 + 7x^2 + 3x = x(2x^2 + 7x + 3) = x(x + 3)(2x + 1)$.

Completing the square (Extended)



A suspension bridge cable hangs in a parabola – the graph of a quadratic.

Completing the square 配方法 rewrites $x^2 + bx + c$ as $(x + p)^2 + q$. Take half of the x-coefficient, square it, then balance.

> **Worked example.** Write $x^2 + 6x + 1$ in completed-square form.

Half of 6 is 3, and $3^2 = 9$:

$$x^2 + 6x + 1 = (x+3)^2 - 9 + 1 = (x+3)^2 - 8.$$

When the x^2 has a coefficient, take it out of the first two terms first:

$$2x^2 + 8x + 3 = 2(x^2 + 4x) + 3 = 2\big((x+2)^2 - 4\big) + 3 = 2(x+2)^2 - 5.$$

Algebraic fractions (Extended)

An **algebraic fraction** 分式 has algebra on the top or bottom. Add and subtract using a common denominator; multiply and divide as with ordinary fractions.

Worked examples.

$$\frac{x}{3} + \frac{x-4}{2} = \frac{2x}{6} + \frac{3(x-4)}{6} = \frac{2x + 3x - 12}{6} = \frac{5x - 12}{6}.$$

$$\frac{3a}{4} \div \frac{9a}{10} = \frac{3a}{4} \times \frac{10}{9a} = \frac{30a}{36a} = \frac{5}{6}.$$

To simplify a **rational expression** 有理式, factorise the top and bottom, then cancel common brackets.

$$\frac{x^2 - 2x}{x^2 - 5x + 6} = \frac{x(x-2)}{(x-2)(x-3)} = \frac{x}{x-3}.$$

Indices in algebra

The laws of **indices** 指数 work with letters too: $a^m \times a^n = a^{m+n}$, $a^m \div a^n = a^{m-n}$, and $(a^m)^n = a^{mn}$.

Worked examples.

$$(5x^3)^2 = 25x^6, \qquad 12a^5 \div 3a^{-2} = 4a^7, \qquad 6x^7y^4 \times 5x^{-5}y = 30x^2y^5.$$

You can also solve simple index equations by writing both sides with the same **base** 底数.

> **Worked example.** Solve $2^x = 32$. Since $32 = 2^5$, you get $x = 5$.

Equations

An **equation** 方程 says two expressions are equal. To solve a **linear** 一次 equation, do the same operation to both sides until the **unknown** 未知数 is alone.

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$$2x + 3 = 11 \qquad \text{(start)}$$

$$2x = 8 \qquad \text{subtract 3 from both sides}$$

$$x = 4 \qquad \text{divide both sides by 2}$$

Solve by doing the same to both sides

Worked example. Solve $5 - 2x = 3(x + 7)$.

$$5 - 2x = 3x + 21 \;\Rightarrow\; 5 - 21 = 3x + 2x \;\Rightarrow\; -16 = 5x \;\Rightarrow\; x = -\tfrac{16}{5}.$$

Common mistake. Moving a term across the equals sign without changing its sign. Do the same thing to both sides instead: "subtract 21 from both sides" makes the sign change automatic.

■ Fractional equations (Extended)

A **fractional equation** 分式方程 has the unknown in a denominator. Multiply both sides by the denominator to clear it.

Worked example. Solve $\dfrac{x}{2x + 1} = 4$.

$$x = 4(2x + 1) = 8x + 4 \;\Rightarrow\; -7x = 4 \;\Rightarrow\; x = -\tfrac{4}{7}.$$

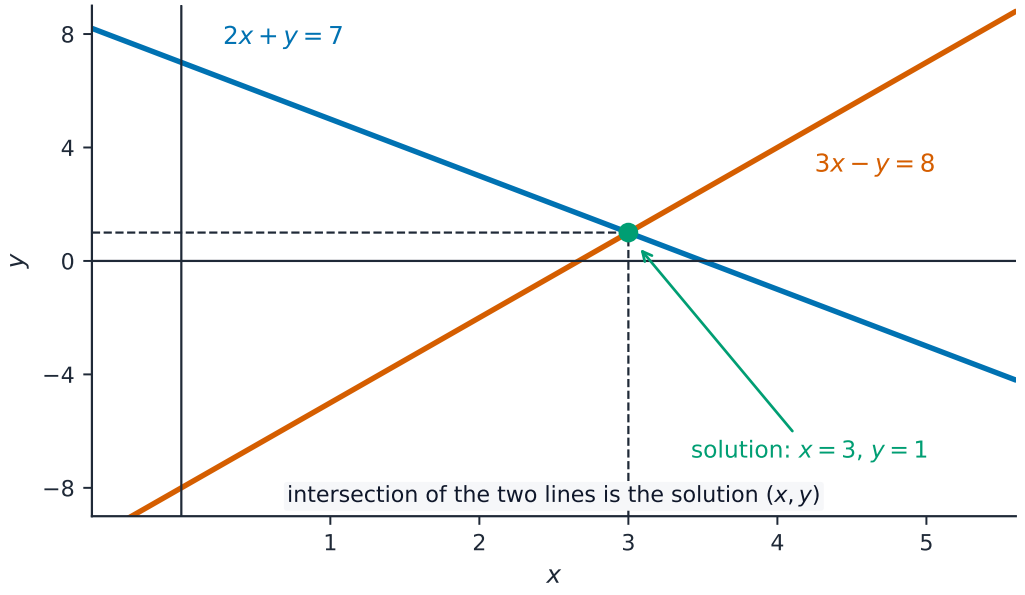
■ Simultaneous equations

Simultaneous equations 联立方程 are two equations solved together. For two linear equations, add or subtract to remove one letter.

Worked example. Solve $2x + y = 7$ and $3x - y = 8$.

Adding removes y: $5x = 15$, so $x = 3$. Then $y = 7 - 2(3) = 1$.

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The solution of simultaneous equations is where their graphs cross

For one linear and one **quadratic** equation (**Extended**), substitute the linear into the curve.

Worked example. Solve $y = x + 2$ and $y = x^2$.

$$x^2 = x + 2 \;\Rightarrow\; x^2 - x - 2 = 0 \;\Rightarrow\; (x-2)(x+1) = 0,$$

so $x = 2$ (giving $y = 4$) or $x = -1$ (giving $y = 1$).

■ Solving quadratic equations (Extended)

There are three methods.

- **By factorising:** $x^2 + 5x + 6 = 0 \Rightarrow (x+2)(x+3) = 0 \Rightarrow x = -2$ or $x = -3$.
- **By completing the square:** $x^2 + 6x + 1 = 0 \Rightarrow (x+3)^2 = 8 \Rightarrow x + 3 = \pm 2\sqrt{2} \Rightarrow x = -3 \pm 2\sqrt{2}$.
- **By the quadratic formula** 求根公式, which is given in the exam:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Worked example (formula). Solve $2x^2 + 3x - 1 = 0$. Here $a = 2$, $b = 3$, $c = -1$:

$$x = \frac{-3 \pm \sqrt{9 + 8}}{4} = \frac{-3 \pm \sqrt{17}}{4}.$$

■ Changing the subject

To **change the subject** 公式变形 of a formula means to rearrange it so a chosen letter is alone on one side.

> **Worked example.** Make r the subject of $A = \pi r^2$ (**Extended**, because of the power).

$$r^2 = \frac{A}{\pi} \;\Rightarrow\; r = \sqrt{\frac{A}{\pi}}.$$

When the letter appears twice (**Extended**), collect those terms and factorise. To make x the subject of $y = \dfrac{x+1}{x-1}$:

$$y(x-1) = x+1 \;\Rightarrow\; yx - x = 1 + y \;\Rightarrow\; x(y-1) = 1+y \;\Rightarrow\; x = \frac{1+y}{y-1}.$$

Inequalities

An **inequality** 不等式 uses $<$, $>$, $\leqslant$ or $\geqslant$. Solve it like an equation, but **reverse the sign** if you multiply or divide by a negative number.

> **Worked example.** Solve $-3 \leqslant 3x - 2 < 7$.

Add 2 to all parts, then divide by 3:

$$-1 \leqslant 3x < 9 \;\Rightarrow\; -\tfrac{1}{3} \leqslant x < 3.$$

> **Worked example.** Solve $5 - 2x > 11$.

$$-2x > 6 \;\Rightarrow\; x < -3.$$

Dividing by -2 reverses the sign. Check with $x = -4$: $5 + 8 = 13 > 11$, which is correct.

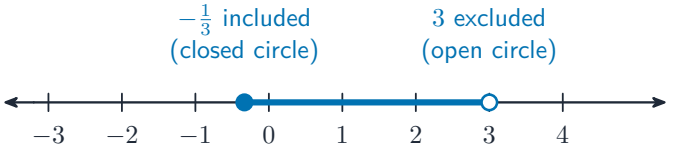
Common mistake. Forgetting to reverse the sign when dividing by a negative number, which here would give the wrong answer $x > -3$.

> **Worked example (integer solutions).** List the integers n that satisfy $-3 \leqslant 2n + 1 < 9$.

Subtract 1: $-4 \leqslant 2n < 8$. Divide by 2: $-2 \leqslant n < 4$. So $n = -2, -1, 0, 1, 2, 3$. The end -2 is included ($\leqslant$), but 4 is not ($<$).

On a **number line** 数轴, use an open circle for $<$ or $>$ (value not included) and a closed circle for $\leqslant$ or $\geqslant$ (value included).

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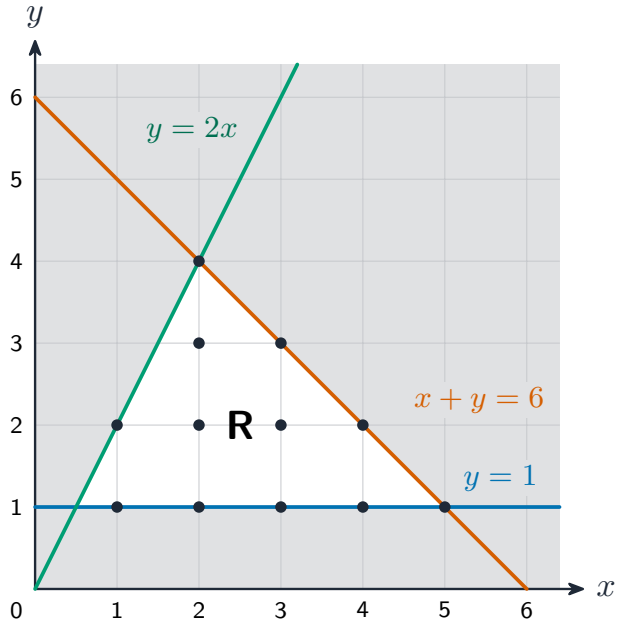
$-\frac{1}{3} \leqslant x < 3$: a closed circle includes the end value, an open circle excludes it.

■ Regions (Extended)

An inequality in two letters describes a **region** 区域 of the graph. Draw the boundary line (broken for $<$ or $>$, solid for $\leqslant$ or $\geqslant$) and shade the unwanted side. You may also be asked to list the inequalities that define a given region.

> **Worked example.** On a grid, show the region R that satisfies $y \geqslant 1$, $x + y \leqslant 6$ and $y \leqslant 2x$. How many points in R have whole-number coordinates?

Draw each boundary as a solid line, because every sign includes "equal to". Find the wanted side with a test point that is not on the line, such as $(3, 2)$: $2 \geqslant 1$, $3+2 = 5 \leqslant 6$ and $2 \leqslant 6$ are all true, so $(3, 2)$ is in R. Shade the other side of each line.



The region R: $y \geq 1$, $x + y \leq 6$ and $y \leq 2x$, with the unwanted side of each line shaded; the dots are the 12 points in R with whole-number coordinates

Count row by row, including the points on the lines (all three signs include equality): 5 points on $y = 1$, 4 on $y = 2$, 2 on $y = 3$ and 1 on $y = 4$, so 12 points. If a sign were $<$ or $>$, the line would be broken and points on it would **not** count.

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Romanesco broccoli: self-similar spirals form a natural number pattern.

A **sequence** 数列 is a list of numbers that follow a rule. The **term-to-term rule** 递推规则 tells you how to get the next term from the one before.

To find the rule for the term in position n (the nth **term**), look at how the terms change.

Linear sequence (the terms go up by the same amount each time). The difference is the multiple of n.

> **Worked example.** Find the nth term of $2, 5, 8, 11, \ldots$

The terms go up by 3, so start with $3n$. Since $3 \times 1 = 3$ but the first term is 2, subtract 1: the nth term is $3n - 1$.

Quadratic sequence (the differences themselves change by the same amount). The second difference equals $2\times$ the coefficient of n^2.

> **Worked example.** Find the nth term of $2, 5, 10, 17, \ldots$

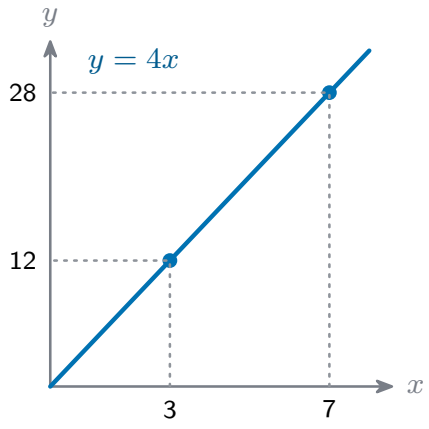
First differences are $3, 5, 7$; the second difference is 2, so the n^2 part is $1n^2$. Subtracting n^2 $(1, 4, 9, 16)$ from the sequence leaves $1, 1, 1, 1$. So the nth term is $n^2 + 1$.

A **cubic** 三次 sequence such as $1, 8, 27, 64, \ldots$ has nth term n^3. An **exponential sequence** 指数数列 such as $2, 6, 18, 54, \ldots$ multiplies by a fixed number each time; here the nth term is $2 \times 3^{n-1}$.

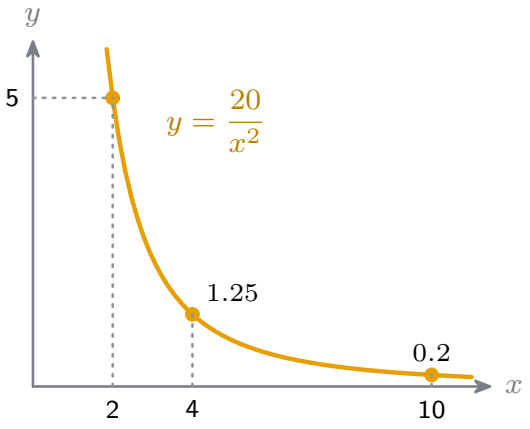
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Direct and inverse proportion (Extended)

Two quantities are in **direct proportion** 正比例 if one is always a fixed multiple of the other: $y \propto x$ means $y = kx$, where k is a **constant** 常数. They are in **inverse proportion** 反比例 if their product is constant: $y \propto \frac{1}{x}$ means $y = \frac{k}{x}$. The symbol $\propto$ is read "is proportional to". You can also have proportion to a square, square root, cube or cube root.



direct: a straight line through the origin



inverse square: double x, y is divided by 4

Direct proportion is a straight line through the origin; for inverse-square proportion, doubling x from 2 to 4 divides y by 4, from 5 to 1.25

> **Worked example.** y is in direct proportion to x, and $y = 12$ when $x = 3$. Find y when $x = 7$.

First find k: $12 = k \times 3$, so $k = 4$ and $y = 4x$. Then $y = 4 \times 7 = 28$.

> **Worked example (inverse square).** y is inversely proportional to x^2, and $y = 5$ when $x = 2$. Find y when $x = 4$, and find x when $y = 0.2$ ($x > 0$).

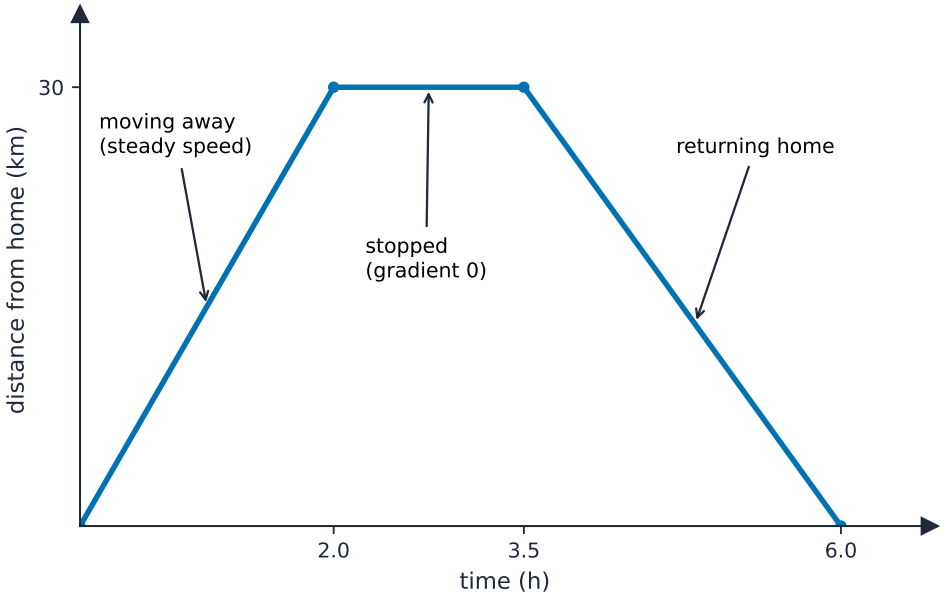
Write $y = \frac{k}{x^2}$. Then $5 = \frac{k}{4}$, so $k = 20$ and $y = \frac{20}{x^2}$. When $x = 4$: $y = 20 \div 16 = 1.25$. When $y = 0.2$: $x^2 = 20 \div 0.2 = 100$, so $x = 10$. Check the pattern: doubling x divides y by $2^2 = 4$, and $5 \div 4 = 1.25$.

Graphs in practical situations

The **gradient** 斜率 (steepness) of a graph shows a **rate of change** 变化率.

- A **travel graph** 行程图 (distance–time graph) has gradient equal to speed; a flat part means the object is not moving.

- A **conversion graph** 换算图 is a straight line used to change between two units (for example, miles and kilometres).



On a distance–time graph the gradient is the speed; a flat section means the object has stopped.

> **Worked example (a journey).** Lena cycles 12 km from home to a lake in 40 minutes, rests there for 20 minutes, then cycles home in 30 minutes. Find her speed on each part in km/h, and her average speed for the whole journey.

Change minutes to hours first: 40 min $= \frac{2}{3}$ h and 30 min $= \frac{1}{2}$ h. Going: $12 \div \frac{2}{3} = 18$ km/h. Resting: 0 km/h (a flat line). Returning: $12 \div \frac{1}{2} = 24$ km/h; the line slopes **down** and is steeper. Average speed = total distance ÷ total time $= 24 \div 1.5 = 16$ km/h. The rest counts in the total time.

■ Speed–time graphs (Extended)

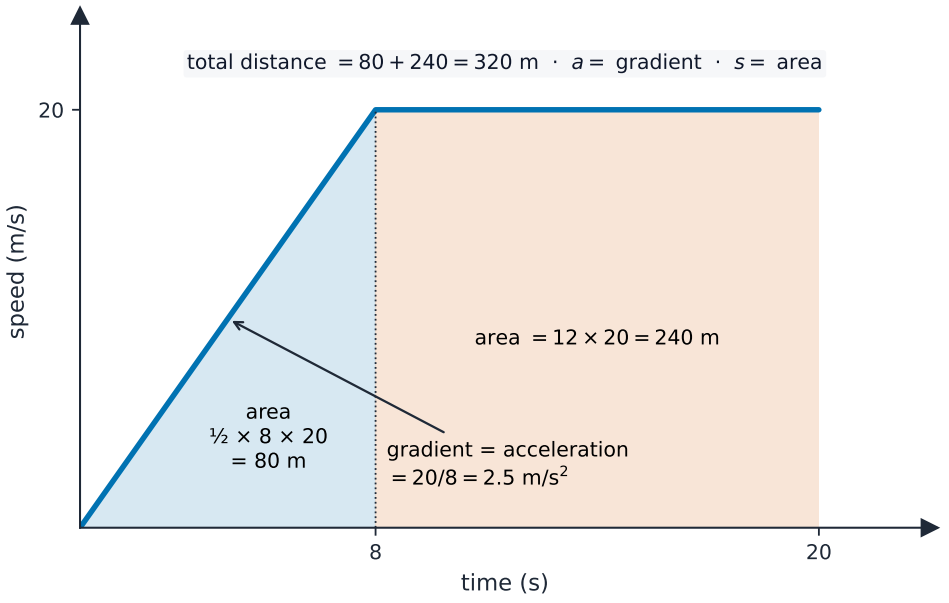
On a speed–time graph the gradient is the **acceleration** 加速度 (or **deceleration** 减速度 if the speed falls), and the **area** 面积 under the graph is the **distance** 距离 travelled.

> **Worked example.** A car speeds up from rest to 20 m/s in 8 s, then stays at 20 m/s for 12 s. Find the acceleration and the total distance.

Acceleration $= \dfrac{20}{8} = 2.5$ m/s^2. The distance is the area: a triangle plus a rectangle,

$$\tfrac{1}{2} \times 8 \times 20 + 12 \times 20 = 80 + 240 = 320 \text{ m}.$$

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On a speed–time graph the gradient is the acceleration and the area underneath is the distance travelled.

Graphs of functions

To draw a graph, make a **table of values** 数值表: choose values of x, work out y, then plot the **coordinates** 坐标 and join them with a smooth curve.

The points where a graph crosses the x-axis (the horizontal **axis** 坐标轴) are the **roots** 根 – the solutions of $y = 0$.

You can solve an equation by reading a graph. The **intersection point** 交点 of a line and a curve gives the solution of the two equations together.

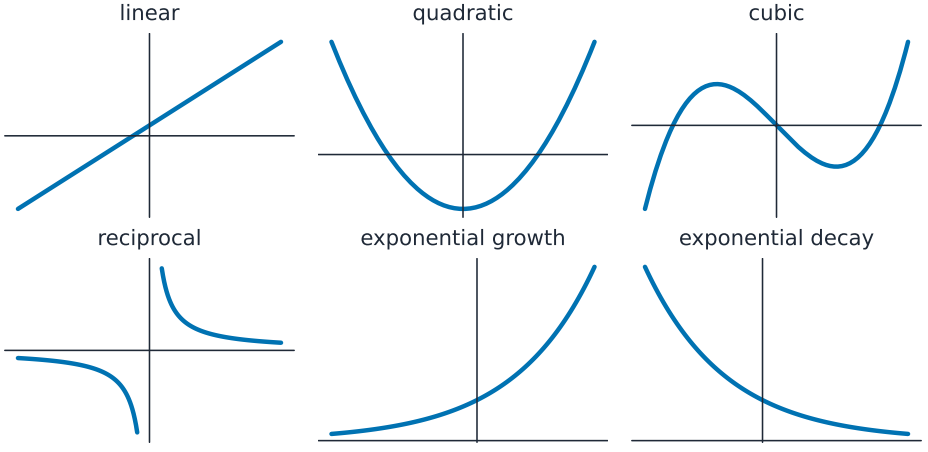
For **exponential growth** 指数增长 and **exponential decay** 指数衰减, consider $y = a\,b^x + c$ with $a > 0$ and $b > 0$, $b \neq 1$. If $b > 1$, the graph grows as x increases and approaches $y = c$ as x decreases. If $0 < b < 1$, it decays towards $y = c$ as x increases.

Sketching curves

A quick sketch should show the right shape and the key features: where it crosses the axes, any **symmetry** 对称, and any line the curve gets close to.

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Function	Shape
linear, $y = mx + c$	straight line; gradient m, y-**intercept** 截距 c
quadratic, $y = ax^2 + bx + c$	a **parabola** 抛物线 (U-shape if $a > 0$, ∩-shape if $a < 0$)
cubic, $y = ax^3 + bx + c$	an S-shaped curve
reciprocal, $y = \dfrac{a}{x} + b$	two separate curves
exponential, $y = a\,r^x + b$	fast growth or decay

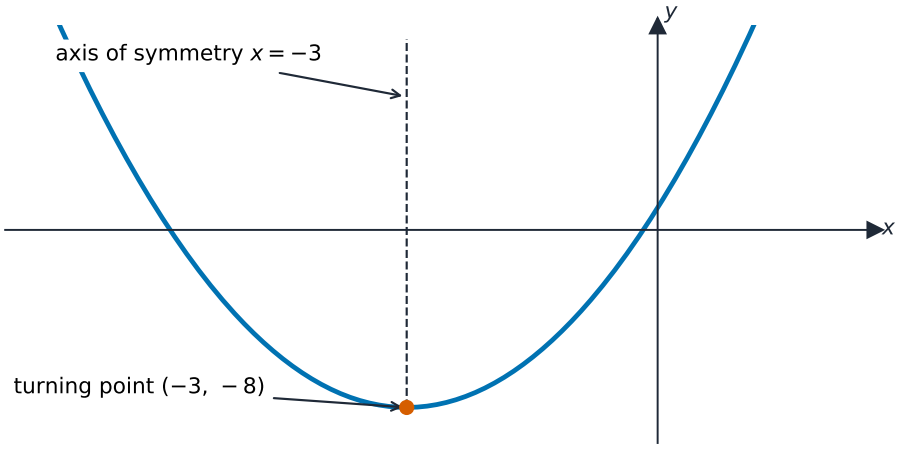


recognise $y = x^2$, $y = 1/x$, $y = 2^x$ from shape

The basic graph shapes; knowing each shape lets you sketch quickly from the equation.

For a parabola, completing the square gives the **turning point** 转折点 (the lowest or highest point). For example $y = (x + 3)^2 - 8$ has its turning point at $(-3, -8)$.

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$$y = (x + 3)^2 - 8$$

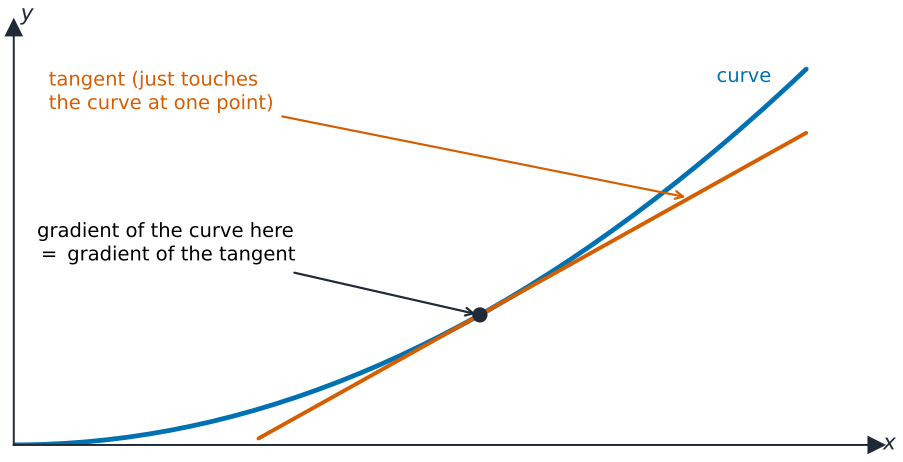
Completing the square, $y = (x + 3)^2 - 8$, shows the turning point $(-3, -8)$ and the line of symmetry $x = -3$.

An **asymptote** 渐近线 is a line that the curve approaches in a stated limit. A curve may cross a horizontal asymptote elsewhere. In these examples the curve does not meet the stated line – for example, the x-axis for $y = \dfrac{a}{x}$, or the line $y = b$ for $y = a\,r^x + b$.

Differentiation (Extended)

Differentiation 微分 finds the gradient of a curve where that gradient exists. You can estimate it by drawing a **tangent** 切线 (a line that just touches the curve) and measuring its gradient.

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$$\frac{dy}{dx} = \text{gradient of the tangent}$$

The gradient of a curve at a point equals the gradient of the tangent there – what differentiation finds.

The exact rule: if $y = ax^n$, then the **derivative** 导数 is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = a\,n\,x^{\,n-1}.$$

Differentiate a sum term by term.

> **Worked example.** If $y = x^3 + 2x^2 - 5x$, then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 + 4x - 5$.

A **stationary point** 驻点 is where the gradient is zero. It need not be a turning point: $y = x^3$ keeps increasing through its stationary point at the origin. To find stationary points, set $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$.

> **Worked example.** Find the turning point of $y = x^2 - 6x + 5$.
>
> $$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x - 6 = 0 \;\Rightarrow\; x = 3, \quad y = 3^2 - 6(3) + 5 = -4.$$
>
> The turning point is $(3, -4)$. To decide whether a turning point is a **maximum** 最大值 or a **minimum** 最小值, check the sign of the gradient on each side, or use the second derivative (positive means a minimum).

> **Worked example (a cubic).** For $y = x^3 - 3x^2 - 9x + 2$, find the gradient at $x = 2$, then find the turning points and decide which is a maximum.
>
> $$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 6x - 9.$$

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At $x = 2$: $3(4)-12-9 = -9$, so the gradient is -9. For turning points, $3x^2-6x-9 = 0$, so $x^2-2x-3 = 0$ and $(x-3)(x+1) = 0$: $x = 3$ or $x = -1$. Then $y = 27-27-27+2 = -25$ and $y = -1-3+9+2 = 7$. The second derivative is $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6x-6$: at $x = 3$ it is $12 > 0$, so $(3, -25)$ is a **minimum**; at $x = -1$ it is $-12 < 0$, so $(-1, 7)$ is a **maximum**.

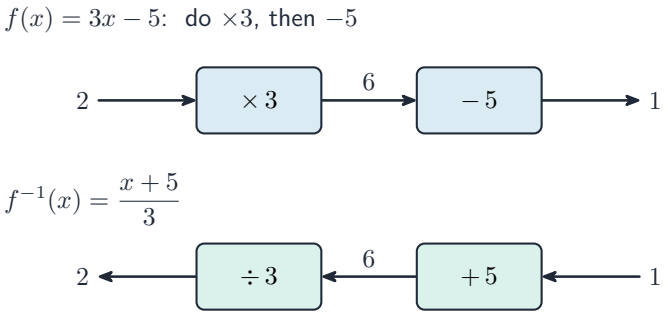
Functions (Extended)

A **function** 函数 turns each input into one output. We write $f(x)$, for example $f(x) = 3x - 5$, so $f(2) = 1$. The set of allowed inputs is the **domain** 定义域; the set of possible outputs is the **range** 值域.

The **inverse function** 反函数 $f^{-1}(x)$ undoes the function. Each output must come from only one allowed input; restrict the original domain if necessary. To find it, write $y = f(x)$, swap the roles, and make x the subject.

Worked example. Find the inverse of $f(x) = 3x - 5$.

$$y = 3x - 5 \;\Rightarrow\; x = \frac{y+5}{3}, \quad \text{so} \quad f^{-1}(x) = \frac{x+5}{3}.$$



A function as a machine; the inverse runs it backwards - reverse the order, undo each step

A **composite function** 复合函数 applies one function after another: $gf(x)$ means "do f first, then g".

Worked example. If $f(x) = 2x$ and $g(x) = x + 3$, then

$$gf(x) = g(2x) = 2x + 3, \qquad fg(x) = f(x + 3) = 2(x + 3) = 2x + 6.$$

Common mistakes. Doing the outside function first ($gf(x)$ starts with f, the one

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nearest to x), and confusing $f^{-1}(x)$ with $\dfrac{1}{f(x)}$. For $f(x) = 3x - 5$, $f^{-1}(1) = 2$ but $\dfrac{1}{f(1)} = -\dfrac{1}{2}$.

Exam tips

- When you **expand brackets**, multiply every term and watch the signs, especially with a minus in front: $-(x - 3) = -x + 3$.
- To solve an equation, do the same thing to both sides. When you multiply or divide an **inequality** by a negative number, *flip* the sign.
- **Factorise fully**: take out the highest common factor first, then look for a difference of two squares or a quadratic pattern.
- A **quadratic** usually has two solutions – give both. Check whether the question wants factorising, the formula, or completing the square.
- When substituting into a formula, put each value in brackets first, so signs and powers come out right.

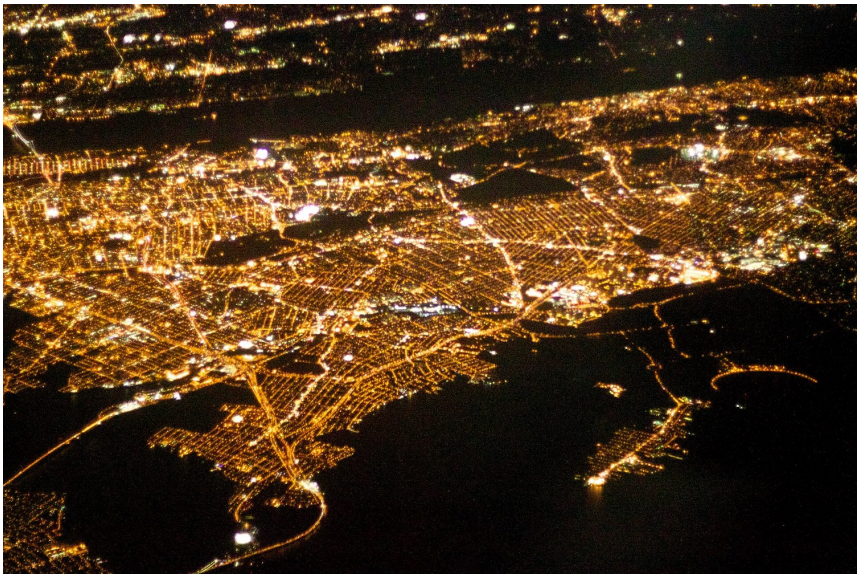
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Coordinate geometry

IGCSE Mathematics

This handout covers Topic 3, Coordinate geometry. Parts marked **(Extended)** are only tested on the Extended papers; everything else is for both levels.

Coordinates

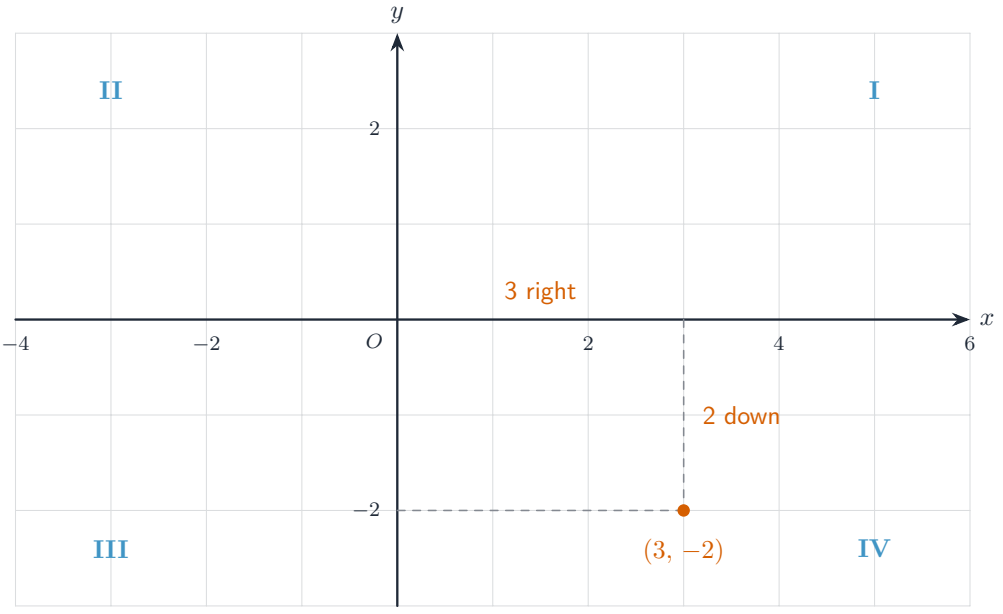


A city street grid: every place is fixed by its coordinates.

A point on a graph is described by its **coordinates** 坐标, sometimes called Cartesian coordinates, written (x, y). The first number is the across value and the second is the up value.

- The two number lines are the **axes** 坐标轴: the **horizontal** 水平 x-axis and the **vertical** 竖直 y-axis.
- They cross at the **origin** 原点, the point $(0, 0)$.
- The axes split the grid into four **quadrants** 象限.

So the point $(3, -2)$ is found by going 3 to the right and 2 down.



The axes meet at the origin O and split the plane into four quadrants; $(3, -2)$ means 3 right then 2 down.

The equation of a straight line

Every non-vertical straight line can be written as

$$y = mx + c,$$

where m is the **gradient** 斜率 (steepness) and c is the **intercept** 截距 – the y-value where the line crosses the y-axis.

- A line like $x = k$ (for example $x = 3$) is **vertical**.
- A line like $y = k$ (for example $y = 3$) is **horizontal**.

Extended: a line may also be given as $ax + by = c$. Rearrange it into $y = mx + c$ to read off the gradient and intercept.

> **Worked example (Extended).** Find the gradient and y-intercept of $5x + 4y = 8$.

$$4y = -5x + 8 \;\Rightarrow\; y = -\tfrac{5}{4}x + 2.$$

So the gradient is $-\frac{5}{4}$ and the y-intercept is 2.

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Gradient



A steep mountain road: gradient measures steepness as rise over run.

The **gradient** measures how steep a line is:

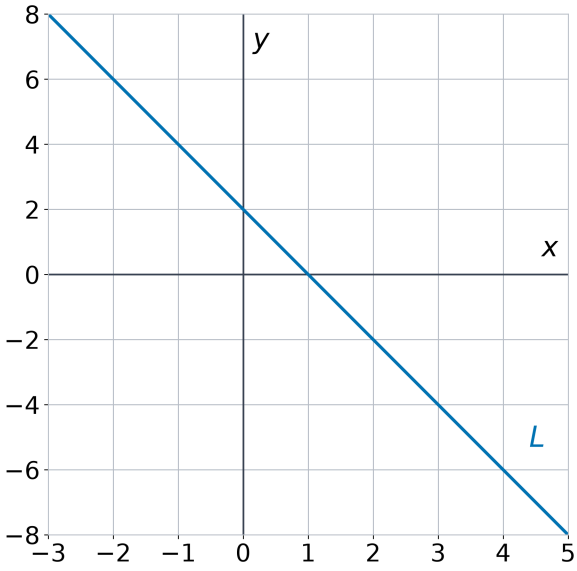
$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{\text{rise}}{\text{run}}.$$

A positive gradient goes up to the right; a negative gradient goes down to the right. Horizontal lines have gradient zero; vertical lines have undefined gradient because the horizontal change is zero.

Core and Extended: from a grid. Choose two clear intersections on the drawn line, preferably far apart. Draw a right-angled triangle and read its horizontal and signed vertical changes in axis units. Check both numbered scales: counting squares alone may be wrong. Core gradient questions use a grid; calculation from two given coordinate pairs without a grid is an Extended demand.

Independent grid task. Use L below. Draw a large gradient triangle, read its changes in axis units and calculate the gradient. Then read the y-intercept and find the simplified equation of the line. Check a different grid point. Explain the trap caused by the two scales.

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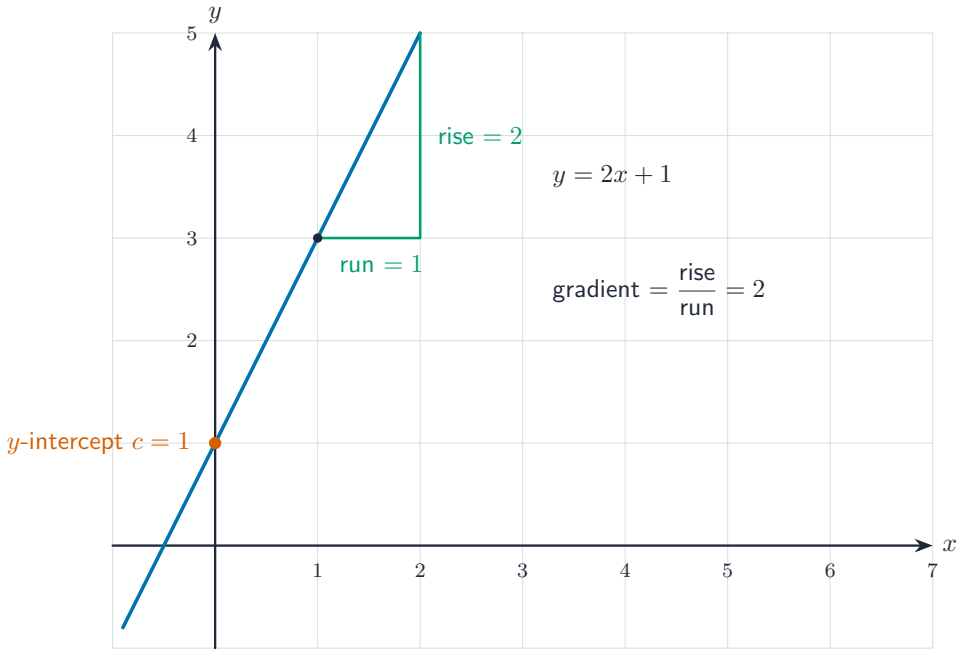
Read the numbered scales, then choose your own triangle. No equation is supplied.

Check after attempting: for example, $(0, 2)$ to $(2, -2)$ gives changes 2 across and -4 vertically, so $m = -2$. The intercept is 2, hence $y = -2x + 2$. The point $(1, 0)$ checks it. There are two large squares across and two down, but the vertical squares represent twice as many units. Counting squares alone would wrongly give -1.

Worked example (Extended: two given coordinate pairs). Find the gradient of the line through $(1, 2)$ and $(4, 11)$.

$$m = \frac{11 - 2}{4 - 1} = \frac{9}{3} = 3.$$

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For $y = mx + c$, the line crosses the y-axis at c and the gradient m is the rise divided by the run.

Drawing a straight-line graph

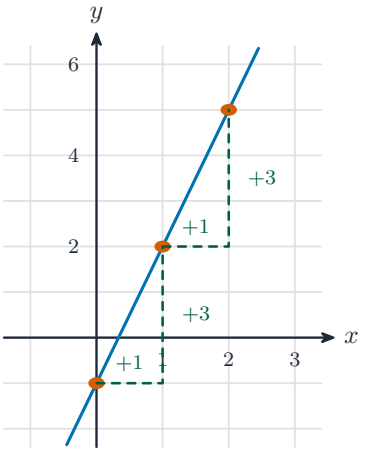
To draw $y = mx + c$, the quickest way is:

1. Mark the intercept c on the y-axis.
2. Use the gradient to step to more points (for $m = 3$, go 1 right and 3 up).
3. Join the points with a straight line.

You can also make a small **table of values** 数值表: choose at least three x-values, work out y, and plot the points. A third point checks the line through the first two. Label both axes and draw with a ruler across the requested interval, not just between nearby points.

Independent drawing task. Draw $y = -2x+3$ for $-2 \le x \le 3$, using $x = -2, 0, 1, 3$. Draw $x = -1$ on the same axes. Your table should give $y = 7, 3, 1, -3$; the second line is vertical. Check the actual drawing against all four points. **Extended:** drawing may also start from an equation such as $3x + 2y = 6$. Rearrange it to $y = -\frac{3}{2}x + 3$, or substitute values directly. Core drawing equations are normally given in $y = mx + c$ form unless a table is supplied.

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drawing $y = 3x - 1$:

1. mark the intercept $c = -1$
2. gradient 3: step 1 right, 3 up, again and again
3. join the points

check with a table: $x = 0, 1, 2 \;\rightarrow\; y = -1, 2, 5$

Drawing a line fast: mark the intercept, then use the gradient to step to new points

Finding the equation of a line

If you know the gradient m and one point on the line, put the point into $y = mx + c$ to find c.

> **Worked example.** A line has gradient 3 and passes through $(1, 2)$. Find its equation.

$$y = 3x + c, \qquad 2 = 3(1) + c, \qquad c = -1.$$

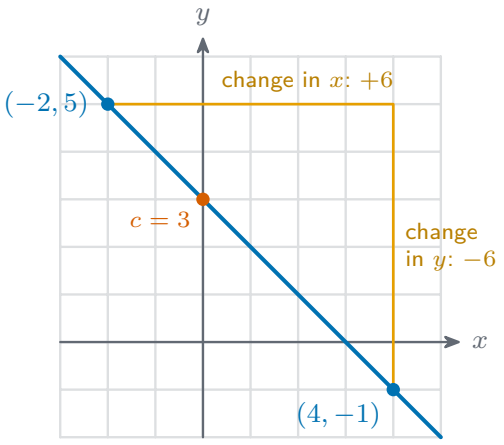
So the equation is $y = 3x - 1$. (Extended: if two coordinate pairs are given without a grid, first calculate the gradient, then do this.)

> **Worked example (Extended: through two points).** Find the equation of the line through $(-2, 5)$ and $(4, -1)$.

$$m = \frac{-1 - 5}{4 - (-2)} = \frac{-6}{6} = -1, \qquad 5 = -1(-2) + c \;\Rightarrow\; c = 3,$$

so $y = -x + 3$. Check with the other point: $-4 + 3 = -1$, which is correct.

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$$m = \frac{-6}{6} = -1, \text{ so } y = -x + 3$$

Change in y over change in x: -6 over 6 gives m = -1, and the line crosses the y-axis at c = 3, so y = -x + 3

Common mistakes. Putting the change in x on top of the gradient fraction (it is $\dfrac{\text{change in } y}{\text{change in } x}$) and losing a minus sign when subtracting a negative coordinate.

For a graph-to-equation question, read c where the drawn line crosses the y-axis and use a large grid triangle to find m. Then write $y = mx + c$ and check another point. A vertical graph through $x = 4$ has equation $x = 4$, not $y = 4$.

Length and midpoint (Extended)

A **line segment** 线段 is the straight piece between two points.

To find the **length** 长度 of the segment between (x_1, y_1) and (x_2, y_2), use **Pythagoras' theorem** 勾股定理 on the horizontal and vertical gaps:

$$\text{length} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

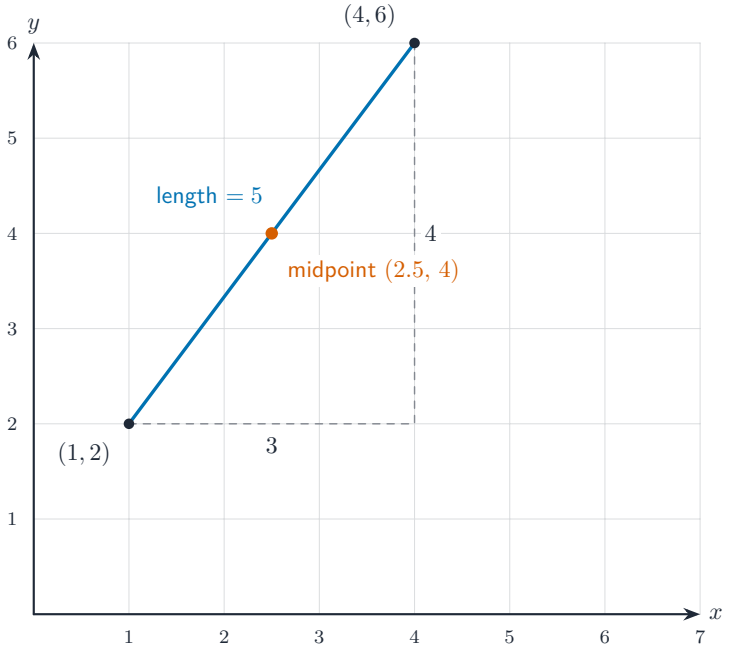
To find the **midpoint** 中点 (the point exactly in the middle), average the coordinates:

$$\text{midpoint} = \left(\frac{x_1 + x_2}{2},\ \frac{y_1 + y_2}{2} \right).$$

> **Worked example.** Find the length and midpoint of the segment from $(1, 2)$ to $(4, 6)$.

$$\text{length} = \sqrt{(4-1)^2 + (6-2)^2} = \sqrt{9 + 16} = \sqrt{25} = 5.$$

$$\text{midpoint} = \left(\frac{1+4}{2},\ \frac{2+6}{2}\right) = (2.5,\ 4).$$



The horizontal gap (3) and vertical gap (4) make a right triangle, so the length is $\sqrt{3^2+4^2} = 5$; the midpoint is the average of the coordinates.

■ Signed changes and a missing endpoint (Extended)

For $P(-4, 3)$ and $Q(2, -5)$, the coordinate changes are 6 and -8. The length is $\sqrt{6^2 + (-8)^2} = 10$ and midpoint is $(-1, -1)$. A coordinate change can be negative; a distance cannot. For a horizontal or vertical segment, use the absolute value of the one non-zero coordinate change.

If A and midpoint M are known, recover the other endpoint by $x_B = 2x_M - x_A$ and $y_B = 2y_M - y_A$. For $A(-3, 4)$ and $M(2, -1)$, $B = (7, -6)$. Check by averaging A and B. Their distance is $\sqrt{10^2 + (-10)^2} = \sqrt{200}$, about 14.1. If the question asks for the **exact** length, leave it as a **surd** 根式 in its simplest form: $\sqrt{200} = \sqrt{100 \times 2} = 10\sqrt{2}$.

Parallel lines

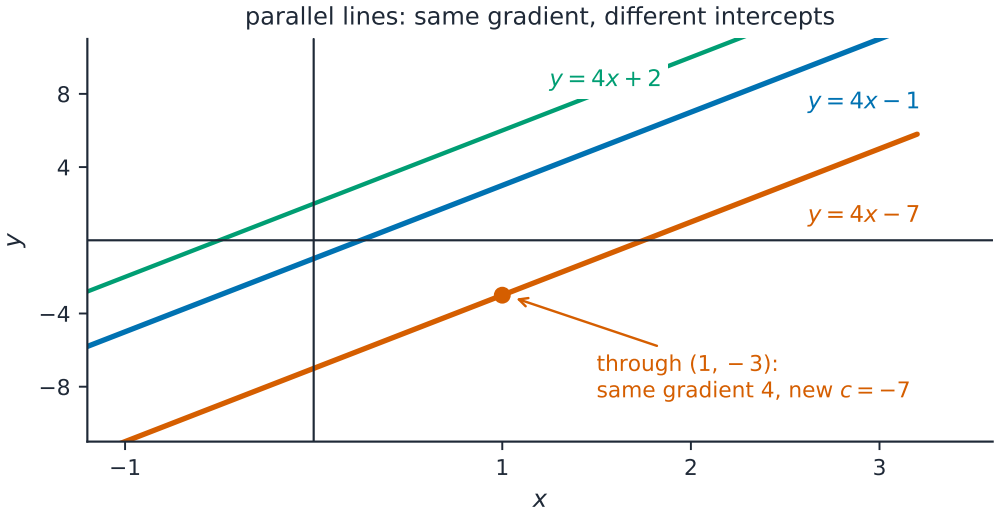
Distinct non-vertical **parallel** 平行 lines never meet and have the **same gradient**. Vertical lines are parallel too, but their gradients are undefined. Equal gradients and equal intercepts describe the same line, not two distinct parallel lines.

> **Worked example.** Find the equation of the line parallel to $y = 4x - 1$ that passes through $(1, -3)$.

The gradient is also 4. Put the point in:

$$-3 = 4(1) + c \;\Rightarrow\; c = -7,$$

so the line is $y = 4x - 7$.



$y - y_1 = m(x - x_1)$ (same m)

Parallel lines: same gradient, different intercepts

■ Where two lines meet (Extended)

Two lines that are not parallel cross at one point. Its coordinates fit **both** equations, so solve them as **simultaneous equations** 联立方程.

Worked example. Find where $y = 2x - 1$ meets $x + y = 8$.

Substitute the first equation into the second:

$$x + (2x - 1) = 8 \;\Rightarrow\; 3x = 9 \;\Rightarrow\; x = 3,$$

then $y = 2(3) - 1 = 5$. The lines meet at $(3, 5)$. Check in the second equation: $3 + 5 = 8$. If the two gradients are equal, there is no crossing point: the lines are parallel.

Perpendicular lines (Extended)

Two lines are **perpendicular** 垂直 if they meet at a **right angle** 直角. When both gradients are finite, their product is -1:

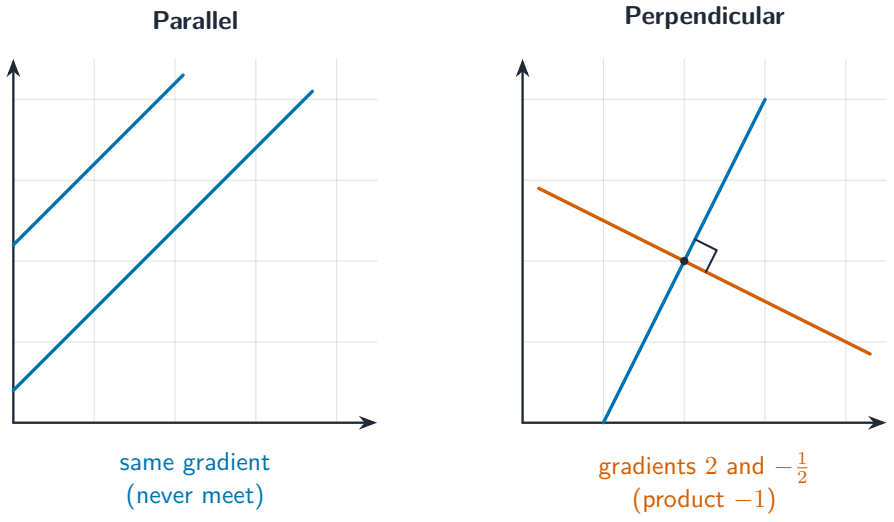
$$m_1 \times m_2 = -1, \qquad \text{so} \qquad m_2 = -\frac{1}{m_1}.$$

In words: flip the fraction and change the sign.

> **Worked example.** Find the gradient of a line perpendicular to $2y = 3x + 1$.

Rearrange: $y = \frac{3}{2}x + \frac{1}{2}$, so the gradient is $\frac{3}{2}$. The perpendicular gradient is $-\frac{2}{3}$.

Common mistake. Changing only the sign ($-\frac{3}{2}$) or only turning the fraction over ($\frac{2}{3}$). The perpendicular gradient needs both: the negative reciprocal. Check that the two gradients multiply to -1.



The non-vertical lines shown have equal gradients when parallel and gradient product -1 when perpendicular.

■ Perpendicular bisector

The **perpendicular bisector** 垂直平分线 of a segment cuts it in half at a right angle. To find its equation: get the midpoint, then use the perpendicular gradient through that midpoint.

> **Worked example.** Find the perpendicular bisector of the segment joining $(-3, 8)$ and $(9, -2)$.

- Midpoint: $\left(\frac{-3+9}{2}, \frac{8+(-2)}{2}\right) = (3, 3)$.
- Gradient of the segment: $\frac{-2-8}{9-(-3)} = \frac{-10}{12} = -\frac{5}{6}$.
- Perpendicular gradient: $\frac{6}{5}$.

Through $(3, 3)$: $3 = \frac{6}{5}(3) + c \Rightarrow c = 3 - \frac{18}{5} = -\frac{3}{5}$. So the bisector is

$$y = \frac{6}{5}x - \frac{3}{5}.$$

Independent bisector task (Extended). Find the perpendicular bisector of

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$A(-4, 1)$ and $B(2, 5)$ in $y = mx + c$ form. Check both the midpoint and the right-angle condition.

Check after attempting: midpoint $(-1, 3)$; segment gradient $2/3$; perpendicular gradient $-3/2$. Substituting the midpoint gives $c = 3/2$, so $y = -\frac{3}{2}x + \frac{3}{2}$. At $x = -1$ this gives 3, and $(2/3)(-3/2) = -1$.

Axis cases (Extended). For endpoints $(-3, 2)$ and $(5, 2)$, the segment is horizontal and its midpoint is $(1, 2)$, so its perpendicular bisector is $x = 1$. For $(4, -5)$ and $(4, 1)$, the bisector is $y = -2$. Handle these orientations directly: a vertical gradient is undefined, and there is no reciprocal of zero.

Exam tips

- The straight line is $y = mx + c$: m is the **gradient** and c is where the line crosses the y-axis.
- Core and Extended: gradient is signed vertical change divided by horizontal change, in grid units. Extended: keep coordinate subtraction in matching order when calculating from two pairs.
- Distinct non-vertical parallel lines have equal gradients. Extended: perpendicular finite gradients have product -1; handle horizontal/vertical pairs directly.
- Extended: midpoint is the coordinate average; distance comes from Pythagoras on coordinate differences. Use both midpoint and perpendicularity to check a bisector.

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Geometry

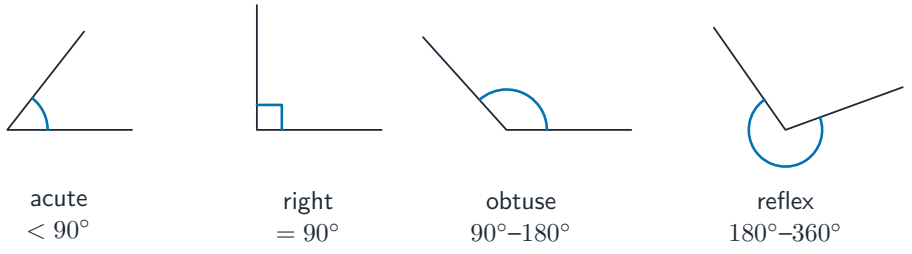
IGCSE Mathematics

This handout covers Topic 4, Geometry. Parts marked **(Extended)** are only tested on the Extended papers; everything else is for both levels. In the exam you must **give reasons** using the correct names below, not just the numbers.

Lines and angles

A corner where two lines meet is a **vertex** 顶点. Two lines are **parallel** 平行 if they never meet, and **perpendicular** 垂直 if they meet at a right angle. Angles are named by their size:

Name	Size
acute angle 锐角	less than 90°
right angle 直角	exactly 90°
obtuse angle 钝角	between 90° and 180°
reflex angle 优角	between 180° and 360°



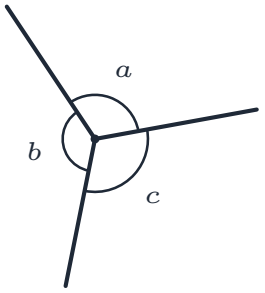
Angles by size: acute (under 90°), right (90°, shown by a square), obtuse (90°–180°) and reflex (180°–360°).

We name an angle with three letters, e.g. angle ABC is the angle at B.

Angle facts

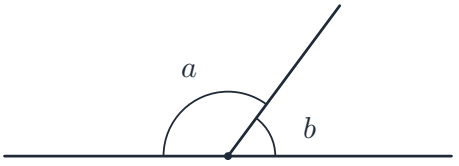
Learn these basic facts. Each one is a valid reason in the exam.

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$a + b + c = 360^\circ$ (round a point)

Angles around a point add up to 360 degrees



$a + b = 180^\circ$ (angles on a straight line)

Angles on a straight line add up to 180 degrees

- Angles at a point add up to 360°.
- Angles on a straight line add up to 180°.
- **Vertically opposite angles** 对顶角 (made by two crossing lines) are equal.

Worked example. Three angles on a straight line are x, 50° and 70°. Find x.

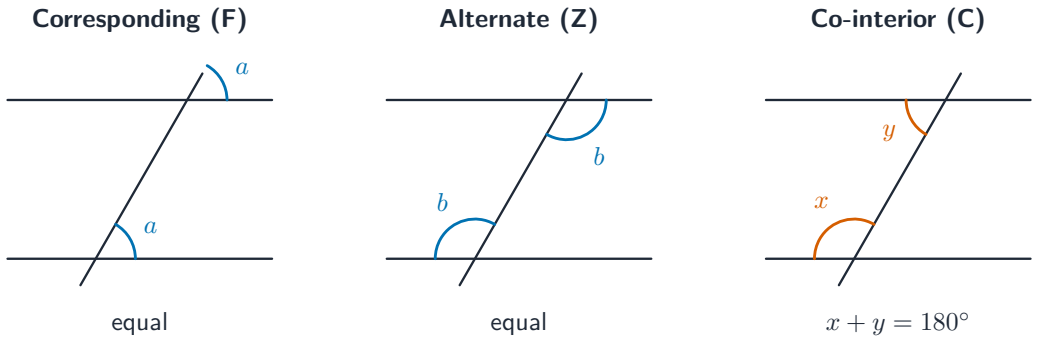
$$x + 50 + 70 = 180 \;\Rightarrow\; x = 60^\circ.$$

Angles in parallel lines

When a line crosses two parallel lines:

- **Corresponding angles** 同位角 (in matching positions, an "F" shape) are equal.
- **Alternate angles** 内错角 (opposite sides of the crossing line, a "Z" shape) are equal.
- **Co-interior angles** 同旁内角 (a "C" shape) add up to 180°; we say they are **supplementary** 互补.

Worked example. A straight line crosses two parallel lines. One angle is 110°. The co-interior angle y satisfies $110 + y = 180$, so $y = 70^\circ$.



Corresponding (F) and alternate (Z) angles are equal; co-interior (C) angles add to $180°$.

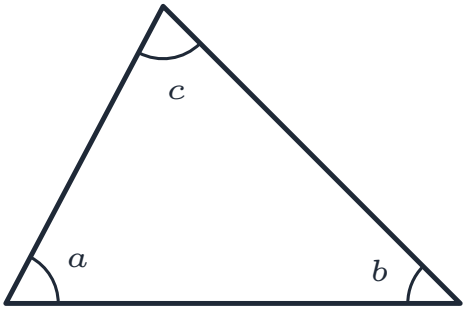
> **Worked example: two steps with reasons.** Lines PQ and RS are parallel, with P and R on the same side. A transversal makes angle $PXY = 65°$ at X on PQ, where Y is the point where the transversal meets RS. Find angle XYS and the obtuse angle XYR.

Angle $XYS = 65°$ (**alternate angles are equal**). Angle $XYR = 180° - 65° = 115°$ (**angles on a straight line add to** $180°$).

Each step gets its own reason, written with the correct name. "Because of the Z" or "parallel lines" alone does not earn the mark.

Triangles

A **triangle** 三角形 has three sides and angles that add up to $180°$.



$$a + b + c = 180°$$

The three angles of a triangle add up to 180 degrees

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Type	Property
equilateral 等边	all sides equal, all angles 60°
isosceles 等腰	two sides equal, two angles equal
scalene 不等边	all sides and angles different
right-angled	has one right angle

Worked example. A triangle has angles x, $2x$ and 90°. Find x.

$$x + 2x + 90 = 180 \;\Rightarrow\; 3x = 90 \;\Rightarrow\; x = 30°.$$

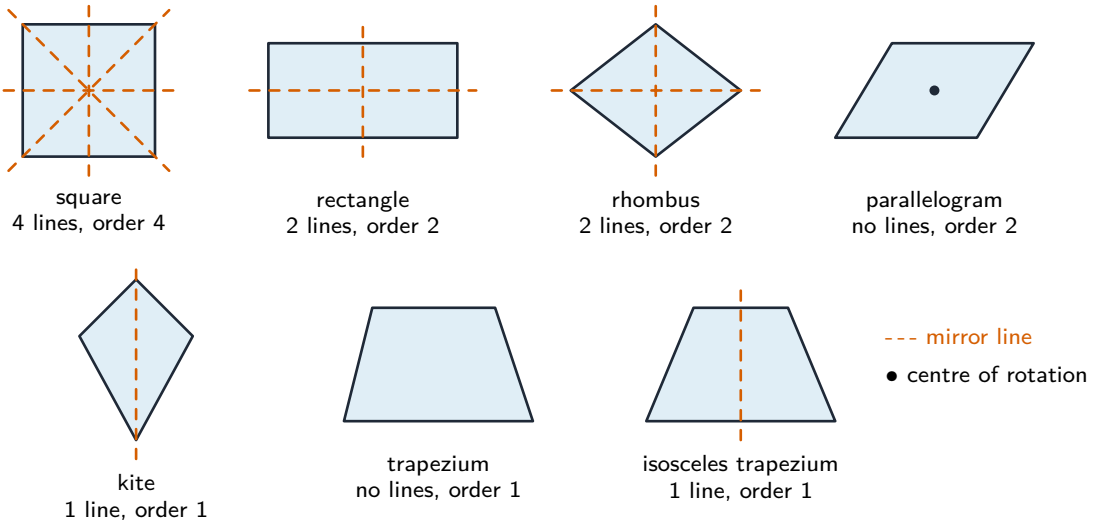
Quadrilaterals

A **quadrilateral** 四边形 has four sides and angles that add up to 360°.

Shape	Property	Mirror lines	Rotation order
square 正方形	four equal sides, four right angles	4	4
rectangle 矩形	opposite sides equal, four right angles	2	2
parallelogram 平行四边形	opposite sides parallel and equal	0	2
rhombus 菱形	four equal sides, opposite sides parallel	2	2
kite 鸢形	two pairs of equal sides next to each other	1	1
trapezium 梯形	one pair of parallel sides	0	1
isosceles trapezium	one pair of parallel sides; the other two sides equal	1	1

"Mirror lines" is the number of lines of symmetry, and "rotation order" is the order of rotational symmetry. Order 1 means the shape fits onto itself only once in a full turn, so it has no rotational symmetry.

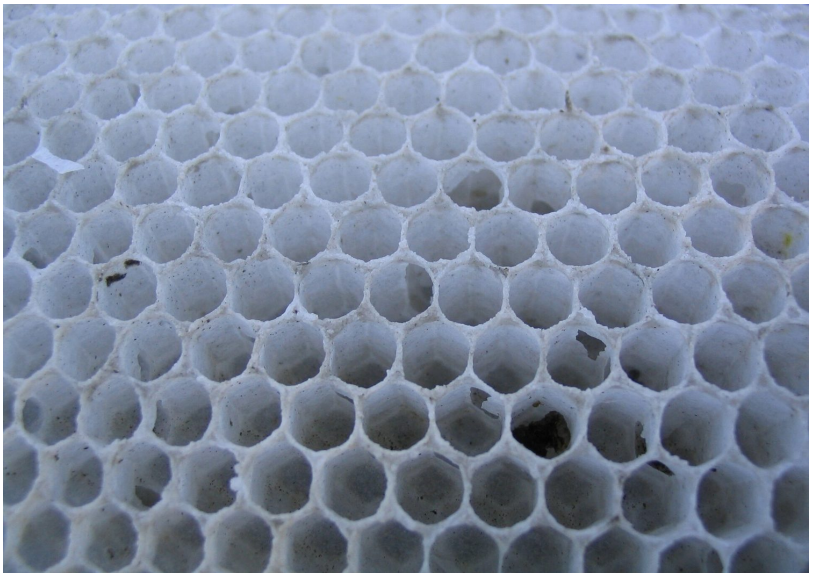
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The table as pictures: dashed lines are mirror lines. A parallelogram has no mirror line but still turns onto itself twice

Worked example. A quadrilateral has four equal sides, two lines of symmetry and rotational symmetry of order 2. Name it. Four equal sides means a square or a rhombus. A square has **four** lines of symmetry and order 4, so the shape is a **rhombus**.

Polygons



A honeycomb tessellates the plane with regular hexagons.

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A **polygon** 多边形 is a shape with straight sides. A **regular polygon** 正多边形 has all sides and all angles equal.

Sides	Name
5	**pentagon** 五边形
6	**hexagon** 六边形
8	**octagon** 八边形
10	**decagon** 十边形

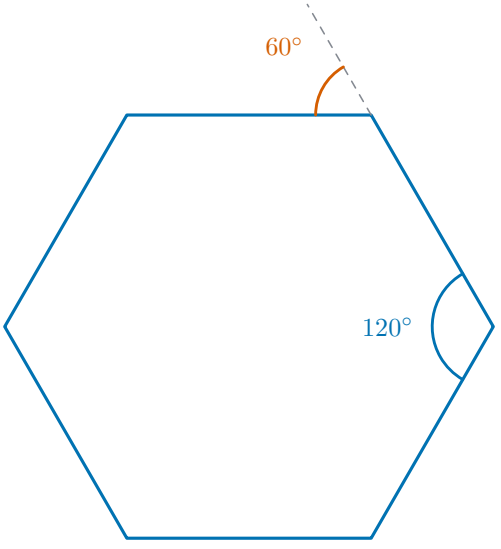
For a polygon with n sides:

$$\text{sum of interior angles} = (n-2) \times 180^\circ, \qquad \text{sum of exterior angles} = 360^\circ.$$

The **interior angle** 内角 and the **exterior angle** 外角 at each corner add up to 180°.

Worked example. Find each interior angle of a regular hexagon.

The exterior angle is $\dfrac{360^\circ}{6} = 60^\circ$, so each interior angle is $180^\circ - 60^\circ = 120^\circ$.



interior + exterior = 180°

At each corner the interior and exterior angles add to 180°*; a regular hexagon has* 60° *exterior and* 120° *interior angles.*

Worked example: find the number of sides. Each interior angle of a regular polygon is 156°. How many sides does it have?

Each exterior angle is $180° - 156° = 24°$, and the exterior angles add to 360°, so there are $360 \div 24 = 15$ sides.

> **Worked example: an irregular polygon.** A pentagon has angles 100°, 110°, 120°, x and $2x$. Find x.

The interior angles of a pentagon add to $(5 - 2) \times 180° = 540°$, so

$$330 + 3x = 540 \quad \Rightarrow \quad 3x = 210 \quad \Rightarrow \quad x = 70°.$$

Symmetry



The Taj Mahal has a clear line of symmetry down its centre.

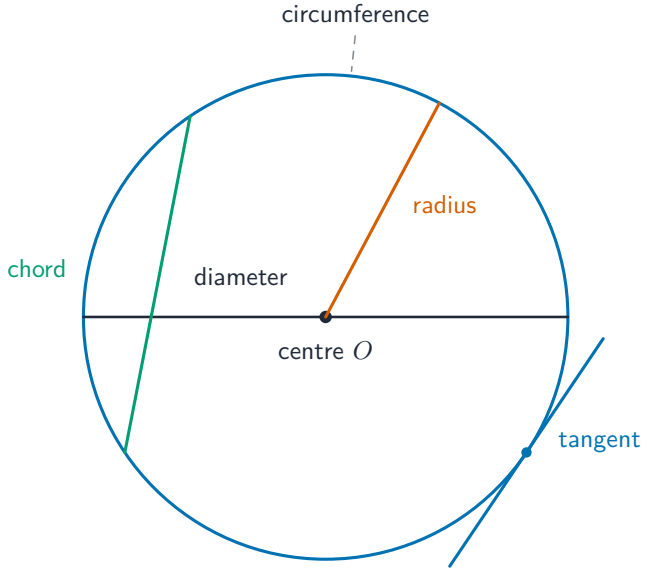
- A shape has **line symmetry** 轴对称 if a mirror line splits it into two matching halves.
- A shape has **rotational symmetry** 旋转对称 if it fits onto itself as you turn it. The order is how many times it fits in one full turn.

For solids (**Extended**), a flat slice that splits the solid into mirror halves is a **plane of symmetry** 对称面, and a line you can spin it around is an **axis of symmetry** 对称轴.

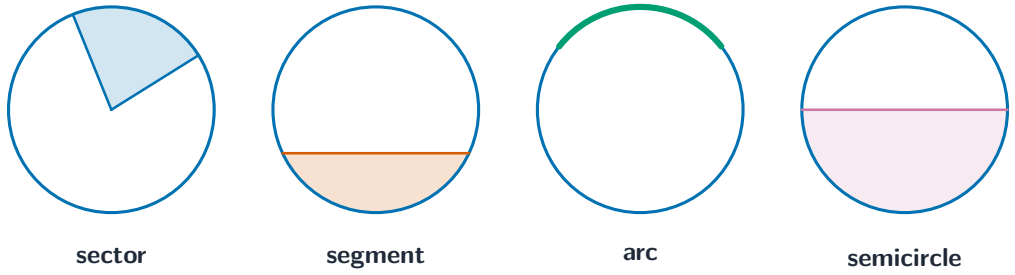
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Circles: the parts

Term	Meaning
circle 圆	all points the same distance from a centre
centre 圆心	the middle point
radius 半径	centre to edge (plural *radii*)
diameter 直径	right across through the centre (= 2× radius)
circumference 圆周	the distance all the way round
chord 弦	a straight line joining two points on the circle
arc 弧	part of the circumference
sector 扇形	a "pizza slice" between two radii
segment 弓形	the region cut off by a chord
semicircle 半圆	half a circle
tangent 切线	a line that touches the circle at one point



The main parts of a circle: a chord joins two points, while a tangent touches at just one.



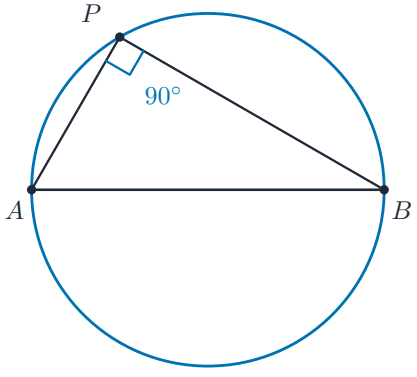
A sector is a slice between two radii, a segment is cut off by a chord, an arc is part of the circumference, and a semicircle is half the circle.

Circle theorems

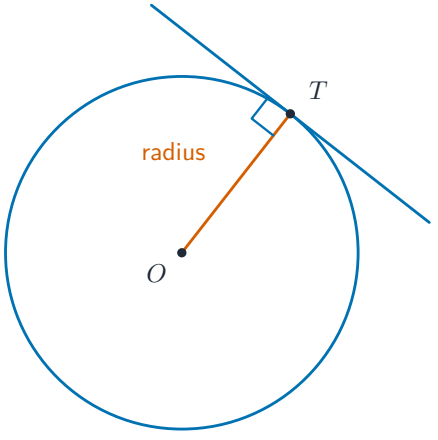
Use these to find unknown angles, always giving the reason.

For both levels:

- The angle in a **semicircle** is 90°.
- The angle between a **tangent** and a **radius** is 90°.



angle in a semicircle = 90°



tangent meets radius = 90°

Two theorems for both levels: the angle in a semicircle is 90°*, and a tangent meets a radius at* 90°*.*

Extended (theorems I):

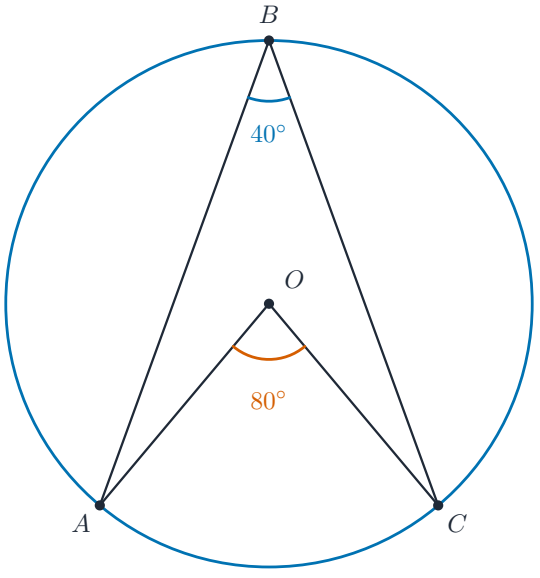
- The angle at the **centre** is twice the angle at the **circumference** (standing on the same arc).
- Angles in the same **segment** are equal.
- Opposite angles of a **cyclic quadrilateral** 圆内接四边形 add up to 180°.

- The **alternate segment theorem** 弦切角定理: the angle between a tangent and a chord equals the angle in the other segment.

Extended (theorems II): equal chords are the same distance from the centre; the perpendicular bisector of a chord passes through the centre; two tangents from the same outside point are equal in length.

> **Worked example.** A, B, C are on a circle. The angle at the circumference ABC is $40°$. Find the angle AOC at the centre O.

The angle at the centre is twice the angle at the circumference: $2 \times 40° = 80°$.



angle at centre = 2× angle at circumference

The angle at the centre ($80°$) is twice the angle at the circumference ($40°$) standing on the same arc.

> **Worked example (both levels).** AB is a diameter of a circle and C is a point on the circle. Angle $CAB = 35°$. Find angle ABC.

Angle $ACB = 90°$ (**angle in a semicircle**). Then angle $ABC = 180° - 90° - 35° = 55°$ (**angles in a triangle add to** $180°$).

> **Worked example (both levels).** PT is a tangent to a circle with centre O, touching it at T. Angle $TOP = 62°$. Find angle OPT.

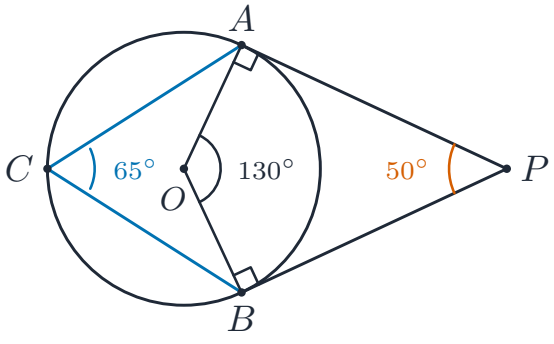
Angle $OTP = 90°$ (**tangent meets radius at** $90°$), so angle $OPT = 180° - 90° - 62° = 28°$ (**angles in a triangle**).

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Worked example (Extended). $ABCD$ is a cyclic quadrilateral with angle $DAB = 105°$. Find angle BCD.

Angle $BCD = 180° - 105° = 75°$ (**opposite angles of a cyclic quadrilateral add to** $180°$).

Worked example (Extended: several steps). Two tangents from a point P touch a circle with centre O at A and B. Angle $APB = 50°$, and C is a point on the major arc. Find angle ACB.



Two right angles from tangent and radius, then the angle at the centre is double the angle at the circumference.

- Angles OAP and OBP are $90°$ (**tangent meets radius at** $90°$).
- Angle $AOB = 360° - 90° - 90° - 50° = 130°$ (**angles in a quadrilateral add to** $360°$).
- Angle $ACB = 130° \div 2 = 65°$ (**angle at the centre is twice the angle at the circumference**).

Worked example (Extended: alternate segment). A tangent touches a circle at A. The angle between the tangent and the chord AB is $58°$. C is a point on the circle in the other segment. Then angle $ACB = 58°$ (**alternate segment theorem**).

Common mistakes. Giving an angle with no reason, or with an informal reason such as "it looks like $90°$"; using the angle-at-the-centre rule when the two angles do not stand on the same arc; and assuming a line is a diameter because it looks like one (the question must say so, or it must pass through O).

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Similar shapes

Two shapes are **similar** 相似 if one is an enlargement of the other: same angles, and all sides multiplied by the same **scale factor** 比例因子 k. (Shapes that are exactly the same size and shape are **congruent** 全等.)



Nesting dolls are similar shapes: each one is the same shape as the others, just multiplied by a scale factor

For similar shapes and solids:

$$\frac{\text{area of } A}{\text{area of } B} = k^2, \qquad \frac{\text{volume of } A}{\text{volume of } B} = k^3.$$

Worked example. Two similar solids have lengths in the ratio $2 : 3$. The smaller has volume $40\ \text{cm}^3$. Find the volume of the larger.

The volume ratio is $2^3 : 3^3 = 8 : 27$. So the larger volume is $40 \times \dfrac{27}{8} = 135\ \text{cm}^3$.

Worked example: lengths (both levels). In triangle ABC, D is on AB and E is on AC, with DE parallel to BC. $AD = 4$ cm, $AB = 10$ cm and $DE = 6$ cm. Find BC.

Triangles ADE and ABC are similar. (**Extended** reason: angle A is shared, and angles ADE and ABC are equal because they are **corresponding angles**.) The scale factor is $\dfrac{AB}{AD} = \dfrac{10}{4} = 2.5$, so

$$BC = 6 \times 2.5 = 15 \text{ cm}.$$

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Worked example: area to length (Extended). Two similar shapes have areas 20 cm^2 and 45 cm^2. The smaller is 8 cm tall. How tall is the larger?

The area ratio is $20 : 45 = 4 : 9$, so the length ratio is $\sqrt{4} : \sqrt{9} = 2 : 3$. The larger is $8 \times \dfrac{3}{2} = 12$ cm tall.

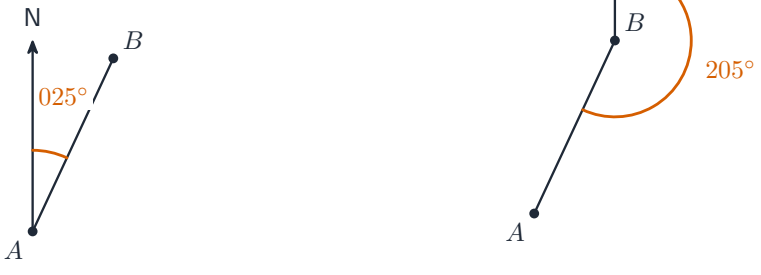
Common mistakes. Multiplying an area by the length scale factor k (use k^2 for area and k^3 for volume), and matching the wrong pairs of sides. Match sides that sit opposite equal angles.

Bearings

A **bearing** 方位角 gives a direction as a three-figure angle, measured **clockwise** 顺时针 from north, from 000° to 360°. So due east is 090° and due south is 180°.

Worked example. The bearing of B from A is 025°. Find the bearing of A from B.

The return (back) bearing differs by 180°: $025° + 180° = 205°$.



Bearing of B from A **Bearing of A from B**

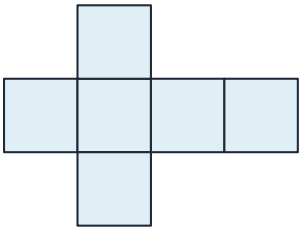
A bearing is measured clockwise from north as three figures; the back bearing differs by 180°.

Constructions, nets and solids

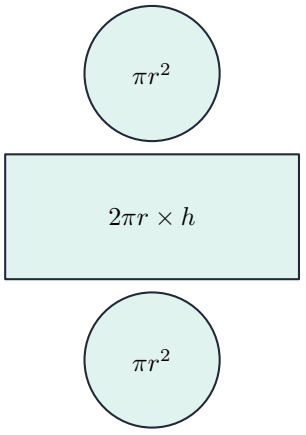
- To **construct** 作图 a triangle from three given sides, draw the base with a ruler, then use a pair of **compasses** 圆规 to mark each other side as an arc. Leave the construction arcs showing.
- A **net** 展开图 is a flat shape that folds up into a solid. You can use a net to work out surface areas.

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- A **scale drawing** 比例图 shows a real object smaller (or larger) by a fixed scale, such as 1 cm to 5 m.



Cube net
6 equal square faces



Cylinder net
2 circles + a rectangle

Unfolding a solid gives its net. Add the areas of the pieces to get the surface area: a cylinder is $2\pi r^2 + 2\pi rh$.

Common **solids** 几何体 and their parts:

Solid	Note
cube 立方体 / **cuboid** 长方体	box shapes
prism 棱柱	the same shape all along its length
cylinder 圆柱	a circular prism
pyramid 棱锥 / **cone** 圆锥	come to a point
sphere 球 / **hemisphere** 半球	a ball / half a ball
frustum 平截头体	a cone or pyramid with the top cut off

A flat side of a solid is a **face** 面, two faces meet at an **edge** 棱, and the whole outside is its **surface** 表面.

Exam tips

- Angles on a straight line add to 180°, around a point to 360°, and in a triangle to 180°. Give a **reason** for every step of an angle question.
- Learn the circle theorems: the angle at the centre is *twice* the angle at the circumference; the angle in a semicircle is 90°; angles in the same segment are equal; opposite angles of a cyclic quadrilateral add to 180°.
- The **exterior angles** of a convex polygon, one at each vertex, add to 360°, and each interior angle + its exterior angle = 180°.
- **Bearings** are measured clockwise from north and always written with three figures (e.g. 072°).

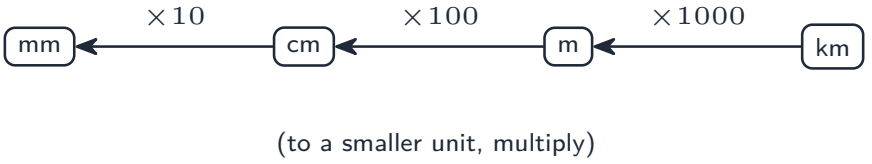
Mensuration

IGCSE Mathematics

This handout covers Topic 5, Mensuration (measuring length, area and volume). The Core and Extended content here is almost the same. In the exam, some formulas are given in the List of formulas, but you should still learn them all.

Units of measure

We use **metric** 公制 units for **mass** 质量 (g, kg), **length** 长度 (mm, cm, m, km), **area** 面积, **volume** 体积 and **capacity** 容量 (ml, litres – the space inside a container).



Changing length units: multiply going down, divide going up

To change units, be careful with squares and cubes:

- Length: 1 m = 100 cm.
- Area: 1 $m^2 = 100^2 = 10\,000$ cm^2.
- Volume: 1 $m^3 = 100^3 = 1\,000\,000$ cm^3.
- Capacity: 1 litre = 1000 cm^3, so 1 m^3 = 1000 litres.

Worked example. Convert 3 m^2 to cm^2.

$$3 \times 10\,000 = 30\,000 \text{ cm}^2.$$

Worked example: volume to capacity. A fish tank is 1.2 m long, 50 cm wide and 40 cm high. How many litres of water does it hold when full?

Change everything to centimetres first: 1.2 m = 120 cm.

$$\text{volume} = 120 \times 50 \times 40 = 240\,000 \text{ cm}^3, \qquad 240\,000 \div 1000 = 240 \text{ litres}.$$

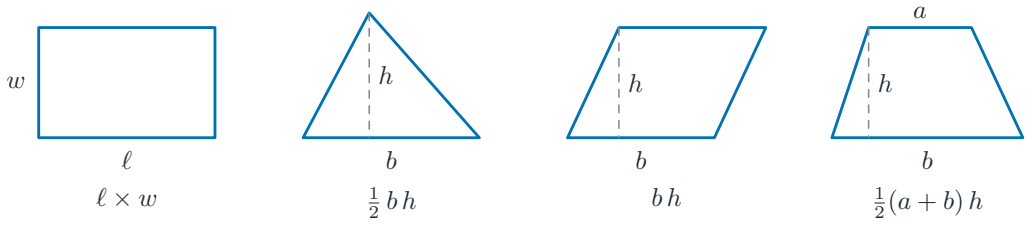
Common mistakes. Dividing by 100 to change cm^2 to m^2 (it is 10 000), and multiplying lengths in different units, such as metres times centimetres.

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Perimeter and area of basic shapes

The **perimeter** 周长 is the distance all the way round a shape. The **area** is the amount of flat space inside it. Here b is the **base** 底 and h is the perpendicular **height** 高.

Shape	Area
rectangle 矩形	length $\times$ width
triangle 三角形	$\frac{1}{2} \times b \times h$
parallelogram 平行四边形	$b \times h$
trapezium 梯形	$\frac{1}{2}(a+b)h$, where a and b are the two parallel sides



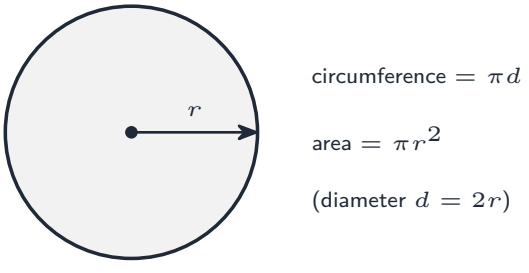
Area of the basic shapes; b is the base, h the perpendicular height, and a, b the two parallel sides of a trapezium.

Worked example. A trapezium has parallel sides 6 cm and 10 cm, and height 4 cm. Find its area.

$$\tfrac{1}{2}(6+10) \times 4 = \tfrac{1}{2} \times 16 \times 4 = 32 \text{ cm}^2.$$

Circles

For a **circle** 圆 with **radius** 半径 r (and **diameter** 直径 $d = 2r$):



A circle: circumference = pi d and area = pi r squared

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$$\text{circumference} = 2\pi r = \pi d, \qquad \text{area} = \pi r^2.$$

The **circumference** 圆周 is the distance round the circle.

> **Worked example.** A circle has radius 7 cm. Find its circumference and area (leave π in the answer).

$$\text{circumference} = 2\pi \times 7 = 14\pi \text{ cm}, \qquad \text{area} = \pi \times 7^2 = 49\pi \text{ cm}^2.$$

> **Worked example: circle and square.** A circle has radius 7 cm. A square has the same perimeter as the circle's circumference. Find the side x of the square in terms of π.

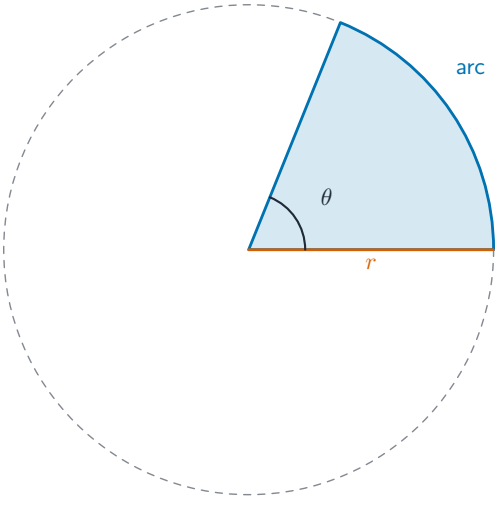
$$4x = 14\pi, \qquad x = \frac{14\pi}{4} = 3.5\pi \text{ cm}.$$

Common mistake. Using the diameter in πr^2. If the question gives the diameter, halve it first.

Arcs and sectors

An **arc** 弧 is part of the circumference. A **sector** 扇形 is a "pizza slice" between two radii. If the sector angle is θ, the arc and sector are that fraction $\dfrac{\theta}{360}$ of the whole circle:

$$\text{arc length} = \frac{\theta}{360} \times 2\pi r, \qquad \text{sector area} = \frac{\theta}{360} \times \pi r^2.$$



A sector is the fraction $\frac{\theta}{360}$ of the whole circle, so its arc and area are that fraction of the circumference and the area.

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A small slice is a **minor sector** 小扇形; the large rest is a **major sector** 大扇形.

> **Worked example.** Find the **arc length** 弧长 and area of a sector with angle $90°$ and radius 8 cm.

The fraction is $\dfrac{90}{360} = \dfrac{1}{4}$, so

$$\text{arc} = \tfrac{1}{4} \times 2\pi \times 8 = 4\pi \text{ cm}, \qquad \text{area} = \tfrac{1}{4} \times \pi \times 8^2 = 16\pi \text{ cm}^2.$$

> **Worked example: perimeter of a major sector (Extended).** A sector of angle $126°$ is cut from a circle of radius 7.5 cm. Find the perimeter of the major sector that is left.

The major sector has angle $360° - 126° = 234°$.

$$\text{arc} = \frac{234}{360} \times 2\pi \times 7.5 = 30.63\ldots \text{ cm}.$$

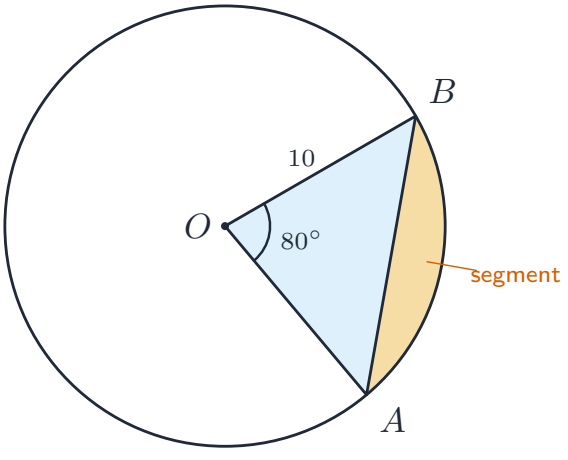
The perimeter is the arc **plus the two radii**: $30.63 + 7.5 + 7.5 = 45.6$ cm (3 s.f.).

> **Worked example: find the angle (Extended).** A sector of radius 6 cm has arc length 11 cm. Find the sector angle.

$$\frac{\theta}{360} \times 2\pi \times 6 = 11 \quad \Rightarrow \quad \theta = \frac{11 \times 360}{12\pi} = 105.0°.$$

A **segment** 弓形 is the part of a circle between a chord and the arc. Its area is the sector area take away the triangle, and the triangle uses $\frac{1}{2}ab\sin C$ from Topic 6.

> **Worked example: area of a segment (Extended).** Find the area of the minor segment cut off by a chord AB when the radius is 10 cm and angle $AOB = 80°$.



Segment = sector take away triangle OAB.

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$$\text{sector} = \frac{80}{360} \times \pi \times 10^2 = 69.81\ldots, \qquad \text{triangle} = \tfrac{1}{2} \times 10 \times 10 \times \sin 80^\circ = 49.24\ldots$$

$$\text{segment} = 69.81 - 49.24 = 20.6 \text{ cm}^2 \text{ (3 s.f.)}.$$

Common mistakes. Giving only the arc as the perimeter of a sector (add the two radii), and using the minor angle when the question asks about the major sector.

Surface area and volume of solids

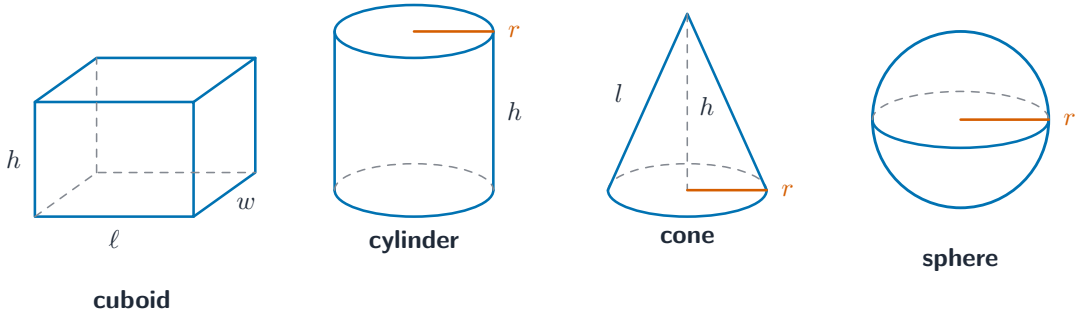


The pyramids of Giza are square-based pyramids – a 3-D solid.

The **surface area** 表面积 is the total area of all the outside faces. The volume is the space inside. For these solids (r = radius, h = height):

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Solid	Volume	Surface area
cuboid 长方体	length × width × height	add the six faces
prism 棱柱	(**cross-section** 横截面 area) × length	–
cylinder 圆柱	$\pi r^2 h$	$2\pi rh$ (**curved surface area** 侧面积) $+ 2\pi r^2$
pyramid 棱锥	$\frac{1}{3}$ × base area × h	–
cone 圆锥	$\frac{1}{3}\pi r^2 h$	πrl (curved) $+ \pi r^2$, where l is the **slant height** 斜高
sphere 球	$\frac{4}{3}\pi r^3$	$4\pi r^2$



Common solids and the dimensions (r, h, ℓ, slant l) used in their volume and surface-area formulas.

Worked example. A cylinder has radius 5 cm and height 10 cm. Find its volume and total surface area (in terms of π).

$$\text{volume} = \pi \times 5^2 \times 10 = 250\pi \text{ cm}^3.$$

$$\text{surface area} = 2\pi(5)(10) + 2\pi(5)^2 = 100\pi + 50\pi = 150\pi \text{ cm}^2.$$

Worked example: a cone. A solid cone has base radius 5 cm and perpendicular height 12 cm. Find its volume and its total surface area in terms of π.

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The curved surface needs the slant height l, which is the hypotenuse of a right-angled triangle with sides r and h:

$$l = \sqrt{5^2 + 12^2} = \sqrt{169} = 13 \text{ cm}.$$

volume $= \frac{1}{3}\pi \times 5^2 \times 12 = 100\pi$ cm^3, surface area $= \pi \times 5 \times 13 + \pi \times 5^2 = 65\pi + 25\pi = 90\pi$ cm^2.

Worked example: working backwards. A cylinder has volume 500 cm^3 and height 8 cm. Find its radius.

$$\pi r^2 \times 8 = 500 \quad \Rightarrow \quad r^2 = \frac{500}{8\pi} = 19.89\ldots \quad \Rightarrow \quad r = 4.46 \text{ cm (3 s.f.)}.$$

Worked example: melting and recasting. A metal sphere of radius 6 cm is melted and made into a cylinder of radius 4 cm. Find the height of the cylinder.

The volume stays the same:

$$\pi \times 4^2 \times h = \tfrac{4}{3}\pi \times 6^3 = 288\pi \quad \Rightarrow \quad 16h = 288 \quad \Rightarrow \quad h = 18 \text{ cm}.$$

Common mistakes. Using the perpendicular height h in πrl (the curved surface needs the slant height l), and giving only the curved surface when the question asks for the total surface area of a solid.

Compound shapes and parts of shapes



Stacked containers are cuboids; volume is length times width times height.

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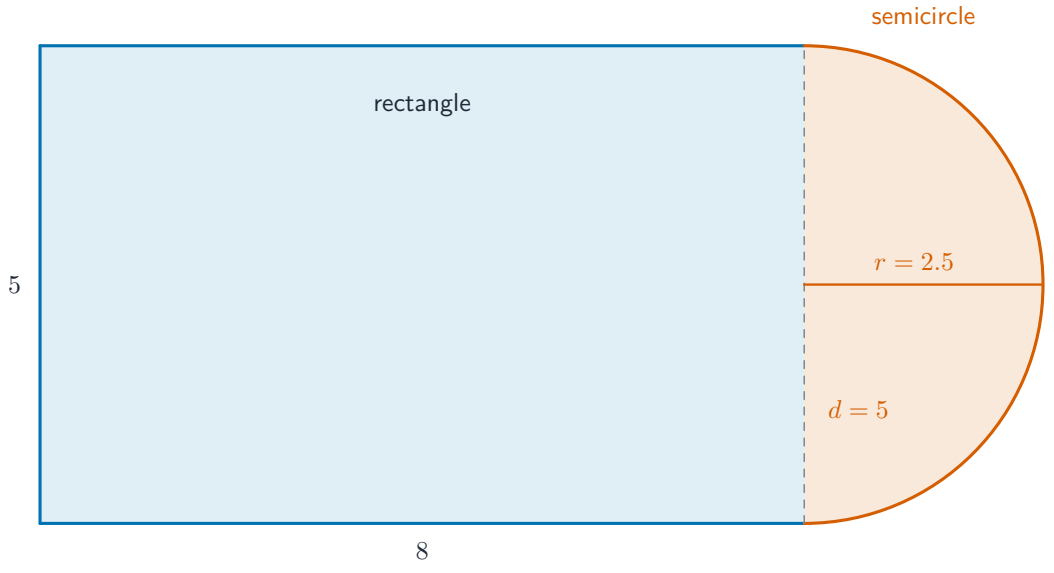
A **compound shape** 组合图形 is made by joining or cutting simple shapes. Split it into parts you know, then add or subtract.

For "parts" of a circle or solid, take the right fraction. For example, a **hemisphere** 半球 (half a sphere) has volume

$$\tfrac{1}{2} \times \tfrac{4}{3}\pi r^3 = \tfrac{2}{3}\pi r^3.$$

A **frustum** 平截头体 is a cone or pyramid with its top cut off parallel to the base; find its volume by subtracting the small top cone from the whole cone.

> **Worked example.** Find the area of a shape made of a rectangle 8 cm × 5 cm with a semicircle of diameter 5 cm on one end.

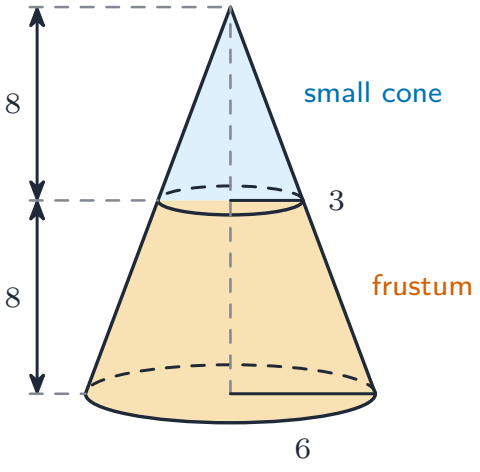


Split a compound shape into parts you know – here a rectangle plus a semicircle – then add the areas.

The semicircle has radius 2.5 cm:

$$\text{area} = 8 \times 5 + \tfrac{1}{2}\pi(2.5)^2 = 40 + 3.125\pi \approx 49.8 \text{ cm}^2.$$

> **Worked example: volume of a frustum (Extended).** A cone has base radius 6 cm and height 16 cm. The top 8 cm is cut off, leaving a frustum. Find the volume of the frustum.



Frustum = whole cone take away the small cone on top.

The small cone is half the height of the whole cone, so its radius is half too: 3 cm.

$$\text{frustum} = \tfrac{1}{3}\pi \times 6^2 \times 16 - \tfrac{1}{3}\pi \times 3^2 \times 8 = 192\pi - 24\pi = 168\pi \approx 528 \text{ cm}^3.$$

Worked example: rate of flow (Extended). Water flows through a pipe with radius 1.5 cm at 20 cm per second into an empty cylindrical tank with radius 40 cm. How long does it take for the water to be 30 cm deep?

Each second, a cylinder of water 20 cm long leaves the pipe:

$$\pi \times 1.5^2 \times 20 = 45\pi \text{ cm}^3 \text{ per second.}$$

The water needed is $\pi \times 40^2 \times 30 = 48\,000\pi$ cm^3, so the time is

$$\frac{48\,000\pi}{45\pi} = 1066.7 \text{ s} \approx 17.8 \text{ minutes.}$$

Worked example: a stone in a tank. A stone is put into a tank 40 cm long and 25 cm wide, and the water rises by 0.6 cm. The stone's density is 2.5 g/cm^3. Find its mass.

The stone's volume is the volume of water it pushes up: $40 \times 25 \times 0.6 = 600$ cm^3. Then mass = density × volume = $2.5 \times 600 = 1500$ g.

Common mistakes. Adding the volumes of the two cones of a frustum instead of subtracting, and mixing the units of a rate (cm per second) with a time given in minutes.

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Exam tips

- Match the formula to the shape: the **area** of a circle is πr^2, the **circumference** is πd (or $2\pi r$) – do not mix them up.
- Keep units consistent, and remember area units are squared and volume units cubed (e.g. $1\ \text{m}^2 = 10\,000\ \text{cm}^2$).
- For an **arc** or **sector**, take the fraction $\frac{\theta}{360}$ of the whole circumference or area.
- Surface area is the total of *all* the faces – draw the net if you are unsure. Leave an answer in terms of π only if the question allows it.

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Trigonometry

IGCSE Mathematics

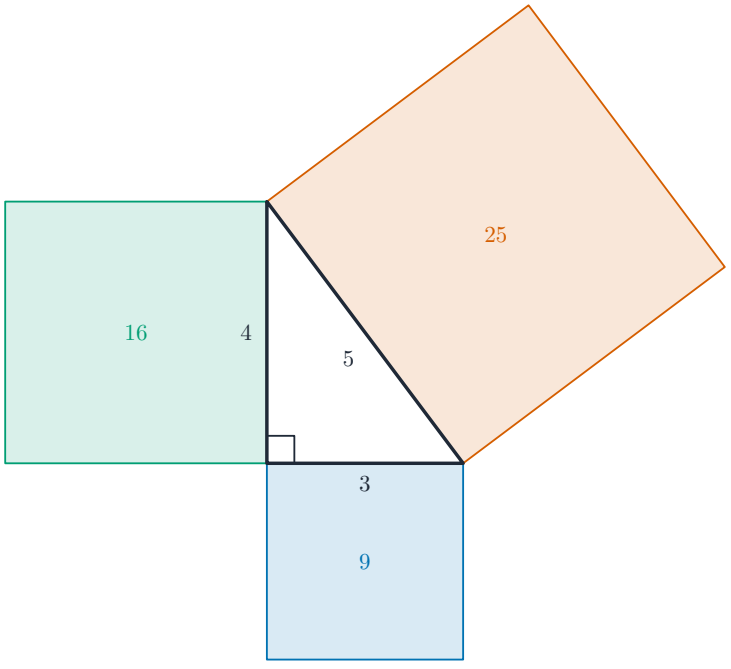
This handout covers Topic 6, Trigonometry. Parts marked **(Extended)** are only tested on the Extended papers; everything else is for both levels. Give angle answers to one decimal place.

Pythagoras' theorem

Pythagoras' theorem 勾股定理 links the three sides of a **right-angled triangle** 直角三角形 (a **triangle** 三角形 with one 90° angle). If the longest side (the **hypotenuse** 斜边, opposite the right angle) is c, then

$$a^2 + b^2 = c^2.$$

Use it to find a missing side.



Pythagoras as areas: the squares on the two shorter sides ($9 + 16$) add up to the square on the hypotenuse (25).

Worked example. A right-angled triangle has a hypotenuse of 13 cm and one short side of 5 cm. Find the other short side.

$$b^2 = 13^2 - 5^2 = 169 - 25 = 144, \qquad b = \sqrt{144} = 12 \text{ cm}.$$

Worked example: a ladder. A ladder 6 m long leans against a vertical wall. Its foot is 1.5 m from the wall. How high up the wall does it reach?

The ladder is the hypotenuse, so subtract:

$$h^2 = 6^2 - 1.5^2 = 36 - 2.25 = 33.75, \qquad h = 5.81 \text{ m (3 s.f.)}.$$

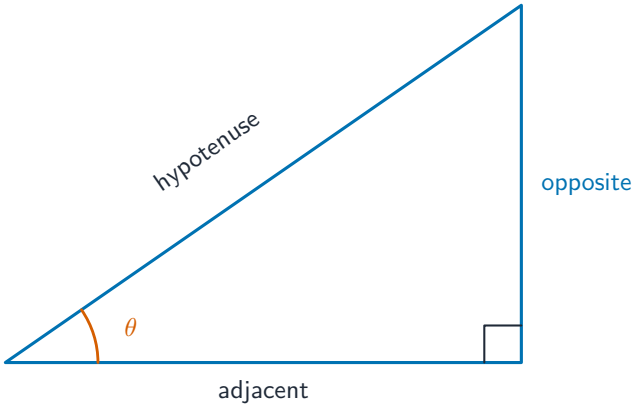
Common mistake. Adding the squares when the hypotenuse is one of the sides you know. Find the hypotenuse first: if you know it, subtract; if you want it, add.

Trigonometry in right-angled triangles

Label the sides from the acute angle you are using: the **opposite** 对边 (across from the angle), the **adjacent** 邻边 (next to the angle, excluding the hypotenuse), and the hypotenuse. The three ratios are the **sine** 正弦, **cosine** 余弦 and **tangent ratio** 正切 (sin, cos, tan):

$$\sin\theta = \frac{\text{opp}}{\text{hyp}}, \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}}, \qquad \tan\theta = \frac{\text{opp}}{\text{adj}}.$$

Remember them as **SOH-CAH-TOA**. To find an angle, use the inverse ($\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$).



Name the sides from the angle θ: the opposite is across from it, the adjacent next to it, and the hypotenuse opposite the right angle (SOH-CAH-TOA).

Worked example (find a side). In a right-angled triangle the hypotenuse is 10 cm and the angle is 30°. Find the opposite side.

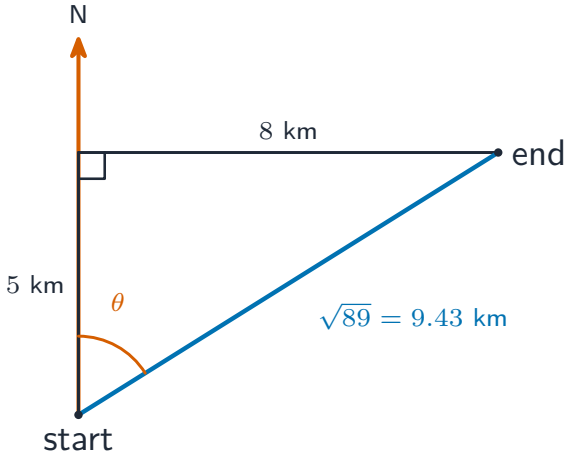
$$\text{opp} = 10 \times \sin 30^\circ = 10 \times 0.5 = 5 \text{ cm}.$$

Worked example (find an angle). The opposite side is 4 cm and the adjacent side is 3 cm.

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$$\tan\theta = \frac{4}{3}, \qquad \theta = \tan^{-1}\left(\frac{4}{3}\right) = 53.1^\circ.$$

Worked example: bearings. Amir walks 5 km due north and then 8 km due east. Find how far he is from his starting point, and his bearing from it.



$\tan\theta = \frac{8}{5}$, bearing 058°

A bearing is measured clockwise from north, so the angle θ sits at the start, between north and the path.

The two walks make a right-angled triangle, so

$$\text{distance} = \sqrt{5^2 + 8^2} = \sqrt{89} = 9.43 \text{ km}.$$

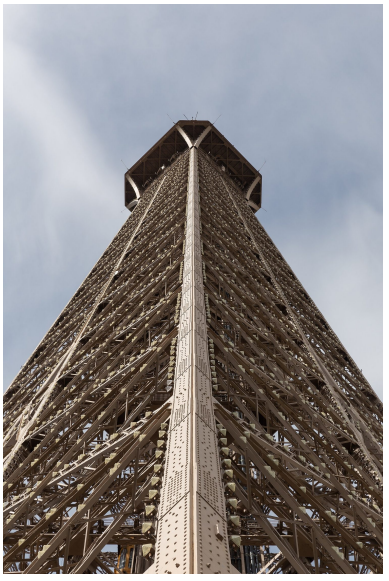
A bearing is measured clockwise from north, so use the angle at the start, between north and his path:

$$\tan\theta = \frac{8}{5}, \qquad \theta = 58.0^\circ, \quad \text{so the bearing is } 058^\circ.$$

Common mistakes. Calling the side next to the right angle the hypotenuse (the hypotenuse is always opposite the right angle), and leaving the calculator in radians: $\sin 30$ should give 0.5.

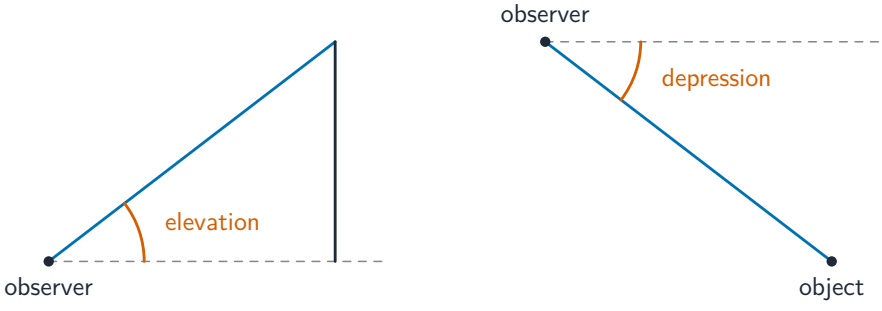
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Angles of elevation and depression (Extended)



Looking up at a tower involves an angle of elevation.

The **angle of elevation** 仰角 is the angle up from the horizontal to an object above you. The **angle of depression** 俯角 is the angle down from the horizontal to an object below you. The shortest distance from a point to a line is the **perpendicular** 垂直 distance.



The angle of elevation looks up from the horizontal; the angle of depression looks down.

Worked example. From a point 50 m from the foot of a tower, the angle of elevation to the top is 40°. Find the height of the tower.

$$\text{height} = 50 \times \tan 40^\circ = 50 \times 0.839 = 42.0 \text{ m}.$$

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Worked example: angle of depression. From the top of a cliff 80 m high, the angle of depression of a boat is 25°. How far is the boat from the foot of the cliff?

The angle of depression from the cliff top equals the angle of elevation from the boat (alternate angles), so at the boat the cliff is the opposite side:

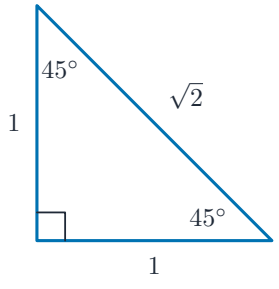
$$\tan 25^\circ = \frac{80}{d}, \qquad d = \frac{80}{\tan 25^\circ} = 172 \text{ m (3 s.f.)}.$$

Common mistake. Putting the angle of depression between the cliff and the line of sight. It is always measured from the **horizontal**.

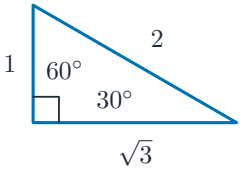
Exact trigonometric values (Extended)

You must know these exact values without a calculator.

x	0°	30°	45°	60°	90°
$\sin x$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan x$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	–



45° **triangle**



30°–60° **triangle**

These two special triangles are where the exact values come from – worth memorising.

Graphs and trigonometric equations (Extended)

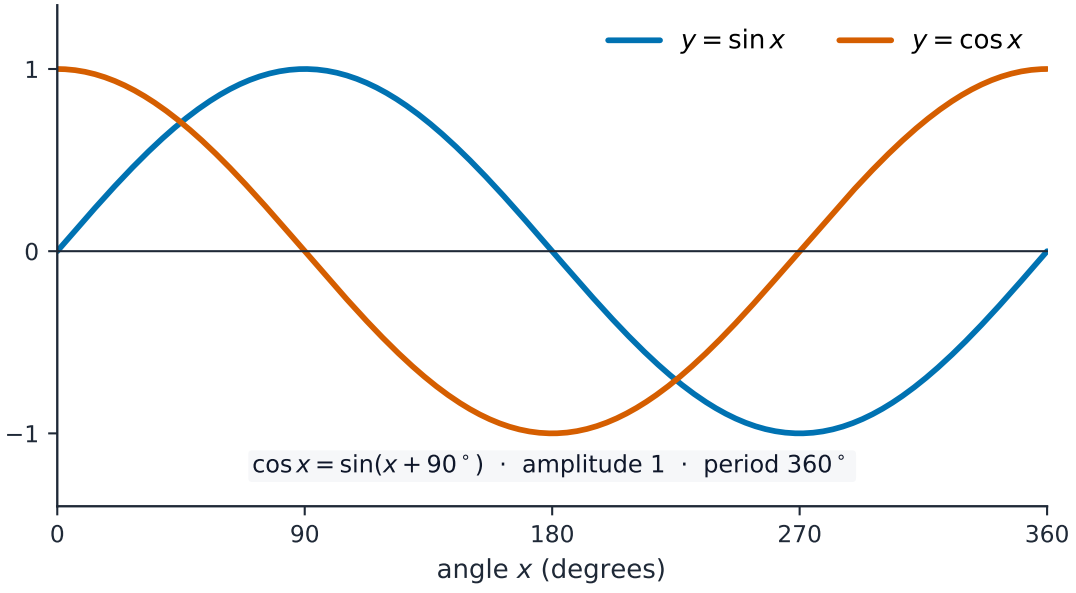
For $0^\circ \leqslant x \leqslant 360^\circ$:

- $y = \sin x$ is a wave starting at 0, peaking at 90°, back to 0 at 180°, down to -1 at 270°.
- $y = \cos x$ is the same wave but starting at 1.
- $y = \tan x$ rises steeply and repeats every 180°.

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A Ferris wheel: a point on the rim traces a sine curve as it turns.



$y = \sin x$ and $y = \cos x$ are smooth waves between -1 and 1; cosine is sine shifted left by $90°$.

A **trigonometric equation** 三角方程 often has more than one answer in this range. Use the graph (or the symmetry of the wave) to find them all.

> **Worked example.** Solve $\sin x = \frac{\sqrt{3}}{2}$ for $0° \leqslant x \leqslant 360°$.

One answer is $x = 60°$. The sine wave is also $\frac{\sqrt{3}}{2}$ at $180° - 60° = 120°$. So $x = 60°$ or $120°$.

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Worked example. Solve $2\cos x + 1 = 0$ for $0° \leqslant x \leqslant 360°$.

$$\cos x = -\tfrac{1}{2} \;\Rightarrow\; x = 120° \text{ or } 240°.$$

Worked example. Solve $\tan x = 1.5$ for $0° \leqslant x \leqslant 360°$.

The calculator gives $x = 56.3°$. The tangent repeats every $180°$, so the other answer is $56.3° + 180° = 236.3°$.

Worked example. Solve $\sin x = -0.4$ for $0° \leqslant x \leqslant 360°$.

The calculator gives $-23.6°$, which is outside the range. Work with $23.6°$: sine is negative between $180°$ and $360°$, so

$$x = 180° + 23.6° = 203.6° \quad \text{or} \quad x = 360° - 23.6° = 336.4°.$$

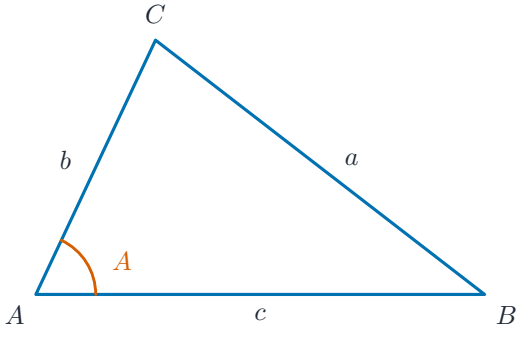
Common mistake. Stopping at the calculator's answer. Sketch the graph and look for every point at the right height in the range.

The sine and cosine rules (Extended)

For **any** triangle (not just right-angled), with sides a, b, c opposite angles A, B, C:

$$\text{sine rule:} \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C},$$

$$\text{cosine rule:} \quad a^2 = b^2 + c^2 - 2bc\cos A.$$



In any triangle, sides a, b, c lie opposite the angles A, B, C of the same letter.

Use the **sine rule** 正弦定理 when you have a side and its opposite angle. Use the **cosine rule** 余弦定理 when you have two sides and the angle between them, or all three sides. With the sine rule, watch for the **ambiguous case** 两解情况, where an angle could be acute or obtuse.

The area of any triangle is

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$$\text{area} = \tfrac{1}{2}ab\sin C.$$

Worked example. A triangle has $b = 7$ cm, $c = 8$ cm and the angle between them $A = 40°$. Find side a.

$$a^2 = 7^2 + 8^2 - 2(7)(8)\cos 40° = 113 - 112 \times 0.766\ldots = 27.20\ldots,$$

$$a = \sqrt{27.20\ldots} = 5.22 \text{ cm (3 s.f.)}.$$

Worked example: cosine rule for an angle. A triangle has sides 5 cm, 7 cm and 8 cm. Find the angle opposite the 7 cm side.

Rearrange the cosine rule so the angle is the subject:

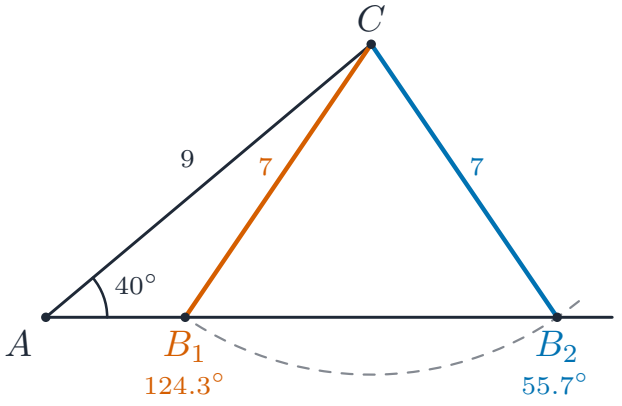
$$\cos\theta = \frac{5^2 + 8^2 - 7^2}{2 \times 5 \times 8} = \frac{40}{80} = 0.5, \qquad \theta = 60°.$$

Worked example: sine rule for a side. In triangle ABC, angle $B = 48°$, angle $C = 71°$ and $AB = 12$ cm. Find AC.

AB is side c (opposite C) and AC is side b (opposite B):

$$\frac{b}{\sin 48°} = \frac{12}{\sin 71°}, \qquad b = \frac{12\sin 48°}{\sin 71°} = 9.43 \text{ cm (3 s.f.)}.$$

Worked example: the ambiguous case. In triangle ABC, $a = 7$ cm, $b = 9$ cm and angle $A = 40°$. Find the two possible values of angle B.



The side of length 7 *can swing to meet the base line in two places, so angle* B *can be* 55.7° *or* 124.3°.

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$$\sin B = \frac{9\sin 40^\circ}{7} = 0.826\ldots, \qquad B = 55.7^\circ.$$

Because $\sin(180^\circ - B) = \sin B$, $B = 180^\circ - 55.7^\circ = 124.3^\circ$ also works. Both fit in a triangle with $A = 40^\circ$, since $40^\circ + 124.3^\circ < 180^\circ$, so there are two different triangles.

> **Worked example: area.** Two sides of a triangle are 8 cm and 11 cm, and the angle between them is 35°. Find the area.

$$\text{area} = \tfrac{1}{2} \times 8 \times 11 \times \sin 35^\circ = 25.2 \text{ cm}^2 \text{ (3 s.f.)}.$$

Common mistakes. Working out $113 - 112$ first in the cosine rule (multiply $112 \times \cos 40^\circ$ first), using the sine rule with a side and an angle that are not opposite each other, and rounding in the middle of a calculation (keep the full calculator value until the end).

Pythagoras and trigonometry in 3D (Extended)

In three dimensions, find a right-angled triangle inside the solid, then use Pythagoras or trigonometry on it. A common task is the angle between a line and a flat surface (a **plane** 平面).

> **Worked example.** A box has a base 6 cm by 8 cm and height 5 cm. Find the angle between a space diagonal and the base.

First the base diagonal: $\sqrt{6^2 + 8^2} = \sqrt{100} = 10$ cm. This diagonal and the height form a right-angled triangle, so the angle θ with the base is

$$\tan\theta = \frac{5}{10} = 0.5, \qquad \theta = \tan^{-1}(0.5) = 26.6^\circ.$$

Exam tips

- Use **Pythagoras** ($a^2 + b^2 = c^2$) when there is no angle; use **SOH-CAH-TOA** when an angle is involved.
- The **hypotenuse** is always opposite the right angle. Label the sides (opposite, adjacent, hypotenuse) relative to the angle you are using.
- To find an angle, use the inverse ($\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$), and check your calculator is set to **degrees**.
- For a triangle with no right angle, use the **sine rule** or the **cosine rule** – the cosine rule when you know two sides and the angle between them.

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Transformations and vectors

IGCSE Mathematics

This handout covers Topic 7, Transformations and vectors. Parts marked **(Extended)** are only tested on the Extended papers; everything else is for both levels. Vectors as a whole topic are Extended.

Transformations



Geometric tiles are built by reflecting, rotating and translating shapes.

A **transformation** 变换 changes the position or size of a shape. The original is the *object* and the result is the *image*. There are four types. When asked to **describe** one, you must name the type and give all the details below.

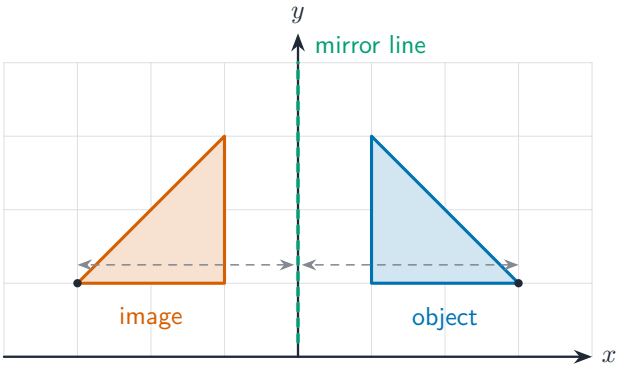
■ Reflection

A **reflection** 反射 flips the shape over a **mirror line** 对称轴. Each image point is the same distance from the line as the object point, on the other side.

- To describe it, give the **equation of the mirror line** (for Core, a horizontal or vertical line; Extended allows any line such as $y = x$).

> **Worked example.** Reflect the point $(3, 2)$ in the y-axis. Only the sign of x changes: the image is $(-3, 2)$. (In the x-axis it would be $(3, -2)$.)

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A reflection flips the shape over a mirror line (here the y-axis); each point and its image are the same distance from the line.

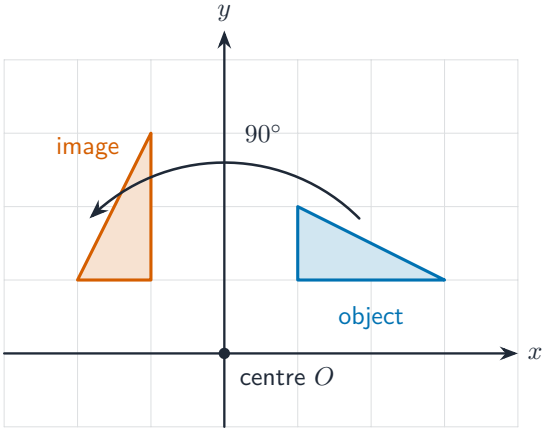
Worked example: other mirror lines. Reflect $(1, 4)$ in the line $x = 3$. The point is 2 units left of the line, so the image is 2 units right of it: $(5, 4)$. (**Extended:** in the line $y = x$ the coordinates swap, so $(1, 4) \to (4, 1)$.)

Rotation

A **rotation** 旋转 turns the shape about a fixed point, the **centre** 中心 of rotation, through multiples of 90°.

- To describe it, give the centre, the angle, and the direction (clockwise or anticlockwise).

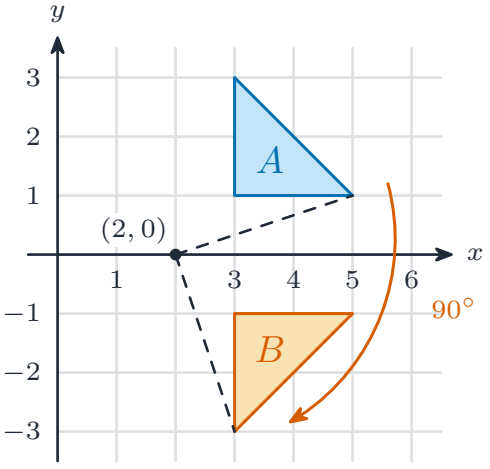
Worked example. Rotate $(3, 1)$ by 90° anticlockwise about the origin. The rule is $(x, y) \to (-y, x)$, so the image is $(-1, 3)$.



A rotation turns the shape about a fixed centre – here 90° anticlockwise about the origin.

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Worked example: describe a rotation. Triangle A has vertices $(3,1)$, $(5,1)$ and $(3,3)$. Triangle B has vertices $(3,-1)$, $(3,-3)$ and $(5,-1)$. Describe the single transformation that maps A onto B.



Each point and its image are the same distance from the centre, and every side has turned through 90° clockwise.

The side from $(3,1)$ to $(5,1)$ points right; its image, from $(3,-1)$ to $(3,-3)$, points down. So the shape has turned 90° clockwise. Try the centre $(2,0)$: it is $\sqrt{2}$ from both $(3,1)$ and its image $(3,-1)$, and $\sqrt{10}$ from both $(5,1)$ and $(3,-3)$. Tracing paper confirms it.

Answer: **rotation, 90° clockwise, centre** $(2,0)$. Three pieces of information give three marks.

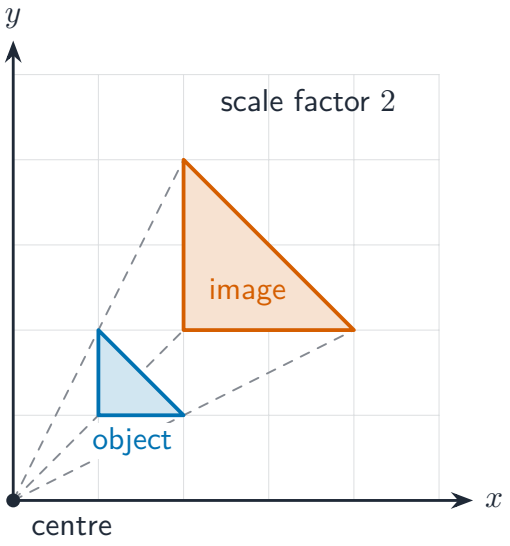
■ Enlargement

An **enlargement** 放大 changes the size by a **scale factor** 比例因子 k, measured from a fixed centre. Each position relative to the centre is multiplied by k; actual distances are multiplied by $|k|$.

- To describe it, give the centre and the scale factor. A **fractional** scale factor (between 0 and 1) makes the shape smaller. For Extended, k may also be **negative** (the image appears on the other side of the centre).

Worked example. Enlarge $(1,2)$ from the origin by scale factor 2. Multiply both coordinates: the image is $(2,4)$.

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An enlargement multiplies every distance from the centre by the scale factor (here 2).

> **Worked example: find the centre and scale factor.** Triangle P has vertices $(1,1)$, $(2,1)$ and $(1,3)$. Triangle Q has vertices $(3,3)$, $(5,3)$ and $(3,7)$. Describe the transformation that maps P onto Q.

The side of length 1 (from $(1,1)$ to $(2,1)$) becomes length 2, so the scale factor is image length $\div$ object length $= 2 \div 1 = 2$. Draw lines from each image vertex back through the matching object vertex; they meet at the centre $(-1,-1)$. Check: from $(-1,-1)$ to $(1,1)$ is $\begin{pmatrix}2\\2\end{pmatrix}$, and twice that reaches $(3,3)$.

Answer: **enlargement, scale factor 2, centre** $(-1,-1)$.

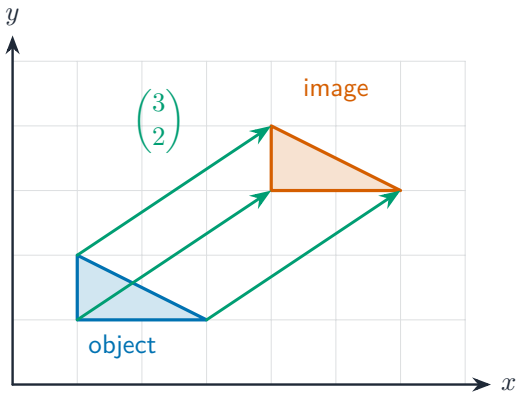
> **Worked example: fractional and negative scale factors.** From the origin, scale factor $\frac{1}{2}$ takes $(4,6)$ to $(2,3)$, closer to the centre. (**Extended:** scale factor -2 takes $(2,1)$ to $(-4,-2)$, on the other side of the centre and twice as far away.)

■ Translation

A **translation** 平移 slides the shape with no turning, by a **vector** 向量 written as a **column vector** 列向量 $\begin{pmatrix}x\\y\end{pmatrix}$ (x across, y up).

> **Worked example.** Translate $(5,3)$ by $\begin{pmatrix}-2\\4\end{pmatrix}$: move 2 left and 4 up to get $(3,7)$.

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A translation slides every point by the same column vector, with no turning.

(**Extended:** a question may ask you to combine two transformations and describe the single transformation that has the same effect.)

> **Worked example: a combination (Extended).** Reflect a shape in the x-axis, then reflect the image in the y-axis. What single transformation has the same effect?

Follow a general point: $(x, y) \to (x, -y) \to (-x, -y)$. Both signs change, which is a **rotation of** 180° **about the origin**. (For 180° no direction is needed.)

Common mistakes. Giving two transformations when the question asks for a **single** one, using words such as "turn" or "move" instead of the proper names, leaving out the direction of a rotation, and writing the scale factor upside down (it is image length ÷ object length).

Vectors in two dimensions (Extended)



Forces like wind are vectors, with both size and direction.

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A vector has both size and direction. It can be written as a column vector, as $\overrightarrow{AB}$ (from A to B), or in bold as $\mathbf{a}$.

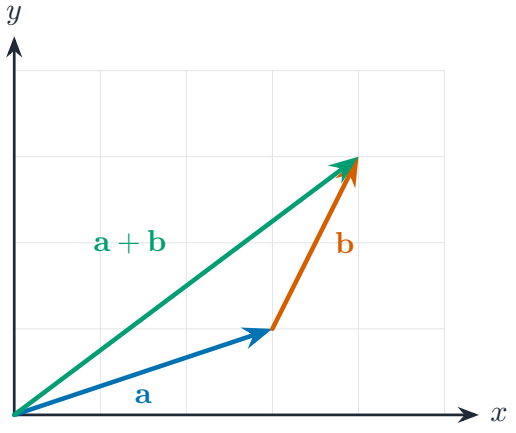
- **Add or subtract** by working on the top and bottom numbers separately.
- **Multiply by a scalar** 标量 (an ordinary number) by multiplying both numbers.

Worked example. If $\mathbf{a} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$, then

$$\mathbf{a} + \mathbf{b} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}, \qquad 3\mathbf{a} = \begin{pmatrix} 9 \\ 3 \end{pmatrix}.$$

Subtracting works the same way, but take care with the signs:

$$2\mathbf{a} - \mathbf{b} = \begin{pmatrix} 6 - 2 \\ 2 - (-4) \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}.$$



Adding by the triangle law: draw $\mathbf{b}$ *from the tip of* $\mathbf{a}$*, and* $\mathbf{a} + \mathbf{b}$ *runs from start to finish.*

Magnitude of a vector (Extended)

The **magnitude** 模 (length) of a vector is found with Pythagoras. For $\begin{pmatrix} x \\ y \end{pmatrix}$,

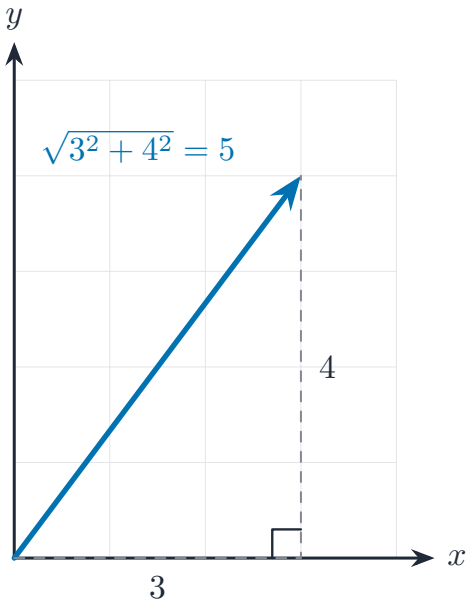
$$\left| \begin{pmatrix} x \\ y \end{pmatrix} \right| = \sqrt{x^2 + y^2}.$$

Worked example. The magnitude of $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ is $\sqrt{3^2 + 4^2} = \sqrt{25} = 5$.

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Worked example: from two points. A is $(2, -1)$ and B is $(7, 11)$. Find $\left|\overrightarrow{AB}\right|$.

$$\overrightarrow{AB} = \begin{pmatrix} 7-2 \\ 11-(-1) \end{pmatrix} = \begin{pmatrix} 5 \\ 12 \end{pmatrix}, \qquad \left|\overrightarrow{AB}\right| = \sqrt{5^2 + 12^2} = \sqrt{169} = 13.$$



The magnitude of $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ *comes from Pythagoras on its horizontal and vertical parts.*

Vector geometry (Extended)

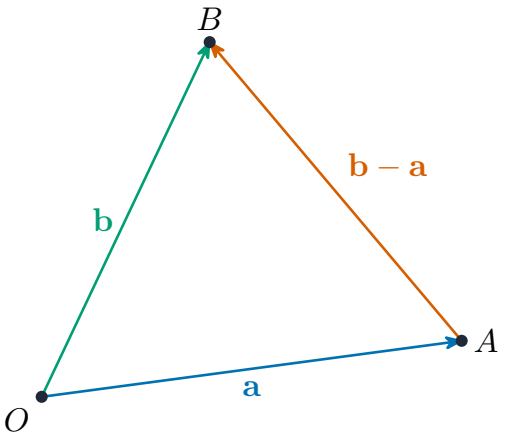
A vector can be drawn as a **directed line segment** 有向线段 (an arrow). The **position vector** 位置向量 of a point is the vector from the origin O to that point.

A key idea: the vector from A to B is

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a},$$

where $\mathbf{a}$ and $\mathbf{b}$ are the position vectors of A and B.

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To go from A to B, travel back along $\mathbf{a}$ *to O then forward along* $\mathbf{b}$*: so* $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$.

Two non-zero vectors are **parallel** 平行 if one is a scalar multiple of the other (for example $\overrightarrow{AB} = 2\,\overrightarrow{CD}$). Three distinct points are **collinear** 共线 (in a straight line) if the connecting vectors are parallel. Any vector in a plane can be expressed using two non-parallel **coplanar** 共面 vectors in that plane.

Worked example. O is the origin, with $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. M is the midpoint of AB. Find $\overrightarrow{OM}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

$$\overrightarrow{OM} = \mathbf{a} + \tfrac{1}{2}\overrightarrow{AB} = \mathbf{a} + \tfrac{1}{2}(\mathbf{b} - \mathbf{a}) = \tfrac{1}{2}(\mathbf{a} + \mathbf{b}).$$

Worked example: a ratio. With the same $\mathbf{a}$ and $\mathbf{b}$, P lies on AB with $AP : PB = 1 : 3$. Find $\overrightarrow{OP}$.

AP is $\frac{1}{4}$ of AB (one part out of $1 + 3 = 4$), so

$$\overrightarrow{OP} = \mathbf{a} + \tfrac{1}{4}(\mathbf{b} - \mathbf{a}) = \tfrac{3}{4}\mathbf{a} + \tfrac{1}{4}\mathbf{b}.$$

Worked example: show that points are collinear. $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = 3\mathbf{b} - 2\mathbf{a}$. Show that A, B and C lie on a straight line.

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}, \qquad \overrightarrow{AC} = (3\mathbf{b} - 2\mathbf{a}) - \mathbf{a} = 3\mathbf{b} - 3\mathbf{a} = 3(\mathbf{b} - \mathbf{a}).$$

So $\overrightarrow{AC} = 3\,\overrightarrow{AB}$: the two vectors are parallel, and they share the point A, so A, B and C are collinear. Both facts are needed for full marks.

Common mistakes. Writing $\overrightarrow{AB} = \mathbf{a} - \mathbf{b}$ (it is end minus start, $\mathbf{b} - \mathbf{a}$), using $\frac{1}{3}$ for the ratio $1 : 3$ (the whole line has 4 parts), and saying "parallel" without the shared point in a collinearity proof.

Exam tips

- Fully **describe** each transformation: a translation (a vector), a reflection (the mirror line), a rotation (centre, angle *and* direction), an enlargement (centre and scale factor).
- A **negative** scale factor turns the image upside down through the centre; a fractional one (between 0 and 1) makes it smaller.
- Add vectors tip-to-tail – add the top numbers, then the bottom numbers. The **magnitude** of a vector comes from Pythagoras on its components.
- $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$ (end minus start). Two vectors are **parallel** if one is a scalar multiple of the other.

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Probability

IGCSE Mathematics

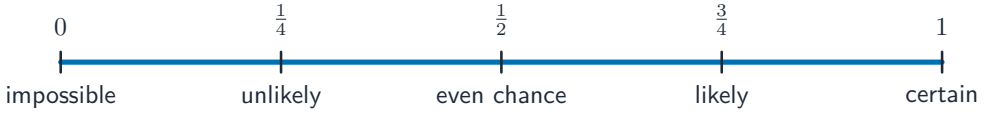
This handout covers Topic 8, Probability. Parts marked **(Extended)** are only tested on the Extended papers; everything else is for both levels.

The probability scale



Dice: the probability scale runs from impossible (0) to certain (1)

Probability 概率 measures how likely an **event** 事件 is. It runs on a scale from 0 (impossible) to 1 (certain), and can be written as a fraction, decimal or percentage. For example, rolling a 7 on an ordinary six-sided die is impossible, while rolling a number from 1 to 6 is certain. An even chance is $\frac{1}{2} = 0.5 = 50\%$. A **fair** 公平 coin has an even chance of heads; merely having two possible results does not guarantee equal chances. A probability of 1 means 100%, not 100.



Probability runs from 0 *(impossible) to* 1 *(certain), with* $\frac{1}{2}$ *an even chance.*

For equally likely **outcomes** 结果,

$$\mathrm{P}(\text{event}) = \frac{\text{number of favourable outcomes}}{\text{total number of outcomes}}.$$

Worked example: count the full total. A bag has 3 red and 5 blue counters. Assume each counter is equally likely to be chosen. Red has probability $\frac{3}{8} = 0.375 = 37.5\%$. Count individual counters, not just the two colour names: the colours are not equally likely.

An ideal fair spinner with 8 equal sectors, 4 blue, 3 red and 1 green, has blue probability $\frac{4}{8} = \frac{1}{2}$ and green probability $\frac{1}{8}$. Three colours do not imply a probability of $\frac{1}{3}$ each. Use favourable divided by total only when the outcomes you count are equally likely.



An ideal fair roulette wheel has equally likely pockets. Equal pocket size alone does not prove that a real wheel is fair.

Read a table. Each of these 30 students uses exactly one travel method. One student is chosen at **random** 随机 with equal chances.

Travel method	Students
Walk	12
Bus	9
Bicycle	6
Car	3
Total	30

The probability of bus is $\frac{9}{30} = \frac{3}{10}$. Use 30 students in the denominator, not the four methods. Not walking includes all three other methods, so its probability is $\frac{9+6+3}{30} = \frac{18}{30} = 0.6$.

An event and "not the event" cover all possibilities without overlap. Their probabilities add to 1. The probability of an event not happening is 1 minus the probability it happens. For example, if a parcel has probability 0.28 of being late, it has probability $1 - 0.28 = 0.72$ of not being late.

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Worked example. A bag has 3 red and 5 blue counters, each equally likely to be chosen. The probability of not red is $1 - \frac{3}{8} = \frac{5}{8}$. If a different bag has 2 red, 3 blue and 5 green counters, not blue includes both red and green, giving $\frac{2+5}{10} = \frac{7}{10}$. "Not blue" does not mean just one other colour.

Check the total. If the only colours are red, blue and green, their probabilities add to 1. Given red 0.35 and blue 0.25, green is $1 - 0.35 - 0.25 = 0.40$. Proposed values 0.4, 0.45 and 0.3 are invalid: they add to 1.15, even though each individual value lies between 0 and 1.

Extended notation. Write $\mathrm{P}(A)$ for the probability of event A, and A' for "not A". Then

$$\mathrm{P}(A') = 1 - \mathrm{P}(A).$$

This notation is not required at Core, where the same calculation can be described in words.

Try it. A box has 6 red, 4 blue and 5 green counters, each equally likely to be chosen. Find the probabilities of blue and not blue. **Answer:** the total is 15, so blue has probability $\frac{4}{15}$ and not blue has probability $\frac{6+5}{15} = \frac{11}{15}$. The total stays the same when the target colour changes.

Worked example: a table with a missing value. A spinner can land on four colours. Find x.

Colour	**red**	**blue**	**green**	**yellow**
Probability	0.2	0.35	x	$2x$

The probabilities add to 1:

$$0.2 + 0.35 + x + 2x = 1 \quad \Rightarrow \quad 3x = 0.45 \quad \Rightarrow \quad x = 0.15.$$

So $\mathrm{P}(\text{green}) = 0.15$ and $\mathrm{P}(\text{yellow}) = 0.3$.

Relative frequency and expected frequency

Use an experiment when a probability is unknown, or to compare results with a theoretical model. The **relative frequency** 相对频率 estimates the probability:

$$\text{relative frequency} = \frac{\text{number of successes}}{\text{total number of trials}}.$$

For example, 84 point-up results in 200 drawing-pin drops give $84/200 = 0.42$. Include all drops in the denominator, not just those that land point-up.

With independent trials under the same conditions, more trials usually give a more reliable estimate; any one estimate can still vary. A **fair** die gives each face equal prob-

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ability. **Bias** 偏倚 favours some outcomes compared with the intended fair model; **random** describes chance-based selection and does not by itself guarantee equal chances.

With stable chances and a consistent method, more trials usually give a more reliable estimate, but an extra trial can move it away from the true probability. More trials do not remove a systematic problem with the equipment or make an unrepresentative sample representative.

A **fair** coin gives heads and tails equal chances. **Bias** means the chances differ from the fair model. **Random** means the individual result is decided by chance; random does not mean equally likely. A biased coin can still give random results. An uneven finite result does not by itself prove that a coin is biased.

The **expected frequency** 期望频数 is a predicted count in n trials, using the same probability for those trials:

$$\text{expected frequency} = \text{P(event)} \times n.$$

> **Worked example.** The probability of rolling a six on a fair die is $\frac{1}{6}$. How many sixes are expected in 300 rolls?

$$\frac{1}{6} \times 300 = 50.$$

The actual count need not be 50. For a fixed probability, expected count grows linearly with trials: doubling the trials doubles the expected count. An expected count can also be non-integer: $0.3 \times 15 = 4.5$, although an actual count must be a whole number.

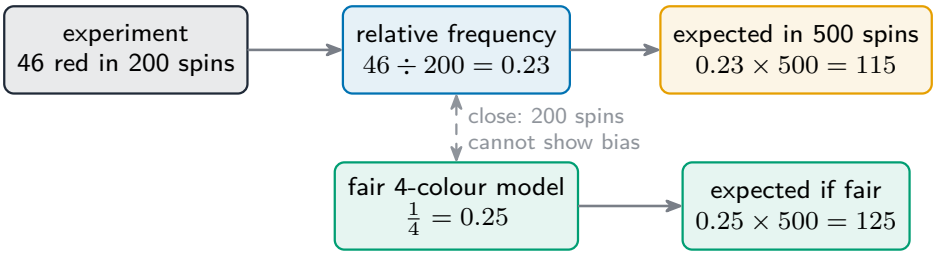
Prediction from results. The drawing-pin estimate 0.42 predicts $0.42 \times 500 = 210$ point-up results in another 500 similar drops. A prediction based on a sample assumes it represents the population and that conditions remain similar. To combine comparable experiments, add successes and trials before dividing; do not take an unweighted average when the trial counts differ.

> **Worked example: relative frequency.** A spinner is spun 200 times and lands on red 46 times. Estimate the probability of red, and how many reds to expect in 500 spins.

$$\text{relative frequency} = \frac{46}{200} = 0.23, \qquad \text{expected number} = 0.23 \times 500 = 115.$$

A fair spinner with 4 equal colours would give 0.25. One experiment of 200 spins is not enough to say this spinner is biased.

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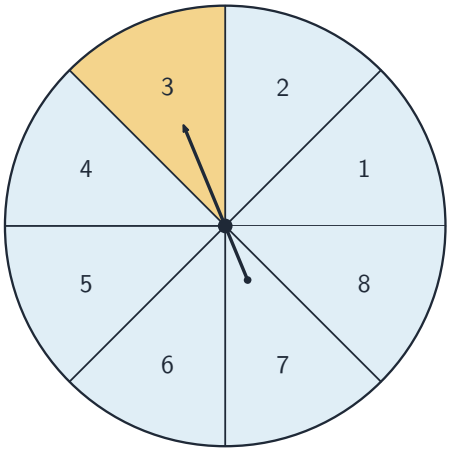


Estimate, then predict: 0.23 times 500 = 115 reds. The fair model would predict 125, and 200 spins cannot tell the two apart

Combined events

For two or more events together (**combined events** 组合事件), check the conditions before choosing a rule:

- **AND** (both happen): **multiply** the probabilities when the events are **independent** 独立事件 (one does not affect the other).
- **OR** (either happens): **add** the probabilities when the events are **mutually exclusive** 互斥 (they cannot both happen).



Eight identical sectors, so every number is **equally likely**.

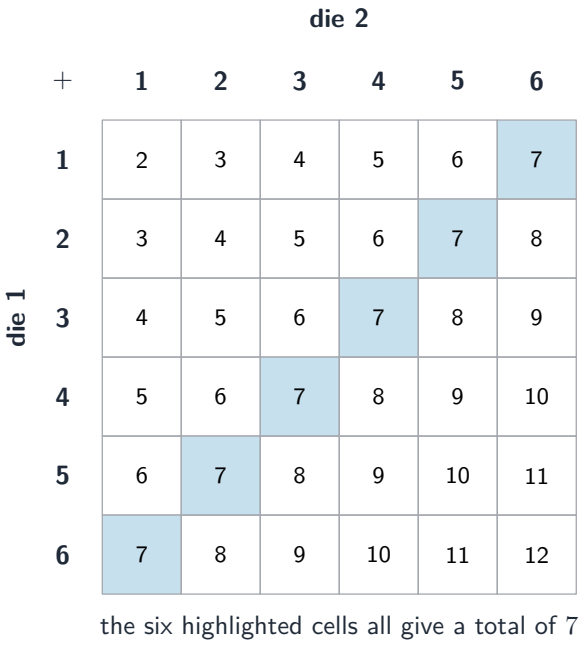
$$\mathrm{P}(3) = \frac{\text{favourable outcomes}}{\text{total outcomes}} = \frac{1}{8}$$

A fair spinner: eight identical sectors, so each number is an equally likely outcome.

Three pictures help you organise the work.

■ Sample space diagrams

A **sample space diagram** 样本空间图 is a table or grid listing every possible outcome – useful for two dice or two spinners. Count the outcomes you want out of the total.



For two independent fair six-sided dice, the 36 ordered-pair cells are equally likely. The totals are not: six cells give 7, but only one gives 2.

Worked example. Two fair dice are thrown. Find the probability that (a) the total is more than 9, (b) at least one die shows a six.

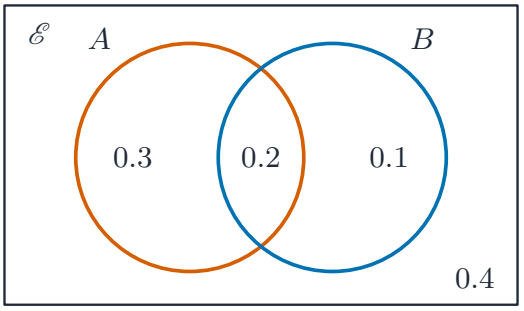
(a) Totals of 10, 11 and 12 fill $3+2+1=6$ cells, so the probability is $\dfrac{6}{36}=\dfrac{1}{6}$.

(b) The six row and the six column hold $6+6=12$ cells, but the cell (6, 6) is in both, so count it once: $\dfrac{11}{36}$.

■ Venn diagrams

A **Venn diagram** 维恩图 sorts outcomes into overlapping sets. From it you can read $\mathrm{P}(A\cap B)$ (in both) and $\mathrm{P}(A\cup B)$ (in either).

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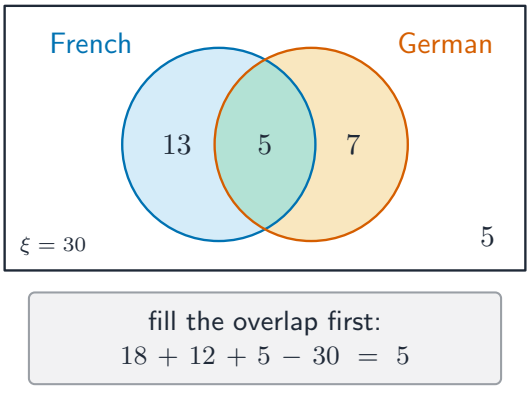
The overlap is $P(A \cap B) = 0.2$; everything inside either circle is $P(A \cup B) = 0.6$; outside both is 0.4.

For these regions, $\mathrm{P}(A) = 0.3 + 0.2 = 0.5$ and $\mathrm{P}(B) = 0.2 + 0.1 = 0.3$. Adding $0.5 + 0.3$ would count the overlap twice. Instead add the three separate regions inside either circle: $0.3 + 0.2 + 0.1 = 0.6$. Here "either" includes both. The outside region is neither, so $0.6 + 0.4 = 1$.

Try it. On one fair six-sided die, find the chance of an even number or a number greater than 4. **Answer:** the selected outcomes are $2, 4, 5, 6$, so the probability is $\frac{4}{6} = \frac{2}{3}$. Counting $\frac{3}{6} + \frac{2}{6}$ would count the 6 twice.

Worked example: fill in a Venn diagram. In a class of 30, 18 study French, 12 study German and 5 study neither. Find the probability that a student picked at random studies both.

$30 - 5 = 25$ students are inside the circles, but $18 + 12 = 30$. The extra $30 - 25 = 5$ students were counted twice, so 5 study both. Fill the overlap first: French only 13, German only 7, both 5, neither 5 (total 30).



Fill the overlap first, then the "only" regions: $18 - 5 = 13$ and $12 - 5 = 7$.

$$\mathrm{P}(\text{both}) = \frac{5}{30} = \frac{1}{6}.$$

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■ Tree diagrams

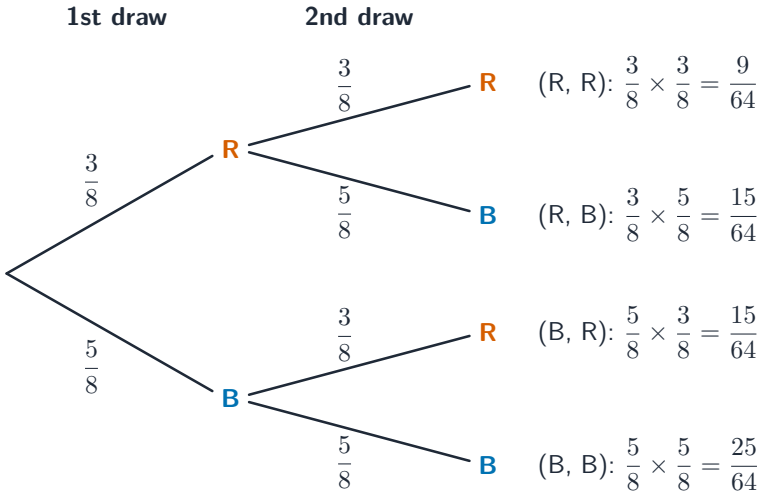
A **tree diagram** 树状图 shows each stage as a set of branches. Write the probability on each branch and the outcome at the end. **Multiply along** the branches, then **add** the paths you want.

> **Worked example (with replacement).** From the bag (3 red, 5 blue), a counter is drawn, replaced, then a second is drawn. This is **with replacement** 有放回, so the chances do not change. Find the probability of two reds.

$$\mathrm{P}(\text{red, red}) = \frac{3}{8} \times \frac{3}{8} = \frac{9}{64}.$$

For **one of each colour**, two paths count, red then blue and blue then red:

$$\mathrm{P}(\text{one of each}) = \frac{3}{8} \times \frac{5}{8} + \frac{5}{8} \times \frac{3}{8} = \frac{30}{64} = \frac{15}{32}.$$



On a tree diagram, multiply the probabilities along the branches; the four outcomes' probabilities add to 1.

Exactly one versus at least one. A different bag contains 2 red and 3 blue counters. Draw twice at random, replacing the first counter and mixing thoroughly. The four paths have probabilities RR $= \frac{4}{25}$, RB $= \frac{6}{25}$, BR $= \frac{6}{25}$ and BB $= \frac{9}{25}$. These add to 1. Exactly one red means RB or BR: $\frac{6}{25} + \frac{6}{25} = \frac{12}{25}$. At least one red also includes RR, so its probability is $\frac{16}{25}$. Equivalently, subtract the no-red path: $1 - \frac{9}{25} = \frac{16}{25}$. On a tree, multiply along a complete path, then add the wanted paths, which cannot happen together in the same pair of draws.

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Worked example (without replacement, Extended). Return to the bag of 3 red and 5 blue. Now the first counter is *not* replaced – drawing **without replacement** 无放回. After one red is taken, 2 reds remain out of 7. After a first blue, all 3 reds would remain out of 7; removing a blue does not reduce the red count. Core combined-event questions use replacement; without replacement is Extended.

$$\mathrm{P}(\text{red, red}) = \frac{3}{8} \times \frac{2}{7} = \frac{6}{56} = \frac{3}{28}.$$

For "**at least one** blue", use the complement: the only path with no blue is red, red, so

$$\mathrm{P}(\text{at least one blue}) = 1 - \frac{3}{28} = \frac{25}{28}.$$

Worked example: three draws (Extended). Three counters are taken from the same bag without replacement. Find the probability that all three are the same colour.

$$\mathrm{P}(\mathrm{RRR}) = \frac{3}{8} \times \frac{2}{7} \times \frac{1}{6} = \frac{6}{336}, \qquad \mathrm{P}(\mathrm{BBB}) = \frac{5}{8} \times \frac{4}{7} \times \frac{3}{6} = \frac{60}{336},$$

$$\mathrm{P}(\text{all the same}) = \frac{66}{336} = \frac{11}{56}.$$

Common mistakes. Adding along a branch instead of multiplying, forgetting the second path in "one of each", and keeping the same denominator on the second draw without replacement (it goes down by one each time).

Conditional probability (Extended)

Conditional probability 条件概率 is the chance of an event given some information. First restrict attention to the people or outcomes that meet the condition, then find the fraction that also meets the target. Use Venn diagrams, tables or tree diagrams; a special conditional-probability formula is not required.

Worked example: a two-way table. A class has these language choices. A student is chosen at random from the French group. Find the chance they also study German.

Students	German	Not German	Total
French	7	11	18
Not French	5	7	12
Total	12	18	30

Only the 18 French students can be selected, so the required probability is $\frac{7}{18}$. Choosing from the whole class instead would give $\frac{7}{30}$ for studying both languages. Reversing the

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condition gives another question: of the 12 German students, 7 study French, so the probability of French given German is $\frac{7}{12}$.



The condition picks the group you divide by: 7 of the 18 French students, 7 of the 12 German students, or 7 of the whole 30

Using a Venn diagram. The same data give French only 11, both 7, German only 5 and neither 7. Given *not French*, look outside the French circle: $5+7=12$ students are available, of whom 5 study German. The required probability is $\frac{5}{12}$. Include neither when the condition selects it.

> **Worked example: a tree without replacement.** A bag contains 3 red and 2 blue counters. Two are drawn at random without replacement. Given that the first is blue, 3 red and 1 blue remain. The probability the second is red is therefore $\frac{3}{4}$. Multiplying $\frac{2}{5}\times\frac{3}{4}=\frac{3}{10}$ would find the probability of *both* first blue and then red, before either draw.

If the second counter is known to be red but the first colour is hidden, two paths meet the condition: RR has probability $\frac{3}{5}\times\frac{2}{4}=\frac{6}{20}$ and BR has probability $\frac{2}{5}\times\frac{3}{4}=\frac{6}{20}$. Their total is $\frac{12}{20}$. The chance the first was blue, given this information, is the BR share: $\frac{6}{20}\div\frac{12}{20}=\frac{1}{2}$. Compare path probabilities; do not assume paths are equally likely just because there are two of them.

> **Try it.** In a class of 40, 20 students swim, 16 play music and 8 do both. Find the probability of music given swimming, then swimming given music. Explain the denominators.

Answer. $\frac{8}{20}=\frac{2}{5}$ and $\frac{8}{16}=\frac{1}{2}$. The overlap stays 8; the condition changes the available group from 20 swimmers to 16 music students.

> **Worked example: from a tree.** Two counters are taken from the bag (3 red, 5 blue) without replacement. Given that they are the same colour, find the probability that they are both red.

From the tree, $\mathrm{P(RR)}=\dfrac{6}{56}$ and $\mathrm{P(BB)}=\dfrac{5}{8}\times\dfrac{4}{7}=\dfrac{20}{56}$. The condition "same colour" keeps only these two paths:

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$$\text{P(both red, given the same colour)} = \frac{6}{6+20} = \frac{6}{26} = \frac{3}{13}.$$

Common mistake. Dividing by the whole total instead of by the part that matches the condition. In the table above, the probability that a German student also studies French is $\frac{7}{12}$, not $\frac{7}{30}$.

Exam tips

- Every probability lies between 0 and 1, and the probabilities of all the outcomes add to 1.
- For "**and**", multiply independent probabilities or use the appropriate conditional probability. For "**or**", add mutually exclusive probabilities; otherwise subtract the overlap. On a tree diagram, multiply along the branches.
- Watch for **without replacement**: the second probability changes because one item has already been removed.
- Expected frequency = probability $\times$ number of trials; it predicts a count, not a guaranteed result. Relative frequency estimates probability from observed results.

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Statistics

IGCSE Mathematics

This handout covers Topic 9, Statistics. Parts marked **(Extended)** are only tested on the Extended papers; everything else is for both levels.

Collecting and showing data



Data is collected from people – a sample drawn from a population.

To organise **statistical** 统计 **data** 数据, use:

- a **tally table** 计数表 – make a mark for each value, then count.
- a **two-way table** 双向表 – sorts data by two features at once (for example, boys/girls against walk/bus).

When you read a diagram, only draw conclusions the data really supports.

Averages and range

Three averages describe a typical value of the data, and the range shows how spread out it is. Each is used for a different purpose:



median $= 3$

mean $= \frac{2+3+3+5+7}{5} = 4$

mode $= 3$ (most common)

range $= 7 - 2 = 5$

Mean, median, mode and range of a small data set

- **mean** 平均数 $= \dfrac{\text{sum of all values}}{\text{how many values}}$.
- **median** 中位数 = the middle value when the data is put in order; for an even number of values, average the two middle values.
- **mode** 众数 = the value that appears most often.
- **range** 极差 = largest value − smallest value (it shows how spread out the data is).

Worked example. Find the mean, median, mode and range of $4, 7, 7, 2, 5$.

Order the data: $2, 4, 5, 7, 7$.

$$\text{mean} = \frac{4+7+7+2+5}{5} = \frac{25}{5} = 5, \quad \text{median} = 5, \quad \text{mode} = 7, \quad \text{range} = 7 - 2 = 5.$$

Worked example: an even number of values. Find the median of $12, 15, 9, 20, 14, 11$.

In order: $9, 11, 12, 14, 15, 20$. There are two middle values, so the median is halfway between them: $\dfrac{12+14}{2} = 13$.

Worked example: a missing value. The mean of five numbers is 8. Four of them are 4, 7, 9 and 10. Find the fifth.

The five numbers add to $5 \times 8 = 40$, and the four known ones add to 30, so the fifth is $40 - 30 = 10$.

Which average to use? The mode suits categories (the most popular colour); the median is not pulled by one extreme value (a few very high salaries); the mean uses every value.

Common mistakes. Finding the median without putting the data in order first, and giving the position $\frac{n+1}{2}$ instead of the value at that position.

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Averages from a frequency table

When data is listed with its **frequency** 频数 (how many times each value occurs), the mean is

$$\text{mean} = \frac{\sum(\text{value} \times \text{frequency})}{\sum \text{frequency}}.$$

Worked example. Values $1, 2, 3$ occur with frequencies $4, 5, 1$. Find the mean.

$$\text{mean} = \frac{1(4) + 2(5) + 3(1)}{4 + 5 + 1} = \frac{17}{10} = 1.7.$$

■ Grouped data (Extended)

For **grouped data** 分组数据, you cannot find the exact mean, so estimate it using the **midpoint** of each group as the value. The **modal class** 众数组 is simply the group with the highest frequency.

Worked example: estimate the mean. The times, t minutes, that 30 students took to travel to school are grouped below.

Time t (min)	$0 < t \leqslant 10$	$10 < t \leqslant 20$	$20 < t \leqslant 30$	$30 < t \leqslant 50$
Frequency	5	12	8	5
Midpoint	5	15	25	40

$$\text{estimated mean} = \frac{5(5) + 15(12) + 25(8) + 40(5)}{30} = \frac{605}{30} = 20.2 \text{ minutes (3 s.f.)}.$$

The modal class is $10 < t \leqslant 20$. The answer is only an estimate, because we do not know the exact times inside each group.

Common mistakes. Multiplying by the end of each class (or the class width) instead of the midpoint, and dividing by the number of classes (4) instead of the total frequency (30).

Measures of spread (Extended)

When data is in order, the **quartiles** 四分位数 cut it into four equal parts. The lower quartile (LQ) is one quarter of the way up; the upper quartile (UQ) is three quarters of the way up. The **interquartile range** 四分位距 measures the spread of the middle half:

$$\text{interquartile range} = \text{UQ} - \text{LQ}.$$

The interquartile range is useful because, unlike the range, it ignores extreme values.

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Worked example: quartiles from a list. Find the median, quartiles and interquartile range of

$$3,\ 5,\ 6,\ 8,\ 9,\ 10,\ 12,\ 13,\ 15,\ 18,\ 20.$$

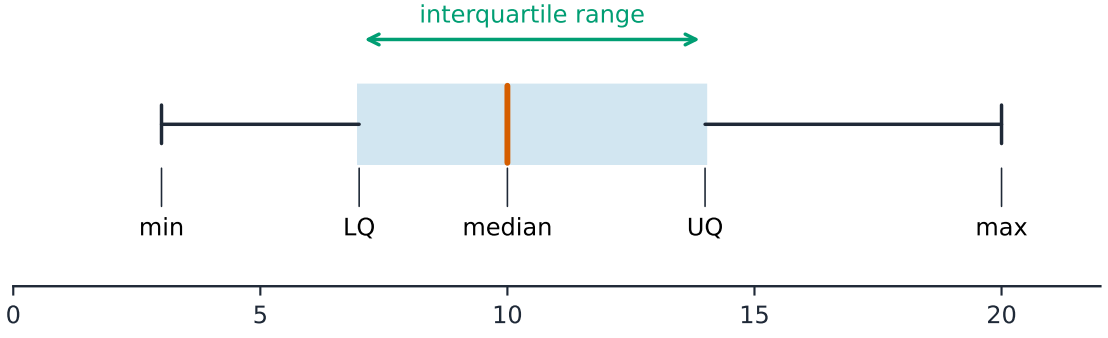
There are $n = 11$ values, already in order. The median is the $\frac{n+1}{2} = 6$th value, 10. The lower quartile is the $\frac{n+1}{4} = 3$rd value, 6, and the upper quartile is the $\frac{3(n+1)}{4} = 9$th value, 15. So the interquartile range is $15 - 6 = 9$.

Worked example: compare two distributions. Two cinemas record their audience numbers each night for a month.

	Mean	Range
Cinema A	78	40
Cinema B	96	18

Make one comparison of the average and one of the spread, and say what each means in context: "On average Cinema B has larger audiences, because its mean is higher. Cinema A's audience numbers vary more, because its range is larger."

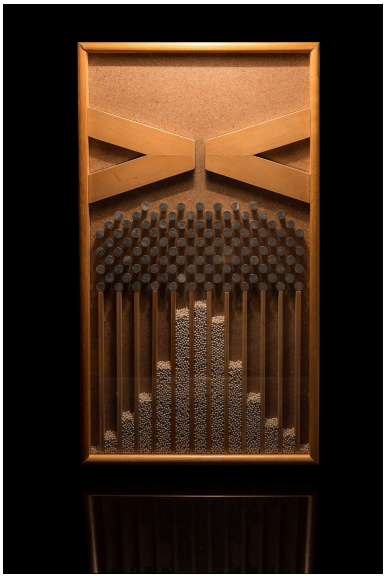
Common mistake. Saying only "B's mean is bigger" without saying what that means for the audiences, or comparing two averages and no spread.



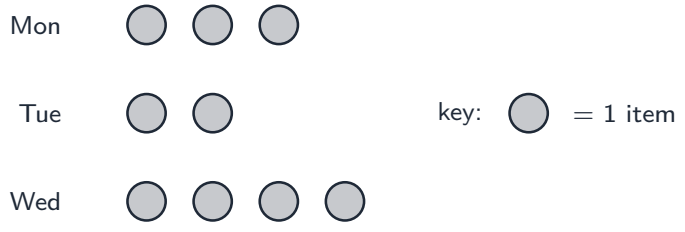
min, Q_1, median, Q_3, max. IQR $= Q_3 - Q_1$

A box-and-whisker plot shows the five-number summary; the box length is the interquartile range.

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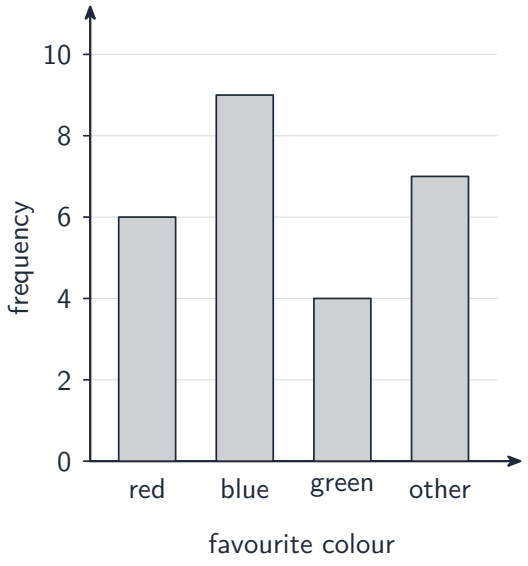


A Galton board shows how data piles up into a distribution.



A pictogram uses a symbol to stand for a number of items

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A bar chart shows the frequency of each group

Chart	What it shows
bar chart 条形图	a bar for each category; height is the frequency
pie chart 饼图	a circle split into slices, each a fraction of 360°
pictogram 象形图	uses a symbol to stand for a number of items
stem-and-leaf diagram 茎叶图	keeps the digits of ordered data, with a key
frequency distribution 频数分布	a table of values and their frequencies

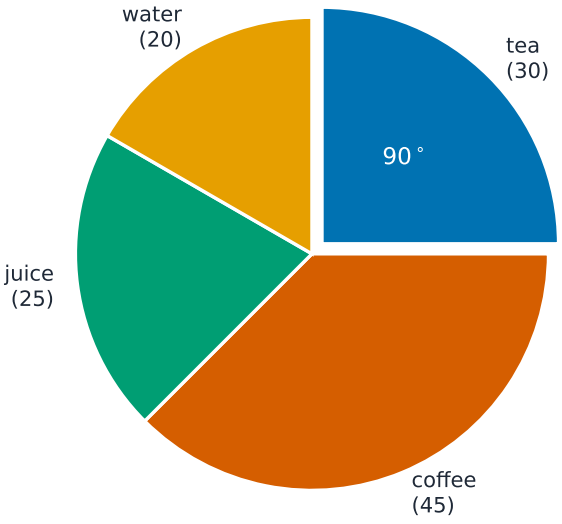
Worked example (pie chart). Out of 120 people, 30 chose tea. Find the angle of the tea slice.

$$\frac{30}{120} \times 360^\circ = 90^\circ.$$

Worked example (pie chart, backwards). A pie chart shows how 450 people travel to work. The bus slice has angle 72°. How many people travel by bus?

$$\frac{72}{360} \times 450 = 90 \text{ people.}$$

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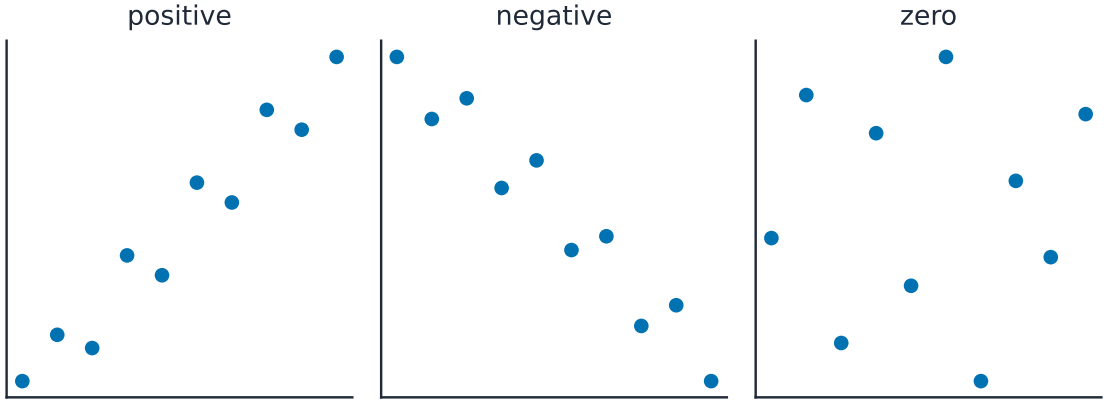


Each slice is its fraction of $360°$*; tea is* $\frac{30}{120} \times 360° = 90°$.

Scatter diagrams and correlation

A **scatter diagram** 散点图 plots pairs of values as points to show whether two things are linked. The link is the **correlation** 相关性:

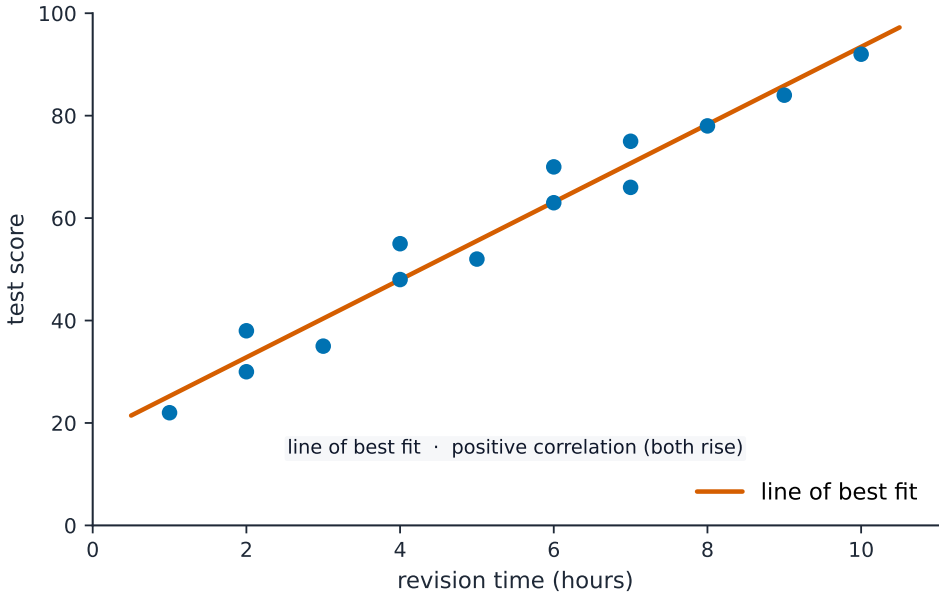
- **positive correlation** 正相关 – as one goes up, the other goes up.
- **negative correlation** 负相关 – as one goes up, the other goes down.
- **zero correlation** 零相关 – no clear straight-line trend; a curved pattern may still exist.



Positive correlation rises together; negative falls as the other rises; zero shows no clear link.

If there is correlation, draw a **line of best fit** 最佳拟合线: one straight ruled line through the middle of the points, with about the same number of points on each side. Use it to predict values.

Predict only **inside** the range of the data (interpolation). Reading the line beyond the plotted points (extrapolation) is unreliable, and correlation does not prove that one thing causes the other.



A line of best fit is one straight ruled line through the middle of the points; use it to predict.

Cumulative frequency diagrams (Extended)

The **cumulative frequency** 累积频数 is a running total of the frequencies. Plot it against the upper end of each class and join the points with a smooth curve.

For the travel times above, the running totals are:

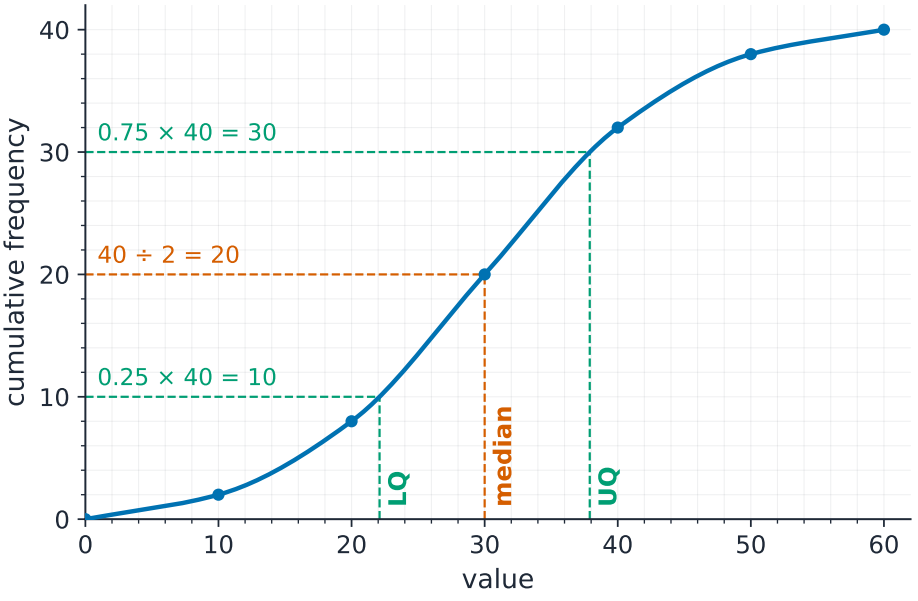
Time t (min)	$t \leqslant 10$	$t \leqslant 20$	$t \leqslant 30$	$t \leqslant 50$
Cumulative frequency	5	17	25	30

Plot $(10, 5)$, $(20, 17)$, $(30, 25)$ and $(50, 30)$, starting from $(0, 0)$. The median is read at $\frac{30}{2} = 15$ on the vertical axis, which falls in the class $10 < t \leqslant 20$.

Common mistake. Plotting at the midpoint of each class. Cumulative frequency is plotted at the **upper end** of the class.

From the curve you can read the median (at half the total), the quartiles (at one quarter and three quarters), and any **percentile** 百分位数 (for example, the 90th percentile is at 90% of the total).

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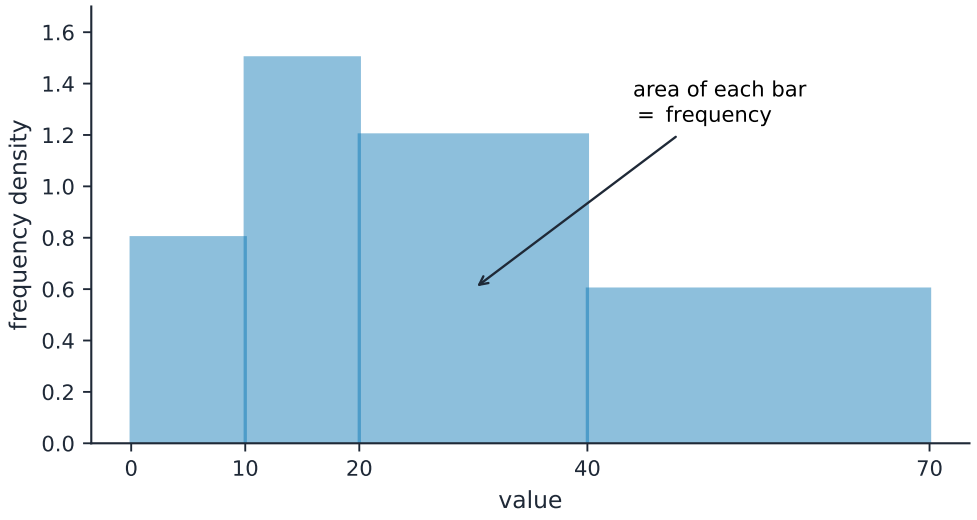
Read the median at half the total, and the quartiles at one quarter and three quarters: across to the curve, then down.

Histograms (Extended)

A **histogram** 直方图 looks like a bar chart, but the bars can have different widths and the area of each bar (not its height) shows the frequency. The vertical axis is the **frequency density** 频数密度:

$$\text{frequency density} = \frac{\text{frequency}}{\text{class width}}.$$

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density = freq / width · area = frequency

With unequal class widths the bar AREA (not its height) is the frequency, so the axis is frequency density.

Worked example. A class has **class width** 组距 10 and frequency 25. Find the frequency density.

$$\text{frequency density} = \frac{25}{10} = 2.5.$$

Worked example: frequency from a bar. A histogram bar covers $20 < h \leqslant 35$ and has frequency density 1.6. How many values are in this class?

$$\text{frequency} = \text{frequency density} \times \text{class width} = 1.6 \times 15 = 24.$$

Worked example: heights of bars. Classes $0 < x \leqslant 10$, $10 < x \leqslant 15$ and $15 < x \leqslant 30$ have frequencies 15, 20 and 18. The bar heights are $15 \div 10 = 1.5$, $20 \div 5 = 4$ and $18 \div 15 = 1.2$. The narrow middle class has the tallest bar, because its values are packed most closely together.

Common mistake. Drawing the frequency as the bar height when the classes have different widths.

Exam tips

- Know the three averages: **mean** (add up ÷ how many), **median** (the middle ordered value, or the mean of the two middle values), **mode** (the most common). The **range** is highest − lowest.

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- From a frequency table the mean is $\frac{\sum fx}{\sum f}$ – divide by the *total frequency*, not the number of rows.
- On a **scatter diagram**, describe the correlation as positive, negative or none, and draw the line of best fit through the mean point.
- For a **histogram** with unequal class widths, the height is the *frequency density* (frequency ÷ class width), not the frequency.

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