

Course Companion

IGCSE Mathematics

Every topic handout in one volume

全部专题讲义合集

exercises.lol

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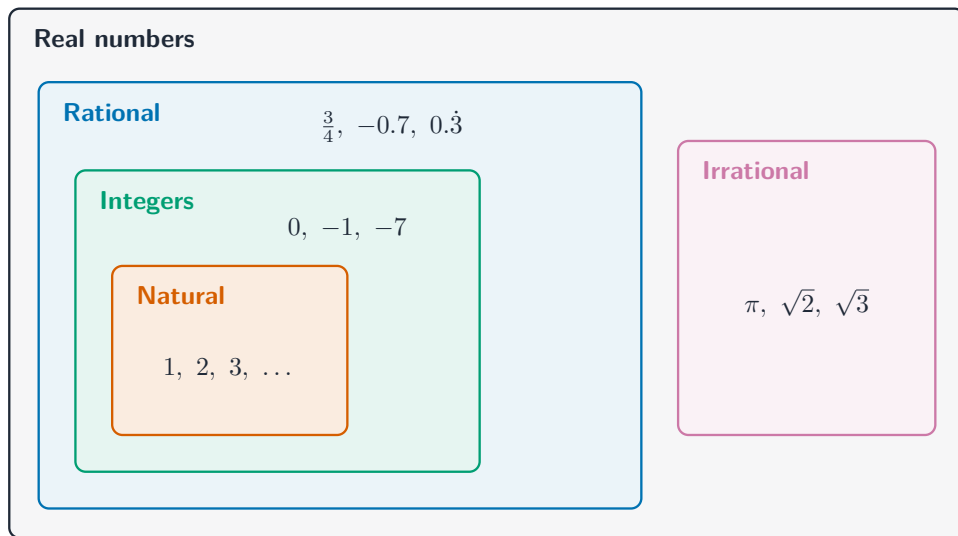
Number

IGCSE Mathematics

This handout covers Topic 1, Number. Cambridge Maths has two levels: Core and Extended. Parts marked **(Extended)** are only tested on the Extended papers; everything else is for both levels.

Types of number

You must know the words for the different kinds of number. Examiners give marks for using them correctly.



Each set sits inside the next: every natural number is an integer, every integer is rational, every rational is real. Irrational numbers are real but not rational.

Counting numbers and integers

- **natural numbers** 自然数—the counting numbers $1, 2, 3, 4, \dots$
- **integers** 整数—whole numbers, positive, negative or zero: $\dots, -2, -1, 0, 1, 2, \dots$

Factors and multiples

- A **factor** 因数 of a number divides into it exactly, leaving no **remainder** 余数. The factors of 18 are $1, 2, 3, 6, 9, 18$.
- A **multiple** 倍数 of a number is that number times an integer. Multiples of 6 are $6, 12, 18, 24, \dots$

- A **common factor** 公因数 of two numbers is a factor of both.
- A **common multiple** 公倍数 of two numbers is a multiple of both.

Prime, square and cube numbers

- A **prime number** 质数 has exactly two factors: 1 and itself. The first primes are 2, 3, 5, 7, 11, 13, ... Note that 1 is **not** prime.
- A **square number** 平方数 is a whole number times itself: 1, 4, 9, 16, 25, ...
- A **cube number** 立方数 uses a whole number three times: 1, 8, 27, 64, ...

Rational, irrational and reciprocal

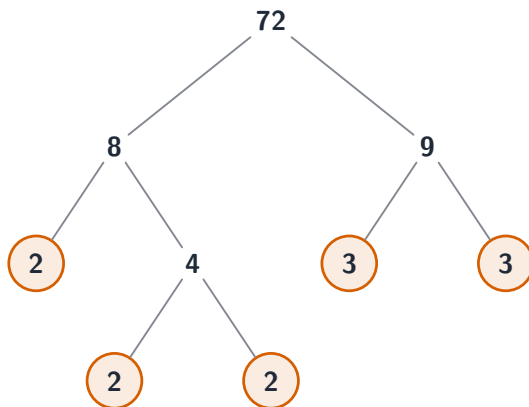
- A **rational number** 有理数 can be written as a **fraction** 分数 $\frac{a}{b}$ of two integers. Examples: $\frac{3}{4}$, 5, 0.7.
- An **irrational number** 无理数 cannot be written this way. Examples: π and $\sqrt{2}$.
- The **reciprocal** 倒数 of a number is 1 divided by that number. The reciprocal of 4 is $\frac{1}{4}$; the reciprocal of 0.25 is 4; the reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$.

Prime factors, HCF and LCM

Every integer above 1 is prime, or can be written as a **product** 乘积 of prime numbers. To write a number as a product of its **prime factors** 质因数, keep dividing by the smallest prime that fits.

Worked example. Write 72 as a product of its prime factors.

$$72 = 2 \times 36 = 2 \times 2 \times 18 = 2 \times 2 \times 2 \times 9 = 2 \times 2 \times 2 \times 3 \times 3 = 2^3 \times 3^2.$$



$$72 = 2^3 \times 3^2$$

Keep splitting until every branch ends on a prime (circled); collecting them gives
 $72 = 2^3 \times 3^2$.

The **highest common factor (HCF)** 最大公因数 of two numbers is the largest factor they share. The **lowest common multiple (LCM)** 最小公倍数 is the smallest multiple they share. Prime factors give a quick method.

Worked example. Find the HCF and LCM of 72 and 120.

First write each as a product of primes:

$$72 = 2^3 \times 3^2, \quad 120 = 2^3 \times 3 \times 5.$$

- HCF: take the **lowest** power of each prime that appears in *both*: $2^3 \times 3 = 24$.
- LCM: take the **highest** power of *every* prime that appears: $2^3 \times 3^2 \times 5 = 360$.

Sets and Venn diagrams

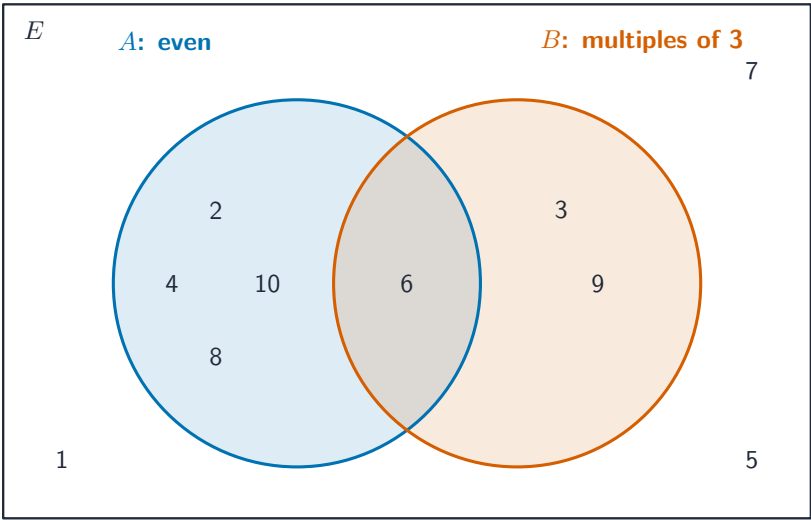
A **set** 集合 is a collection of objects. Each object in the set is an **element** 元素 of the set. You should know this notation:

Symbol	Meaning
$n(A)$	the number of elements in set A
$x \in A$	x is an element of A
$x \notin A$	x is not an element of A
E	the universal set 全集—everything being talked about
A'	the complement 补集 of A —everything not in A
$\emptyset$	the empty set 空集—a set with no elements
$A \subseteq B$	A is a subset 子集 of B —every element of A is also in B
$A \cup B$	the union 并集—elements in A or B or both
$A \cap B$	the intersection 交集—elements in both A and B

A **Venn diagram** 维恩图 draws each set as a circle inside a rectangle (the universal set). Core uses two sets; Extended may use three.

Worked example. $E = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{\text{even numbers}\}$, $B = \{\text{multiples of 3}\}$.

- $A = \{2, 4, 6, 8, 10\}$ and $B = \{3, 6, 9\}$.
- $A \cap B = \{6\}$ —the only number in both.
- $A \cup B = \{2, 3, 4, 6, 8, 9, 10\}$ —the numbers in either set.
- $n(A \cup B) = 7$.



$A \cap B = \{6\}$ is the only number in both circles; $A \cup B$ is everything inside either circle.

You may also see a set written as a rule, e.g. $C = \{x : 1 \leq x \leq 5\}$ means "all values x such that $1 \leq x \leq 5$ ".

Powers and roots

- A **power** 幂 (also called an **index** 指数, plural *indices*) shows how many times to multiply a number by itself: $2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32$.
- A **square root** 平方根 of a number gives that number when squared: $\sqrt{169} = 13$ because $13^2 = 169$.
- A **cube root** 立方根 works the same way for cubes: $\sqrt[3]{8} = 2$ because $2^3 = 8$.

You should be able to recall the squares from 1^2 to 15^2 (and their roots), and the cubes of 1, 2, 3, 4, 5 and 10.

Worked example. Work out $5^2 \times \sqrt[3]{8}$.

$$5^2 \times \sqrt[3]{8} = 25 \times 2 = 50.$$

The laws of indices

When you multiply or divide powers of the **same base** 底数, use these rules:

$$a^m \times a^n = a^{m+n}, \quad a^m \div a^n = a^{m-n}, \quad (a^m)^n = a^{mn}.$$

Some special powers:

$$a^0 = 1, \quad a^{-n} = \frac{1}{a^n}, \quad a^{\frac{1}{n}} = \sqrt[n]{a}, \quad a^{\frac{m}{n}} = (\sqrt[n]{a})^m.$$

(Fractional powers like $a^{\frac{m}{n}}$ are **Extended**.)

Worked examples.

- $2^{-3} \times 2^4 = 2^{-3+4} = 2^1 = 2$.
- $(2^3)^2 = 2^6 = 64$.
- $2^3 \div 2^4 = 2^{3-4} = 2^{-1} = \frac{1}{2}$.
- $7^{-2} = \frac{1}{7^2} = \frac{1}{49}$.
- $81^{\frac{1}{2}} = \sqrt{81} = 9$.
- $8^{-\frac{2}{3}} = \frac{1}{8^{\frac{2}{3}}} = \frac{1}{(\sqrt[3]{8})^2} = \frac{1}{2^2} = \frac{1}{4}$.

multiply the powers

$$a^m \times a^n = a^{m+n}$$

add:

$$2^{-3} \times 2^4 = 2^1 = 2$$

divide the powers

$$a^m \div a^n = a^{m-n}$$

subtract:

$$2^3 \div 2^4 = 2^{-1} = \frac{1}{2}$$

power of a power

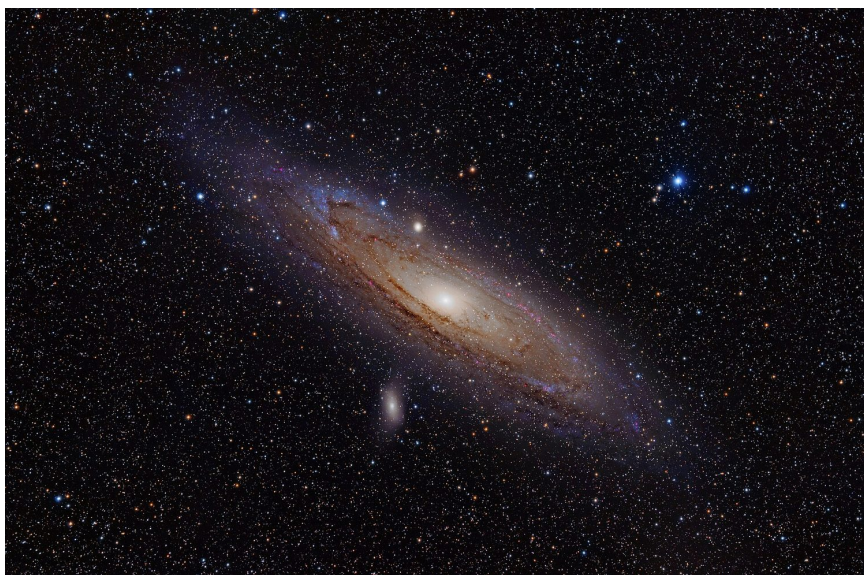
$$(a^m)^n = a^{mn}$$

multiply:

$$(2^3)^2 = 2^6 = 64$$

The three index laws: multiply adds the powers, divide subtracts them, a power of a power multiplies them

Standard form



A galaxy: huge distances are written compactly in standard form.

Standard form 科学记数法 writes a number as $A \times 10^n$, where $1 \leq A < 10$ and n is an integer. It is used for very large or very small numbers.

To convert, count how many places the decimal point moves:

- $4\,500\,000 = 4.5 \times 10^6$ —the point moves 6 places left, so the power is positive.
- $0.00072 = 7.2 \times 10^{-4}$ —the point moves 4 places right, so the power is negative.

Worked example. Work out $(3 \times 10^5) \times (2 \times 10^{-2})$.

Multiply the front numbers and add the powers:

$$3 \times 2 = 6, \quad 10^5 \times 10^{-2} = 10^3, \quad \text{so the answer is } 6 \times 10^3.$$

Standard form: $A \times 10^n$, **with** $1 \leq A < 10$

6 500 000



6.5×10^6

point moves 6 places

0.00042



4.2×10^{-4}

point moves 4 places

standard form: $a \times 10^n$ with $1 \leq a < 10$

Standard form: a number written as $a \times 10^n$ with $1 \leq a < 10$

Surds (Extended)

A **surd** 根式 is a root that is irrational, such as $\sqrt{5}$. Leave it in exact form instead of rounding. Two useful rules:

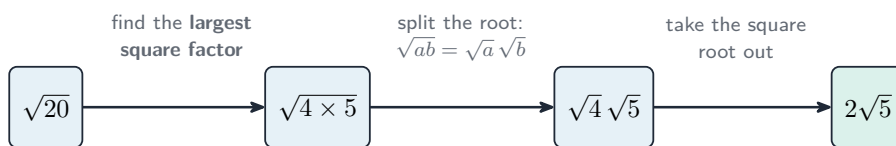
$$\sqrt{a} \times \sqrt{b} = \sqrt{ab}, \quad \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}.$$

Simplify a surd by taking out the largest square factor.

Worked example. Simplify $\sqrt{20}$ and $\sqrt{200} - \sqrt{32}$.

$$\sqrt{20} = \sqrt{4 \times 5} = \sqrt{4} \sqrt{5} = 2\sqrt{5}.$$

$$\sqrt{200} - \sqrt{32} = \sqrt{100 \times 2} - \sqrt{16 \times 2} = 10\sqrt{2} - 4\sqrt{2} = 6\sqrt{2}.$$



picking a smaller square factor ($\sqrt{20} = \sqrt{2 \times 10}$) gets stuck -
always take out the **largest** square: 4, 9, 16, 25, ...

Simplifying a surd: take out the largest square factor

To **rationalise the denominator** 分母有理化 means to remove a surd from the bottom of a fraction (the **denominator** 分母). Multiply the top and bottom by a value that clears the surd.

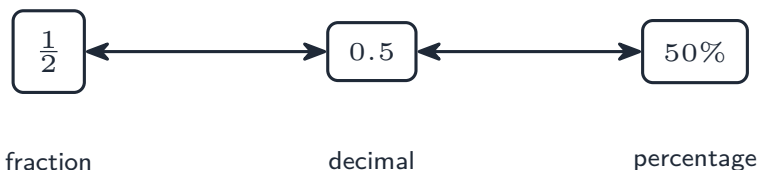
Worked example. Rationalise $\frac{10}{\sqrt{5}}$ and $\frac{1}{-1 + \sqrt{3}}$.

$$\frac{10}{\sqrt{5}} = \frac{10}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} = \frac{10\sqrt{5}}{5} = 2\sqrt{5}.$$

$$\frac{1}{-1 + \sqrt{3}} = \frac{1}{\sqrt{3} - 1} \times \frac{\sqrt{3} + 1}{\sqrt{3} + 1} = \frac{\sqrt{3} + 1}{3 - 1} = \frac{1 + \sqrt{3}}{2}.$$

Fractions, decimals and percentages

A fraction has a **numerator** 分子 (the top) and a **denominator** (the bottom).



One value three ways: a fraction, a decimal and a percentage

- **proper fraction** 真分数: numerator smaller than denominator, e.g. $\frac{3}{4}$.
- **improper fraction** 假分数: numerator the same or larger, e.g. $\frac{7}{4}$.
- **mixed number** 带分数: a whole number plus a fraction, e.g. $1\frac{3}{4}$.

Change between improper and mixed: $\frac{7}{4} = 1\frac{3}{4}$ because $7 \div 4 = 1$ remainder 3.

A **decimal** 小数 uses place value after a point. A **percentage** 百分比 means "out of 100", so $37\% = \frac{37}{100} = 0.37$.

Converting between forms

To change	Method	Example
fraction $\rightarrow$ decimal	divide top by bottom	$\frac{3}{8} = 3 \div 8 = 0.375$
decimal $\rightarrow$ percentage	multiply by 100	$0.07 = 7\%$
percentage $\rightarrow$ fraction	put over 100, then simplify	$7\% = \frac{7}{100}$
percentage $\rightarrow$ decimal	divide by 100	$34\% = 0.34$

Write a fraction in its **simplest form** 最简形式 by dividing the top and bottom by their HCF: $\frac{18}{24} = \frac{3}{4}$ (both divided by 6).

Recurring decimals (Extended)

A **recurring decimal** 循环小数 repeats the same digits forever. Dots mark the repeating part: $0.1\dot{7} = 0.1777\dots$ and $0.\dot{1}2\dot{3} = 0.123123\dots$

To turn a recurring decimal into a fraction, multiply so the repeating parts line up, then subtract.

Worked example. Write $0.1\dot{7}$ as a fraction.

Let $x = 0.1777\dots$ Only the 7 repeats, so use $10x$ and $100x$:

$$100x = 17.777\dots, \quad 10x = 1.777\dots$$

$$100x - 10x = 16, \quad 90x = 16, \quad x = \frac{16}{90} = \frac{8}{45}.$$

Write $x = 0.1\dot{7}$ as a fraction:

$100x = 17.$

$10x = 1.$

$90x = 16$

$\text{so } x = \frac{16}{90} = \frac{8}{45}$

777...

777...

subtract: the tails **cancel** - no dots left

Line up the repeating tails, subtract, and the tails cancel

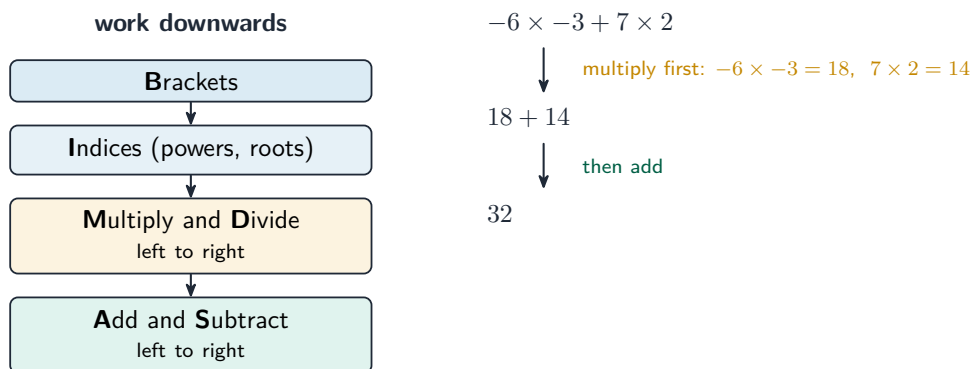
The four operations

Order of operations

Work in this order —the **order of operations** 运算顺序: brackets first, then indices (powers and roots), then multiply and divide (left to right), then add and subtract (left to right).

Worked example. Work out $-6 \times -3 + 7 \times 2$.

Do the multiplications first: $-6 \times -3 = 18$ and $7 \times 2 = 14$. Then add: $18 + 14 = 32$.



The order of operations, applied to the worked example

Negative numbers

- Adding a negative: $5 + (-3) = 5 - 3 = 2$.
- Subtracting a negative: $5 - (-3) = 5 + 3 = 8$.
- Multiplying or dividing: same signs give a positive; different signs give a negative.
So $-6 \times -3 = 18$ but $-12 \div 4 = -3$.

A change in temperature from -5°C to 3°C is a rise of 8°C .

Calculating with fractions

- **Multiply:** multiply the tops, multiply the bottoms: $\frac{2}{3} \times \frac{4}{5} = \frac{8}{15}$.
- **Divide:** multiply by the reciprocal of the second fraction: $\frac{2}{3} \div \frac{4}{5} = \frac{2}{3} \times \frac{5}{4} = \frac{10}{12} = \frac{5}{6}$.
- **Add or subtract:** use a common denominator (the LCM of the bottoms).

Worked example. Work out $1\frac{7}{15} - \frac{4}{5}$, giving the answer in its simplest form.

Change the mixed number to an improper fraction, then use denominator 15:

$$1\frac{7}{15} = \frac{22}{15}, \quad \frac{4}{5} = \frac{12}{15}, \quad \frac{22}{15} - \frac{12}{15} = \frac{10}{15} = \frac{2}{3}.$$

Ordering

Use these symbols to compare numbers by **magnitude** 大小 (size):

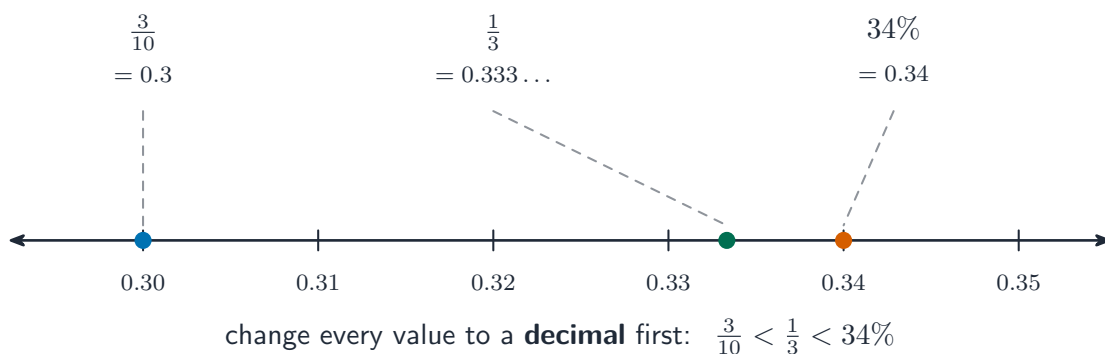
Symbol	Meaning
=	is equal to
≠	is not equal to
>	is greater than
<	is less than
≥	is greater than or equal to
≤	is less than or equal to

To put a mixed list in order, change every value to a decimal first.

Worked example. Put 34%, $\frac{1}{3}$ and $\frac{3}{10}$ in order, smallest first.

As decimals: 34% = 0.34, $\frac{1}{3} = 0.333\dots$, $\frac{3}{10} = 0.3$. So the order is

$$\frac{3}{10} < \frac{1}{3} < 34\%.$$



Convert every value to a decimal, then place them on a number line

Percentages



Shops use percentages for discounts, sales tax and profit margins.

Find a percentage of an amount. 15% of $\$80 = 0.15 \times 80 = \12 .

Write one amount as a percentage of another. A score of 18 out of 25 is $\frac{18}{25} \times 100\% = 72\%$.

Percentage increase or decrease uses

$$\text{percentage change} = \frac{\text{change}}{\text{original amount}} \times 100\%.$$

Worked example. A price rises from $\$40$ to $\$50$. Find the percentage increase.

The change is $\$10$, so $\frac{10}{40} \times 100\% = 25\%$ increase.

A quick way is a **multiplier** 乘数. To increase by 15% , multiply by 1.15 ; to decrease by 15% , multiply by 0.85 .

Simple and compound interest

Interest 利息 is money paid for borrowing or for saving. The **principal** 本金 is the starting amount.

- **Simple interest** 单利 pays the same amount each year, worked out on the principal only:

$$I = \frac{P \times r \times t}{100},$$

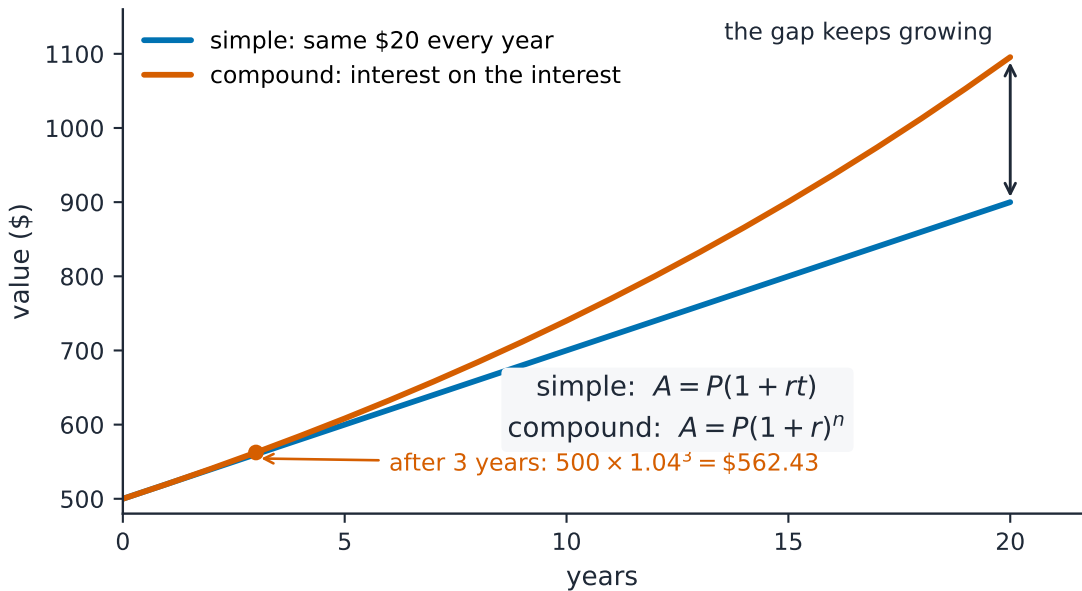
where P is the principal, r is the rate per year (as a percentage) and t is the number of years.

- **Compound interest** 复利 adds the interest on each year, so the next year earns interest on a larger total:

$$\text{final value} = P \left(1 + \frac{r}{100} \right)^t.$$

Worked example. Find the value of \$500 saved at 4% compound interest for 3 years.

$$500 \times 1.04^3 = 500 \times 1.124864 = \$562.43 \text{ (to the nearest cent).}$$



Simple interest grows in a straight line; compound interest grows faster every year

Reverse percentages (Extended)

A **reverse percentage** 逆百分比 problem gives the amount **after** a change and asks for the original. To solve it, divide by the multiplier —do not just take the percentage off.

Worked example. A coat costs \$60 after a 20% increase. Find the original price.

\$60 is 120% of the original, so the original price is $60 \div 1.2 = \$50$.

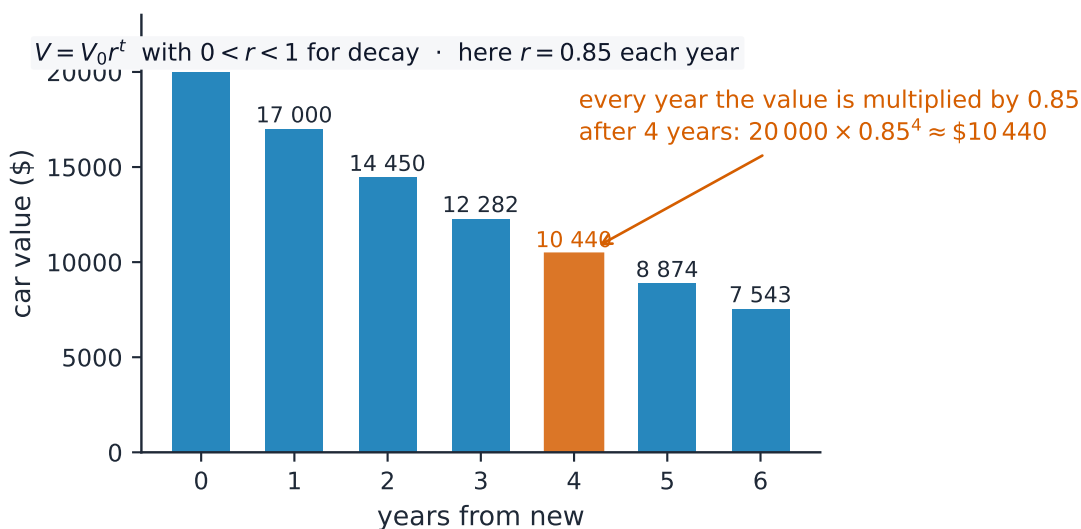
Exponential growth and decay (Extended)

When a quantity changes by the same percentage in each time period, it shows **exponential growth** 指数增长 (it gets bigger) or **exponential decay** 指数衰减 (it gets smaller). Use the compound formula. **Depreciation** 折旧, where something like a car loses value each year, is decay.

Worked example. A car worth \$20 000 loses 15% of its value each year. Find its value after 4 years.

The multiplier is 0.85, so

$$20\,000 \times 0.85^4 = 20\,000 \times 0.522 \dots = \$10\,440 \text{ (to the nearest dollar).}$$



Exponential decay: the car loses 15 percent of its current value every year

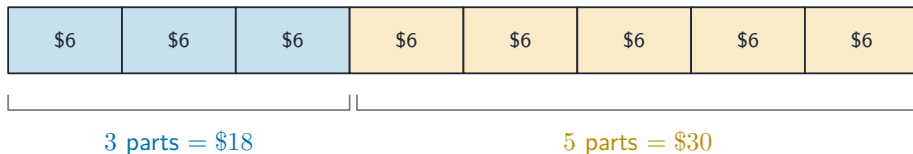
Ratio and proportion

A **ratio** 比 compares quantities, written like $a : b$. Simplify it like a fraction by dividing by the HCF: $20 : 30 : 40 = 2 : 3 : 4$.

Dividing in a ratio. Share \$48 in the ratio 3 : 5.

The total number of parts is $3 + 5 = 8$. One part is $48 \div 8 = \$6$. So the shares are $3 \times 6 = \$18$ and $5 \times 6 = \$30$.

Share \$48 in the ratio 3 : 5 - $3 + 5 = 8$ parts, one part = $48 \div 8 = \$6$



A bar model for sharing 48 dollars in the ratio 3 to 5

Proportion 比例 means two ratios are equal. Use it for recipes, map **scales** 比例尺 and finding best value.

Worked example. 3 pens cost \$1.80. Find the cost of 7 pens.

One pen costs $1.80 \div 3 = \$0.60$. So 7 pens cost $7 \times 0.60 = \$4.20$.

Rates and average speed

A **rate** 比率 compares two quantities measured in different units, such as price per kilogram, or distance per hour.

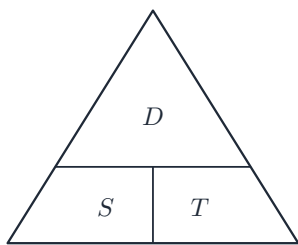
Average speed 平均速度 uses

$$\text{average speed} = \frac{\text{total distance}}{\text{total time}}.$$

Worked example. A cyclist travels 45 km in 3 hours 45 minutes. Find the average speed.

First change the time to hours: $3 \text{ h } 45 \text{ min} = 3.75 \text{ h}$. Then

$$\text{average speed} = \frac{45}{3.75} = 12 \text{ km/h}.$$



cover the one
you want

$$S = \frac{D}{T} \quad (\text{cover } S: D \text{ over } T)$$

$$D = S \times T \quad (\text{cover } D: S \text{ next to } T)$$

$$T = \frac{D}{S} \quad (\text{cover } T: D \text{ over } S)$$

$$\text{cyclist: } 45 \text{ km in } 3 \text{ h } 45 \text{ min} = 3.75 \text{ h} \Rightarrow S = \frac{45}{3.75} = 12 \text{ km/h}$$

The distance-speed-time triangle: cover the one you want

Other rates work the same way. **Density** 密度 is found from **mass** 质量 and **volume** 体积:

$$\text{density} = \frac{\text{mass}}{\text{volume}}.$$

Other examples are flow rate, fuel consumption and **population density** 人口密度. If a rate needs a special formula (such as **pressure** 压强), the question will give it to you.

Rounding and estimation

Rounding

- **decimal places (d.p.)** 小数位: digits counted after the point. 3.14159 to 2 d.p. is 3.14.
- **significant figures (s.f.)** 有效数字: digits counted from the first non-zero digit. 5764 to 1 s.f. is 6000; 0.004067 to 2 s.f. is 0.0041.

Rule: look at the next digit. If it is 5 or more, round up; if it is less, round down.

Estimation

To **estimate** 估算 an answer, round every number to 1 s.f., then calculate.

Worked example. Estimate $\frac{41.3}{9.79 \times 0.765}$.

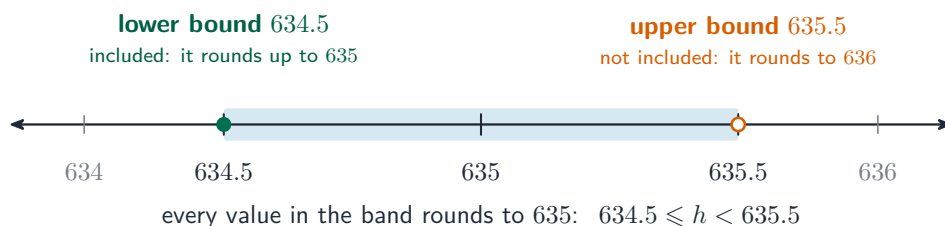
$$\frac{41.3}{9.79 \times 0.765} \approx \frac{40}{10 \times 0.8} = \frac{40}{8} = 5.$$

Limits of accuracy

A rounded value could really be anything that rounds to it. The smallest possible value is the **lower bound** 下界; the largest is the **upper bound** 上界. For a value rounded to the nearest unit, the bounds lie half a unit on each side.

Worked example. A height h is 635 m, correct to the nearest metre. Give the bounds.

$$634.5 \leq h < 635.5.$$



Everything in the band rounds to 635: the lower bound is included, the upper bound is not

So the lower bound is 634.5 m and the upper bound is 635.5 m.

Bounds in calculations (Extended)

Combine the bounds to get the bound you want.

Worked example. A rectangle is 8 cm by 5 cm, each side to the nearest cm. Find the largest possible **area** 面积.

Use the upper bounds of both sides: $8.5 \times 5.5 = 46.75 \text{ cm}^2$. (The smallest area uses the lower bounds: $7.5 \times 4.5 = 33.75 \text{ cm}^2$.)

For a divided quantity such as speed = $\frac{\text{distance}}{\text{time}}$, the **largest** speed comes from the **largest** distance divided by the **smallest** time.

Time

- 60 seconds = 1 minute, 60 minutes = 1 hour, 24 hours = 1 day, and 1 year = 365 days.
- The 24-hour clock writes a time as four digits: 3.15 p.m. is 15 15.

Worked example. A film starts at 19 35 and lasts 70 minutes. Find the time it finishes.

70 min = 1 h 10 min. Adding 1 hour gives 20 35; adding 10 minutes gives 20 45.

For timetables and **time zone** 时区 problems, add or subtract the time difference between the places.

Money

Work with money as ordinary decimals, but give answers to 2 d.p. (so \$4.8 is written \$4.80).

Currency conversion. Use the **exchange rate** 汇率 as a multiplier.

Worked example. The exchange rate is \$1 = €0.92. Convert \$150 to euros, and convert €138 back to dollars.

$$150 \times 0.92 = \text{€}138, \quad 138 \div 0.92 = \$150.$$

Using a calculator

- Do not round part-way through a calculation. Keep the full value and round only the final answer.
- Enter time as a decimal of an hour: 2 hours 30 minutes is 2.5 hours, not 2.30.
- Read the display in context: in money, 4.8 means \$4.80; in time, 3.25 hours means 3 hours 15 minutes.

Exam tips

- Follow **BIDMAS** (brackets, indices, division/multiplication, addition/subtraction) in order, and remember a negative times a negative is positive.
- In **standard form** the number in front is between 1 and 10; a small number (like 0.0004) has a *negative* power of 10.
- A percentage *change* is worked out on the **original** amount. For a reverse percentage, divide by the multiplier (e.g. $\div 1.2$ undoes a 20% rise).
- Do not round part-way through —keep the full value and round only at the end, to the accuracy the question asks for (decimal places or significant figures).
- For **limits of accuracy**, a value rounded to the nearest whole number can be up to 0.5 either side (so 8 means $7.5 \leq x < 8.5$).

Algebra and graphs

IGCSE Mathematics

This handout covers Topic 2, Algebra and graphs. Parts marked **(Extended)** are only tested on the Extended papers; everything else is for both levels.

Working with algebra

In **algebra** 代数 we use letters to stand for numbers. A letter whose value can change is a **variable** 变量. To **substitute** 代入 means to put a number in place of a letter.

Worked example. Find the value of $3x^2 - 2y$ when $x = 4$ and $y = 5$.

$$3 \times 4^2 - 2 \times 5 = 3 \times 16 - 10 = 48 - 10 = 38.$$

Simplifying and expanding

A **term** 项 is a single part of an **expression** 表达式, such as $5a$ or $-9b$. **Like terms** 同类项 have exactly the same letters; you may add or subtract them. The number in front of the letter is the **coefficient** 系数.

	x	2	
3	$3x$	6	

$$3(x + 2) = 3x + 6$$

Expanding a bracket with an area model

$$\begin{array}{c} 3a + 2a + 4b - b \\ \downarrow \text{collect like terms} \\ 5a + 3b \end{array}$$

Collecting like terms: add terms with the same letter

Worked example. Simplify $2a^2 + 3ab - 1 + 5a^2 - 9ab + 4$.

Collect like terms: $2a^2 + 5a^2 = 7a^2$, $3ab - 9ab = -6ab$, $-1 + 4 = 3$. So the answer is

$$7a^2 - 6ab + 3.$$

To **expand** 展开 means to multiply out **brackets** 括号. Multiply every term inside by the term outside; for two brackets, multiply every term in the first by every term in the second.

Worked examples.

$$3x(2x - 4y) = 6x^2 - 12xy.$$

$$(2x + 1)(x - 4) = 2x^2 - 8x + x - 4 = 2x^2 - 7x - 4.$$

For three brackets (**Extended**), expand two first, then multiply by the third:

$$(x - 2)(x + 3)(2x + 1) = (x^2 + x - 6)(2x + 1) = 2x^3 + 3x^2 - 11x - 6.$$

like terms: same letters/powers · add coefficients only

$$2a^2 \quad + 3ab \quad - 1 \quad + 5a^2 \quad - 9ab \quad + 4$$



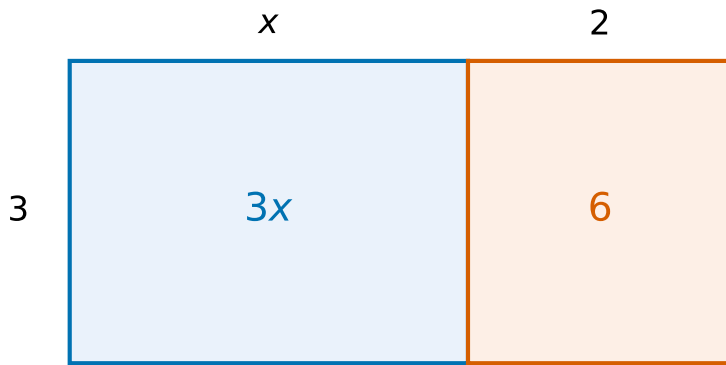
group like terms

$$7a^2 \quad - 6ab \quad + 3$$

terms that share a colour are like terms

Like terms share the same letters and powers —add coefficients only

$$3(x + 2) = 3x + 6$$



$$a(b + c) = ab + ac \quad \cdot \text{ area model of expansion}$$

Expand by distributing: $a(b+c)=ab+ac$ (area model)

Factorising

To **factorise** 因式分解 is the opposite of expanding: write the expression as a product of brackets. Always take out the **common factor** 公因式 first.

Worked example. $9x^2 + 15xy = 3x(3x + 5y)$, because $3x$ divides both terms.

The following patterns are **Extended**.

Grouping (four terms): take a common factor from each pair.

$$xy + 2x + 3y + 6 = x(y + 2) + 3(y + 2) = (x + 3)(y + 2).$$

Difference of two squares 平方差: $a^2 - b^2 = (a + b)(a - b)$.

$$9x^2 - 16 = (3x + 4)(3x - 4).$$

Perfect square 完全平方: $a^2 + 2ab + b^2 = (a + b)^2$.

$$x^2 + 6x + 9 = (x + 3)^2.$$

Quadratic 二次 expressions $ax^2 + bx + c$: find two numbers that multiply to $a \times c$ and add to b , then split the middle term.

Worked example. Factorise $2x^2 + 7x + 3$.

Here $a \times c = 6$ and $b = 7$. The numbers 1 and 6 work. Split and group:

$$2x^2 + x + 6x + 3 = x(2x + 1) + 3(2x + 1) = (x + 3)(2x + 1).$$

For $ax^3 + bx^2 + cx$, take out the common x first: $2x^3 + 7x^2 + 3x = x(2x^2 + 7x + 3) = x(x + 3)(2x + 1)$.

Completing the square (Extended)



A suspension bridge cable hangs in a parabola — the graph of a quadratic.

Completing the square 配方法 rewrites $x^2 + bx + c$ as $(x + p)^2 + q$. Take half of the x -coefficient, square it, then balance.

Worked example. Write $x^2 + 6x + 1$ in completed-square form.

Half of 6 is 3, and $3^2 = 9$:

$$x^2 + 6x + 1 = (x + 3)^2 - 9 + 1 = (x + 3)^2 - 8.$$

When the x^2 has a coefficient, take it out of the first two terms first:

$$2x^2 + 8x + 3 = 2(x^2 + 4x) + 3 = 2((x + 2)^2 - 4) + 3 = 2(x + 2)^2 - 5.$$

Algebraic fractions (Extended)

An **algebraic fraction** 分式 has algebra on the top or bottom. Add and subtract using a common denominator; multiply and divide as with ordinary fractions.

Worked examples.

$$\frac{x}{3} + \frac{x-4}{2} = \frac{2x}{6} + \frac{3(x-4)}{6} = \frac{2x+3x-12}{6} = \frac{5x-12}{6}.$$

$$\frac{3a}{4} \div \frac{9a}{10} = \frac{3a}{4} \times \frac{10}{9a} = \frac{30a}{36a} = \frac{5}{6}.$$

To simplify a **rational expression** 有理式, factorise the top and bottom, then cancel common brackets.

$$\frac{x^2-2x}{x^2-5x+6} = \frac{x(x-2)}{(x-2)(x-3)} = \frac{x}{x-3}.$$

Indices in algebra

The laws of **indices** 指数 work with letters too: $a^m \times a^n = a^{m+n}$, $a^m \div a^n = a^{m-n}$, and $(a^m)^n = a^{mn}$.

Worked examples.

$$(5x^3)^2 = 25x^6, \quad 12a^5 \div 3a^{-2} = 4a^7, \quad 6x^7y^4 \times 5x^{-5}y = 30x^2y^5.$$

You can also solve simple index equations by writing both sides with the same **base** 底数.

Worked example. Solve $2^x = 32$. Since $32 = 2^5$, you get $x = 5$.

Equations

An **equation** 方程 says two expressions are equal. To solve a **linear** 一次 equation, do the same operation to both sides until the **unknown** 未知数 is alone.

$$2x + 3 = 11 \quad (\text{start})$$

$$2x = 8 \quad \text{subtract 3 from both sides}$$

$$x = 4 \quad \text{divide both sides by 2}$$

Solve by doing the same to both sides

Worked example. Solve $5 - 2x = 3(x + 7)$.

$$5 - 2x = 3x + 21 \Rightarrow 5 - 21 = 3x + 2x \Rightarrow -16 = 5x \Rightarrow x = -\frac{16}{5}.$$

Fractional equations (Extended)

A **fractional equation** 分式方程 has the unknown in a denominator. Multiply both sides by the denominator to clear it.

Worked example. Solve $\frac{x}{2x+1} = 4$.

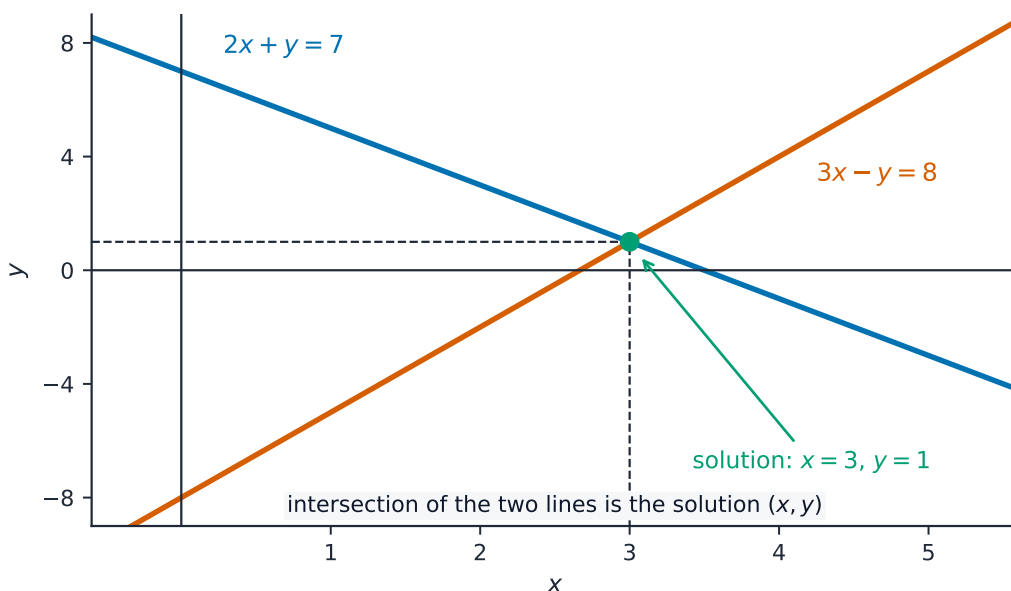
$$x = 4(2x + 1) = 8x + 4 \Rightarrow -7x = 4 \Rightarrow x = -\frac{4}{7}.$$

Simultaneous equations

Simultaneous equations 联立方程 are two equations solved together. For two linear equations, add or subtract to remove one letter.

Worked example. Solve $2x + y = 7$ and $3x - y = 8$.

Adding removes y : $5x = 15$, so $x = 3$. Then $y = 7 - 2(3) = 1$.



The solution of simultaneous equations is where their graphs cross

For one linear and one **quadratic** equation (**Extended**), substitute the linear into the curve.

Worked example. Solve $y = x + 2$ and $y = x^2$.

$$x^2 = x + 2 \Rightarrow x^2 - x - 2 = 0 \Rightarrow (x - 2)(x + 1) = 0,$$

so $x = 2$ (giving $y = 4$) or $x = -1$ (giving $y = 1$).

Solving quadratic equations (Extended)

There are three methods.

- **By factorising:** $x^2 + 5x + 6 = 0 \Rightarrow (x + 2)(x + 3) = 0 \Rightarrow x = -2$ or $x = -3$.
- **By completing the square:** $x^2 + 6x + 1 = 0 \Rightarrow (x + 3)^2 = 8 \Rightarrow x + 3 = \pm 2\sqrt{2} \Rightarrow x = -3 \pm 2\sqrt{2}$.

- **By the quadratic formula** 求根公式, which is given in the exam:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Worked example (formula). Solve $2x^2 + 3x - 1 = 0$. Here $a = 2$, $b = 3$, $c = -1$:

$$x = \frac{-3 \pm \sqrt{9 + 8}}{4} = \frac{-3 \pm \sqrt{17}}{4}.$$

Changing the subject

To **change the subject** 公式变形 of a formula means to rearrange it so a chosen letter is alone on one side.

Worked example. Make r the subject of $A = \pi r^2$ (**Extended**, because of the power).

$$r^2 = \frac{A}{\pi} \Rightarrow r = \sqrt{\frac{A}{\pi}}.$$

When the letter appears twice (**Extended**), collect those terms and factorise. To make x the subject of $y = \frac{x+1}{x-1}$:

$$y(x-1) = x+1 \Rightarrow yx - x = 1+y \Rightarrow x(y-1) = 1+y \Rightarrow x = \frac{1+y}{y-1}.$$

Inequalities

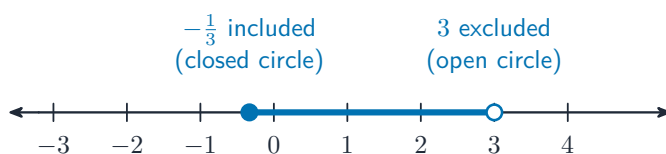
An **inequality** 不等式 uses $<$, $>$, $\leq$ or $\geq$. Solve it like an equation, but **reverse the sign** if you multiply or divide by a negative number.

Worked example. Solve $-3 \leq 3x - 2 < 7$.

Add 2 to all parts, then divide by 3:

$$-1 \leq 3x < 9 \Rightarrow -\frac{1}{3} \leq x < 3.$$

On a **number line** 数轴, use an open circle for $<$ or $>$ (value not included) and a closed circle for $\leq$ or $\geq$ (value included).

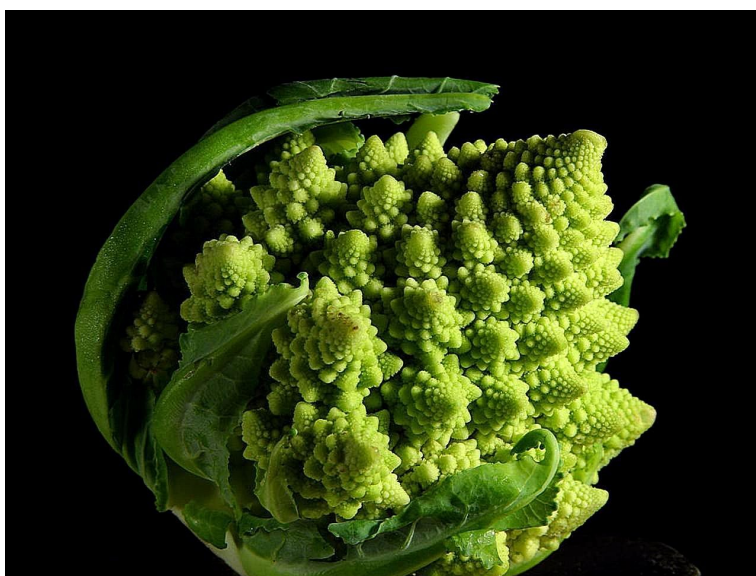


$-\frac{1}{3} \leq x < 3$: a closed circle includes the end value, an open circle excludes it.

Regions (Extended)

An inequality in two letters describes a **region** 区域 of the graph. Draw the boundary line (broken for $<$ or $>$, solid for $\leq$ or $\geq$) and shade the unwanted side. You may also be asked to list the inequalities that define a given region.

Sequences



Romanesco broccoli: self-similar spirals form a natural number pattern.

A **sequence** 数列 is a list of numbers that follow a rule. The **term-to-term rule** 递推规则 tells you how to get the next term from the one before.

To find the rule for the term in position n (the n th **term**), look at how the terms change.

Linear sequence (the terms go up by the same amount each time). The difference is the multiple of n .

Worked example. Find the n th term of 2, 5, 8, 11, ...

The terms go up by 3, so start with $3n$. Since $3 \times 1 = 3$ but the first term is 2, subtract 1: the n th term is $3n - 1$.

Quadratic sequence (the differences themselves change by the same amount). The second difference equals $2 \times$ the coefficient of n^2 .

Worked example. Find the n th term of 2, 5, 10, 17, ...

First differences are 3, 5, 7; the second difference is 2, so the n^2 part is $1n^2$. Subtracting n^2 (1, 4, 9, 16) from the sequence leaves 1, 1, 1, 1. So the n th term is $n^2 + 1$.

A **cubic** 三次 sequence such as 1, 8, 27, 64, ... has n th term n^3 . An **exponential sequence** 指数数列 such as 2, 6, 18, 54, ... multiplies by a fixed number each time; here the n th term is $2 \times 3^{n-1}$.

Direct and inverse proportion (Extended)

Two quantities are in **direct proportion** 正比例 if one is always a fixed multiple of the other: $y \propto x$ means $y = kx$, where k is a **constant** 常数. They are in **inverse proportion** 反比例 if one rises as the other falls: $y \propto \frac{1}{x}$ means $y = \frac{k}{x}$. The symbol $\propto$ is read "is proportional to". You can also have proportion to a square, square root, cube or cube root.

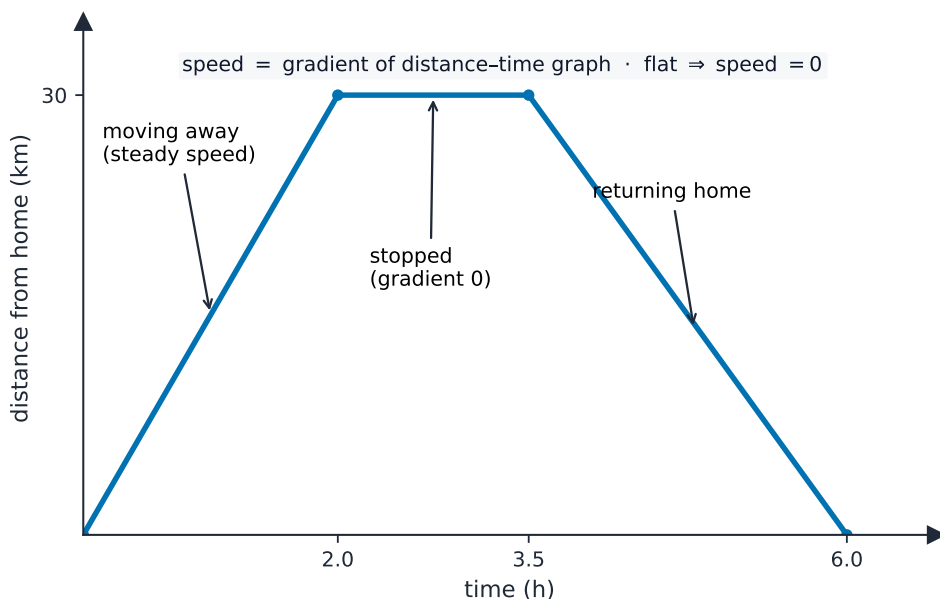
Worked example. y is in direct proportion to x , and $y = 12$ when $x = 3$. Find y when $x = 7$.

First find k : $12 = k \times 3$, so $k = 4$ and $y = 4x$. Then $y = 4 \times 7 = 28$.

Graphs in practical situations

The **gradient** 斜率 (steepness) of a graph shows a **rate of change** 变化率.

- A **travel graph** 行程图 (distance–time graph) has gradient equal to speed; a flat part means the object is not moving.
- A **conversion graph** 换算图 is a straight line used to change between two units (for example, miles and kilometres).



On a distance–time graph the gradient is the speed; a flat section means the object has stopped.

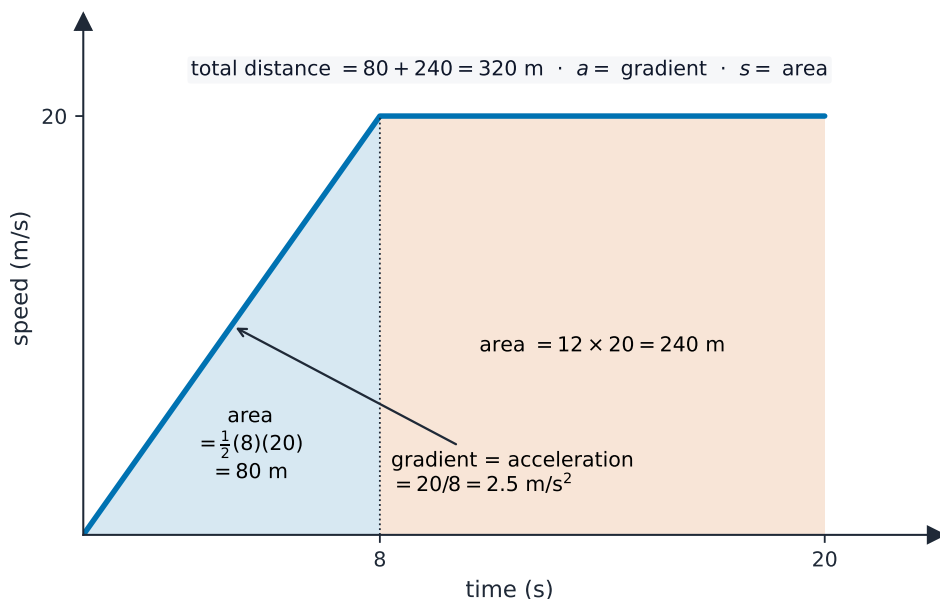
Speed–time graphs (Extended)

On a speed–time graph the gradient is the **acceleration** 加速度 (or **deceleration** 减速度 if the speed falls), and the **area** 面积 under the graph is the **distance** 距离 travelled.

Worked example. A car speeds up from rest to 20 m/s in 8 s, then stays at 20 m/s for 12 s. Find the acceleration and the total distance.

Acceleration = $\frac{20}{8} = 2.5 \text{ m/s}^2$. The distance is the area: a triangle plus a rectangle,

$$\frac{1}{2} \times 8 \times 20 + 12 \times 20 = 80 + 240 = 320 \text{ m.}$$



On a speed–time graph the gradient is the acceleration and the area underneath is the distance travelled.

Graphs of functions

To draw a graph, make a **table of values** 数值表: choose values of x , work out y , then plot the **coordinates** 坐标 and join them with a smooth curve.

The points where a graph crosses the x -axis (the horizontal **axis** 坐标轴) are the **roots** 根—the solutions of $y = 0$.

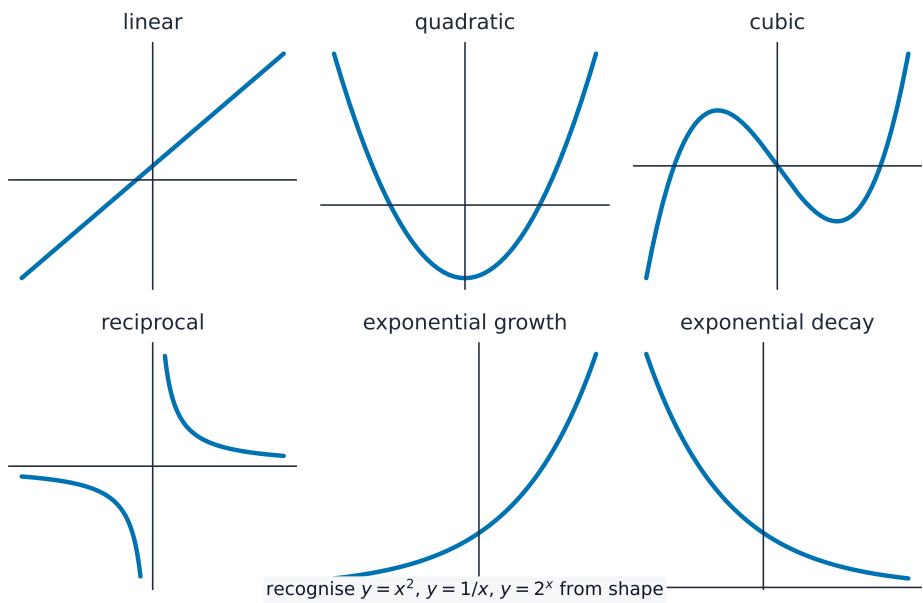
You can solve an equation by reading a graph. The **intersection point** 交点 of a line and a curve gives the solution of the two equations together.

For **exponential growth** 指数增长 and **exponential decay** 指数衰减, the graph of $y = ab^x + c$ rises (or falls) faster and faster and flattens towards a horizontal line.

Sketching curves

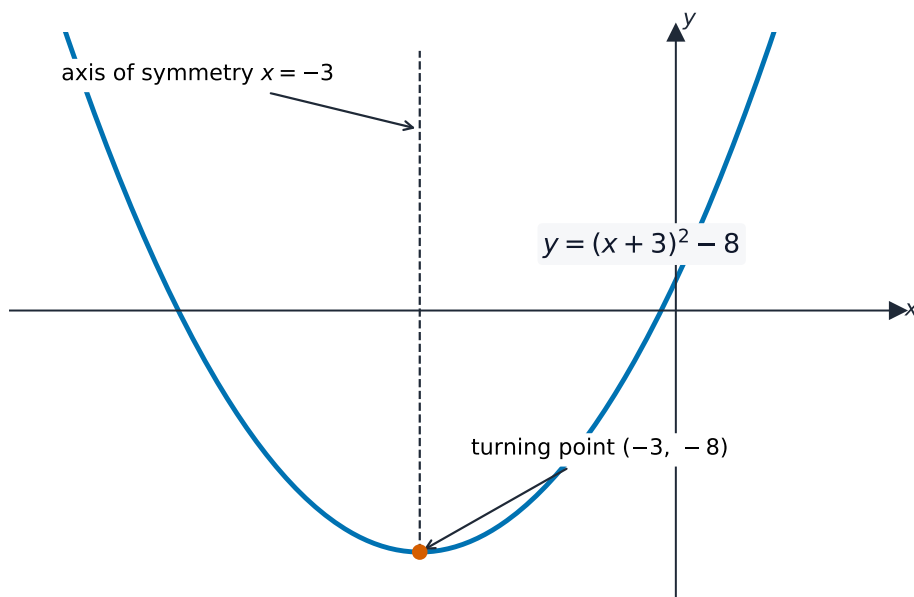
A quick sketch should show the right shape and the key features: where it crosses the axes, any **symmetry** 对称, and any line the curve gets close to.

Function	Shape
linear, $y = mx + c$	straight line; gradient m , y-intercept 截距 c
quadratic, $y = ax^2 + bx + c$	a parabola 抛物线 (U-shape if $a > 0$, $\cap$ -shape if $a < 0$)
cubic, $y = ax^3 + bx + c$	an S-shaped curve
reciprocal, $y = \frac{a}{x} + b$	two separate curves
exponential, $y = ar^x + b$	fast growth or decay



The basic graph shapes; knowing each shape lets you sketch quickly from the equation.

For a parabola, completing the square gives the **turning point** 转折点 (the lowest or highest point). For example $y = (x + 3)^2 - 8$ has its turning point at $(-3, -8)$.

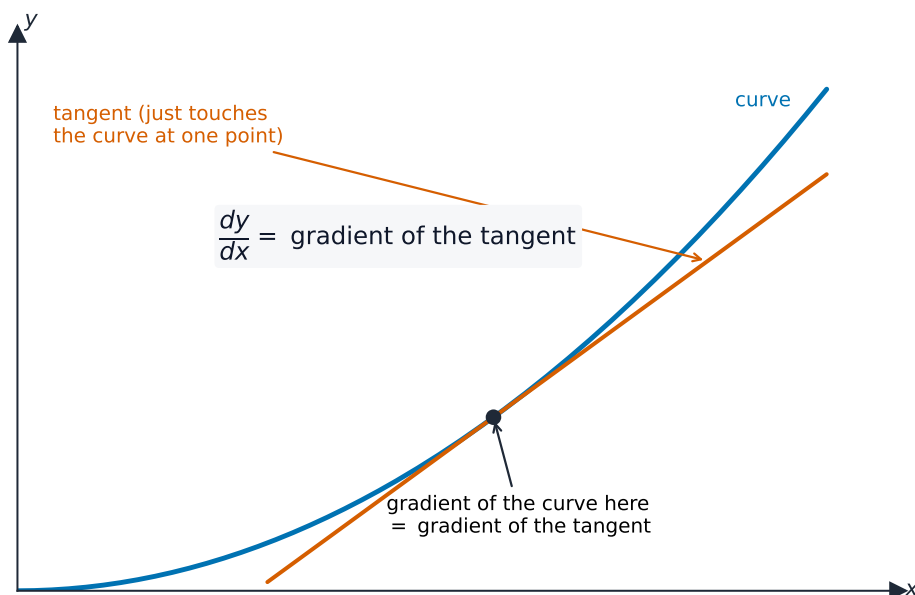


Completing the square, $y = (x + 3)^2 - 8$, shows the turning point $(-3, -8)$ and the line of symmetry $x = -3$.

An **asymptote** 渐近线 is a line that the curve gets closer and closer to but never touches—for example, the x -axis for $y = \frac{a}{x}$, or the line $y = b$ for $y = ar^x + b$.

Differentiation (Extended)

Differentiation 微分 finds the gradient of a curve at any point. You can estimate it by drawing a **tangent** 切线 (a line that just touches the curve) and measuring its gradient.



The gradient of a curve at a point equals the gradient of the tangent there — what differentiation finds.

The exact rule: if $y = ax^n$, then the **derivative** 导数 is

$$\frac{dy}{dx} = a n x^{n-1}.$$

Differentiate a sum term by term.

Worked example. If $y = x^3 + 2x^2 - 5x$, then $\frac{dy}{dx} = 3x^2 + 4x - 5$.

A **stationary point** 驻点 (turning point) is where the gradient is zero, so set $\frac{dy}{dx} = 0$.

Worked example. Find the turning point of $y = x^2 - 6x + 5$.

$$\frac{dy}{dx} = 2x - 6 = 0 \Rightarrow x = 3, \quad y = 3^2 - 6(3) + 5 = -4.$$

The turning point is $(3, -4)$. To decide whether a turning point is a **maximum** 最大值 or a **minimum** 最小值, check the sign of the gradient on each side, or use the

second derivative (positive means a minimum).

Functions (Extended)

A **function** 函数 turns each input into one output. We write $f(x)$, for example $f(x) = 3x - 5$, so $f(2) = 1$. The set of allowed inputs is the **domain** 定义域; the set of possible outputs is the **range** 值域.

The **inverse function** 反函数 $f^{-1}(x)$ undoes the function. To find it, write $y = f(x)$, swap the roles, and make x the subject.

Worked example. Find the inverse of $f(x) = 3x - 5$.

$$y = 3x - 5 \Rightarrow x = \frac{y + 5}{3}, \quad \text{so} \quad f^{-1}(x) = \frac{x + 5}{3}.$$

$f(x) = 3x - 5$: do $\times 3$, then -5



$f^{-1}(x) = \frac{x + 5}{3}$: **reverse the order, undo each step**



the inverse takes the output back to the input: $f(2) = 1$, so $f^{-1}(1) = 2$

A function as a machine; the inverse runs it backwards - reverse the order, undo each step

A **composite function** 复合函数 applies one function after another: $gf(x)$ means "do f first, then g ".

Worked example. If $f(x) = 2x$ and $g(x) = x + 3$, then

$$gf(x) = g(2x) = 2x + 3, \quad fg(x) = f(x + 3) = 2(x + 3) = 2x + 6.$$

Exam tips

- When you **expand brackets**, multiply every term and watch the signs, especially with a minus in front: $-(x - 3) = -x + 3$.
- To solve an equation, do the same thing to both sides. When you multiply or divide an **inequality** by a negative number, *flip* the sign.

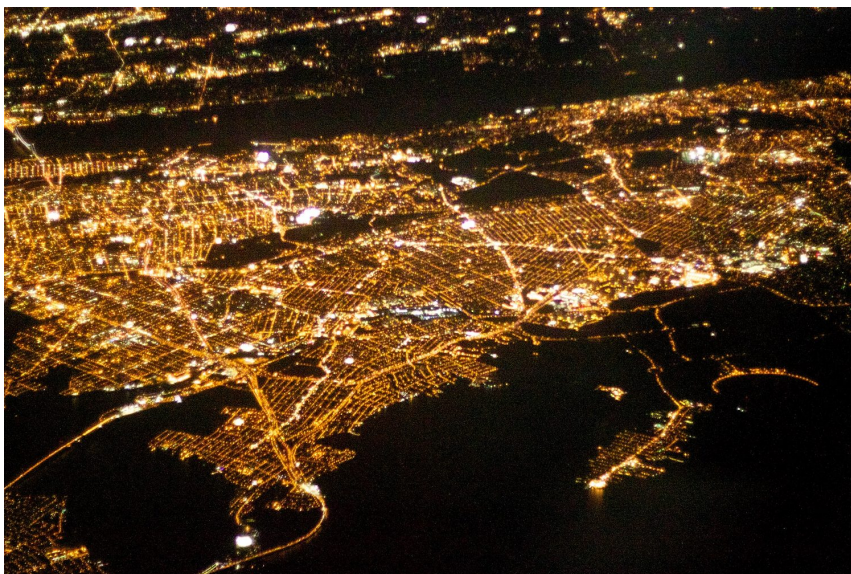
- **Factorise fully:** take out the highest common factor first, then look for a difference of two squares or a quadratic pattern.
- A **quadratic** usually has two solutions —give both. Check whether the question wants factorising, the formula, or completing the square.
- When substituting into a formula, put each value in brackets first, so signs and powers come out right.

Coordinate geometry

IGCSE Mathematics

This handout covers Topic 3, Coordinate geometry. Parts marked (**Extended**) are only tested on the Extended papers; everything else is for both levels.

Coordinates

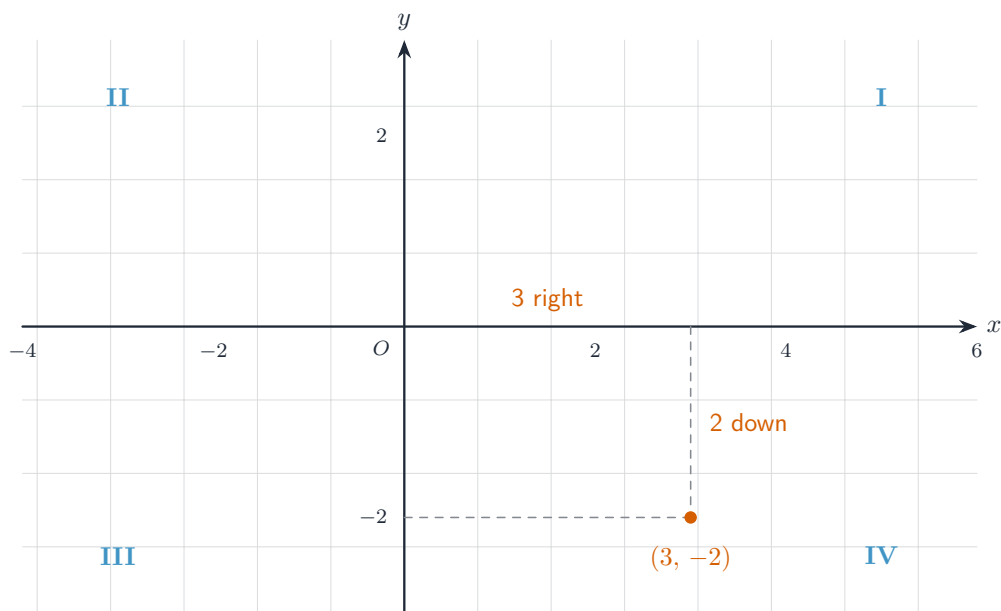


A city street grid: every place is fixed by its coordinates.

A point on a graph is described by its **coordinates** 坐标, sometimes called Cartesian coordinates, written (x, y) . The first number is the across value and the second is the up value.

- The two number lines are the **axes** 坐标轴: the **horizontal** 水平 x -axis and the **vertical** 竖直 y -axis.
- They cross at the **origin** 原点, the point $(0, 0)$.
- The axes split the grid into four **quadrants** 象限.

So the point $(3, -2)$ is found by going 3 to the right and 2 down.



The axes meet at the origin O and split the plane into four quadrants; $(3, -2)$ means 3 right then 2 down.

The equation of a straight line

Most straight lines can be written as

$$y = mx + c,$$

where m is the **gradient** 斜率 (steepness) and c is the **intercept** 截距—the y -value where the line crosses the y -axis.

- A line like $x = k$ (for example $x = 3$) is **vertical**.
- A line like $y = k$ (for example $y = 3$) is **horizontal**.

A line may also be given as $ax + by = c$. Rearrange it into $y = mx + c$ to read off the gradient and intercept.

Worked example. Find the gradient and y -intercept of $5x + 4y = 8$.

$$4y = -5x + 8 \Rightarrow y = -\frac{5}{4}x + 2.$$

So the gradient is $-\frac{5}{4}$ and the y -intercept is 2.

Gradient



A steep mountain road: gradient measures steepness as rise over run.

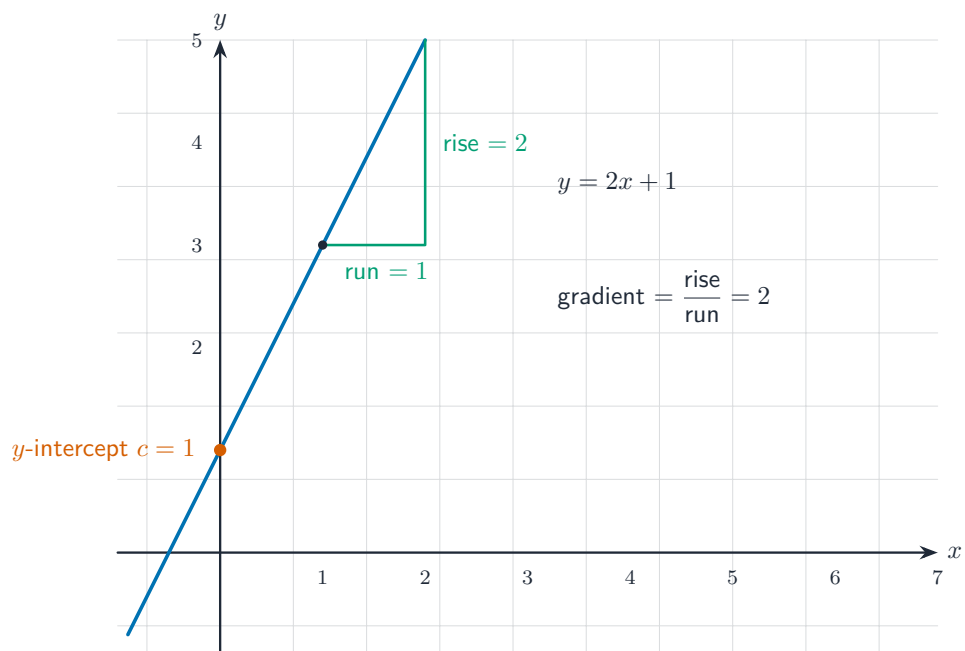
The **gradient** measures how steep a line is:

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{\text{rise}}{\text{run}}.$$

A positive gradient goes up to the right; a negative gradient goes down to the right.

Worked example. Find the gradient of the line through $(1, 2)$ and $(4, 11)$.

$$m = \frac{11 - 2}{4 - 1} = \frac{9}{3} = 3.$$



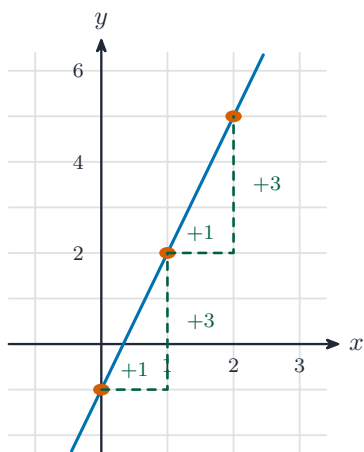
For $y = mx + c$, the line crosses the y -axis at c and the gradient m is the rise divided by the run.

Drawing a straight-line graph

To draw $y = mx + c$, the quickest way is:

1. Mark the intercept c on the y -axis.
2. Use the gradient to step to more points (for $m = 3$, go 1 right and 3 up).
3. Join the points with a straight line.

You can also make a small **table of values** 数值表: choose two or three x -values, work out y , and plot the points.



drawing $y = 3x - 1$:

1. mark the intercept $c = -1$
2. gradient 3: step 1 right, 3 up, again and again
3. join the points

check with a table: $x = 0, 1, 2 \rightarrow y = -1, 2, 5$

Drawing a line fast: mark the intercept, then use the gradient to step to new points

Finding the equation of a line

If you know the gradient m and one point on the line, put the point into $y = mx + c$ to find c .

Worked example. A line has gradient 3 and passes through $(1, 2)$. Find its equation.

$$y = 3x + c, \quad 2 = 3(1) + c, \quad c = -1.$$

So the equation is $y = 3x - 1$. (If you are given two points, first find the gradient, then do this.)

Length and midpoint (Extended)

A **line segment** 线段 is the straight piece between two points.

To find the **length** 长度 of the segment between (x_1, y_1) and (x_2, y_2) , use **Pythagoras' theorem** 勾股定理 on the horizontal and vertical gaps:

$$\text{length} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

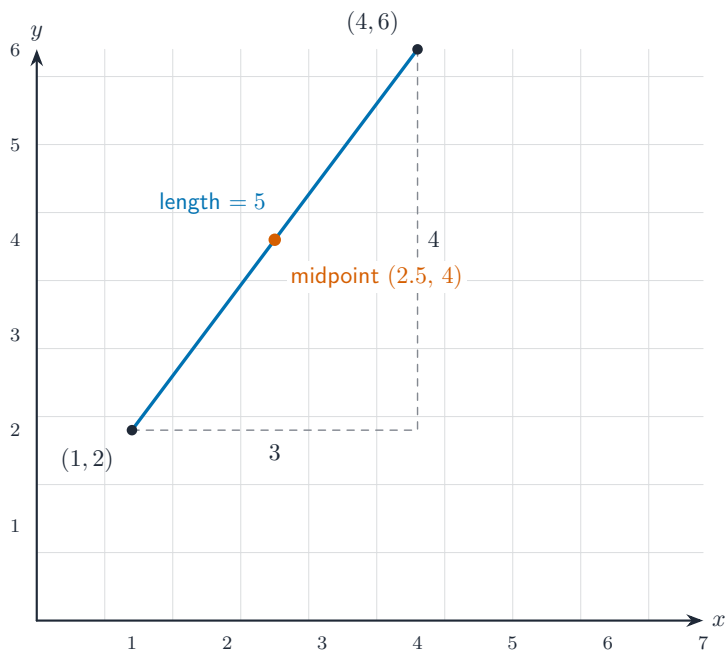
To find the **midpoint** 中点 (the point exactly in the middle), average the coordinates:

$$\text{midpoint} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right).$$

Worked example. Find the length and midpoint of the segment from $(1, 2)$ to $(4, 6)$.

$$\text{length} = \sqrt{(4 - 1)^2 + (6 - 2)^2} = \sqrt{9 + 16} = \sqrt{25} = 5.$$

$$\text{midpoint} = \left(\frac{1 + 4}{2}, \frac{2 + 6}{2} \right) = (2.5, 4).$$



The horizontal gap (3) and vertical gap (4) make a right triangle, so the length is $\sqrt{3^2 + 4^2} = 5$; the midpoint is the average of the coordinates.

Parallel lines

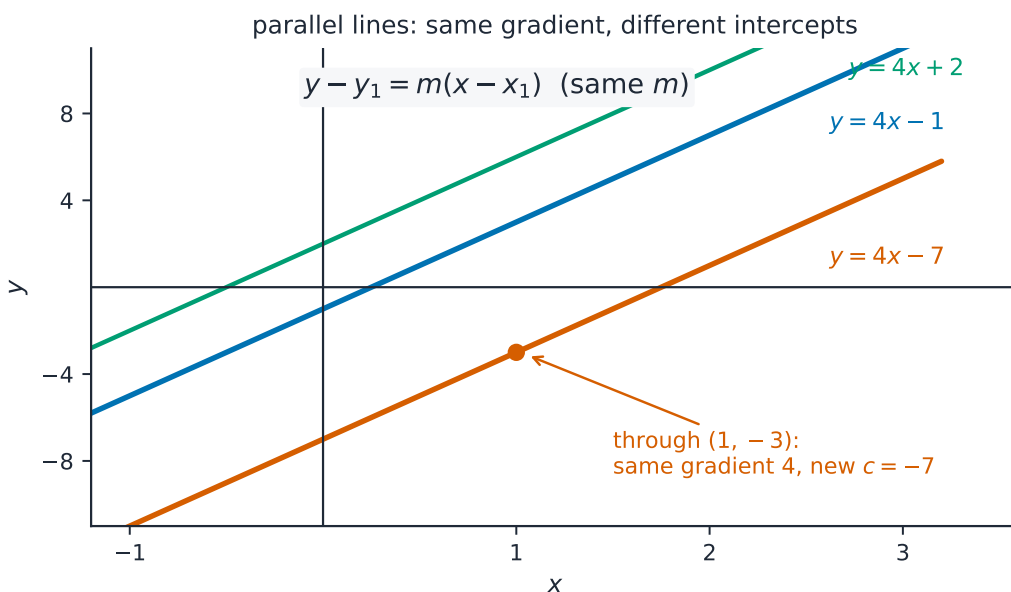
Parallel 平行 lines never meet, so they have the **same gradient**.

Worked example. Find the equation of the line parallel to $y = 4x - 1$ that passes through $(1, -3)$.

The gradient is also 4. Put the point in:

$$-3 = 4(1) + c \Rightarrow c = -7,$$

so the line is $y = 4x - 7$.



Parallel lines: same gradient, different intercepts

Perpendicular lines (Extended)

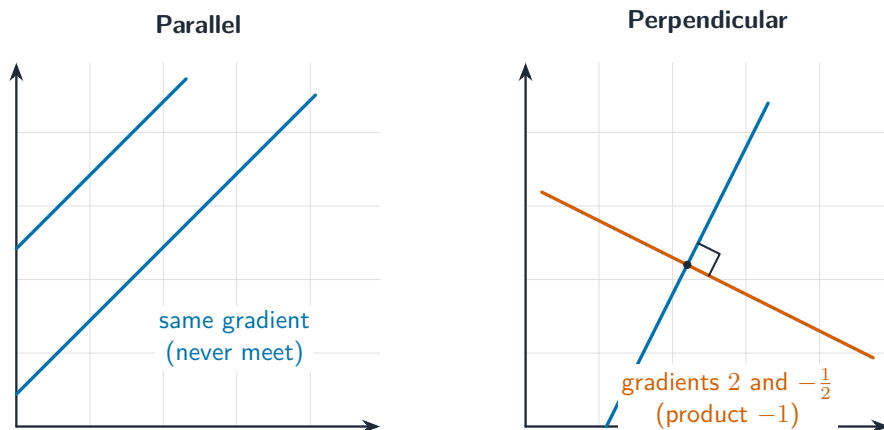
Two lines are **perpendicular** 垂直 if they meet at a **right angle** 直角. Their gradients multiply to -1 :

$$m_1 \times m_2 = -1, \quad \text{so} \quad m_2 = -\frac{1}{m_1}.$$

In words: flip the fraction and change the sign.

Worked example. Find the gradient of a line perpendicular to $2y = 3x + 1$.

Rearrange: $y = \frac{3}{2}x + \frac{1}{2}$, so the gradient is $\frac{3}{2}$. The perpendicular gradient is $-\frac{2}{3}$.



Parallel lines share the same gradient; perpendicular gradients multiply to -1 .

Perpendicular bisector

The **perpendicular bisector** 垂直平分线 of a segment cuts it in half at a right angle. To find its equation: get the midpoint, then use the perpendicular gradient through that midpoint.

Worked example. Find the perpendicular bisector of the segment joining $(-3, 8)$ and $(9, -2)$.

- Midpoint: $\left(\frac{-3+9}{2}, \frac{8+(-2)}{2}\right) = (3, 3)$.
- Gradient of the segment: $\frac{-2-8}{9-(-3)} = \frac{-10}{12} = -\frac{5}{6}$.
- Perpendicular gradient: $\frac{6}{5}$.

Through $(3, 3)$: $3 = \frac{6}{5}(3) + c \Rightarrow c = 3 - \frac{18}{5} = -\frac{3}{5}$. So the bisector is

$$y = \frac{6}{5}x - \frac{3}{5}.$$

Exam tips

- The straight line is $y = mx + c$: m is the **gradient** and c is where the line crosses the y -axis.
- Gradient = (change in y) $\div$ (change in x). Keep the two coordinates in the same order on the top and the bottom.

- **Parallel** lines have the *same* gradient; **perpendicular** lines have gradients that multiply to -1 (the negative reciprocal).
- The midpoint is the average of the coordinates; the distance between two points comes from Pythagoras on the differences.

Geometry

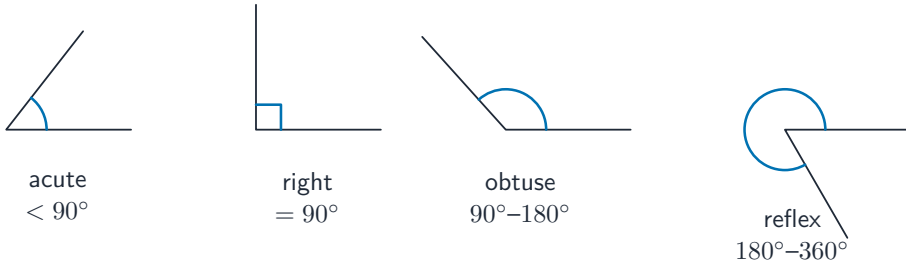
IGCSE Mathematics

This handout covers Topic 4, Geometry. Parts marked **(Extended)** are only tested on the Extended papers; everything else is for both levels. In the exam you must **give reasons** using the correct names below, not just the numbers.

Lines and angles

A corner where two lines meet is a **vertex** 顶点. Two lines are **parallel** 平行 if they never meet, and **perpendicular** 垂直 if they meet at a right angle. Angles are named by their size:

Name	Size
acute angle 锐角	less than 90°
right angle 直角	exactly 90°
obtuse angle 钝角	between 90° and 180°
reflex angle 优角	between 180° and 360°

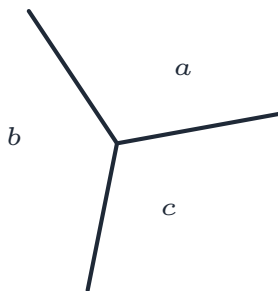


Angles by size: acute (under 90°), right (90° , shown by a square), obtuse ($90^\circ - 180^\circ$) and reflex ($180^\circ - 360^\circ$).

We name an angle with three letters, e.g. angle ABC is the angle at B .

Angle facts

Learn these basic facts. Each one is a valid reason in the exam.



$$a + b + c = 360^\circ \text{ (round a point)}$$

Angles around a point add up to 360 degrees



$$a + b = 180^\circ \text{ (straight line)}$$

Angles on a straight line add up to 180 degrees

- Angles at a point add up to 360° .
- Angles on a straight line add up to 180° .
- **Vertically opposite angles** 对顶角 (made by two crossing lines) are equal.

Worked example. Three angles on a straight line are x , 50° and 70° . Find x .

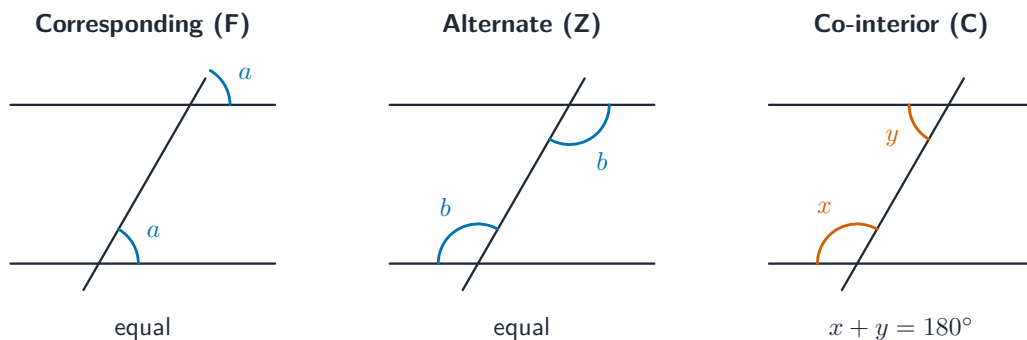
$$x + 50 + 70 = 180 \Rightarrow x = 60^\circ.$$

Angles in parallel lines

When a line crosses two parallel lines:

- **Corresponding angles** 同位角 (in matching positions, an "F" shape) are equal.
- **Alternate angles** 内错角 (opposite sides of the crossing line, a "Z" shape) are equal.
- **Co-interior angles** 同旁内角 (a "C" shape) add up to 180° ; we say they are **supplementary** 互补.

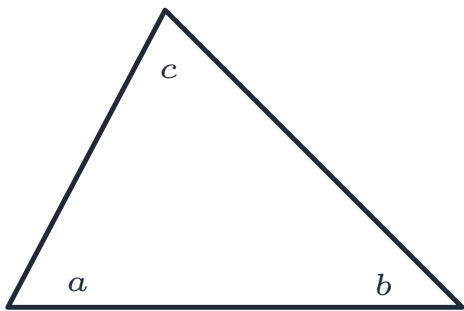
Worked example. A straight line crosses two parallel lines. One angle is 110° . The co-interior angle y satisfies $110 + y = 180$, so $y = 70^\circ$.



Corresponding (F) and alternate (Z) angles are equal; co-interior (C) angles add to 180° .

Triangles

A **triangle** 三角形 has three sides and angles that add up to 180° .



$$a + b + c = 180^\circ$$

The three angles of a triangle add up to 180 degrees

Type	Property
equilateral 等边	all sides equal, all angles 60°
isosceles 等腰	two sides equal, two angles equal
scalene 不等边	all sides and angles different
right-angled	has one right angle

Worked example. A triangle has angles x , $2x$ and 90° . Find x .

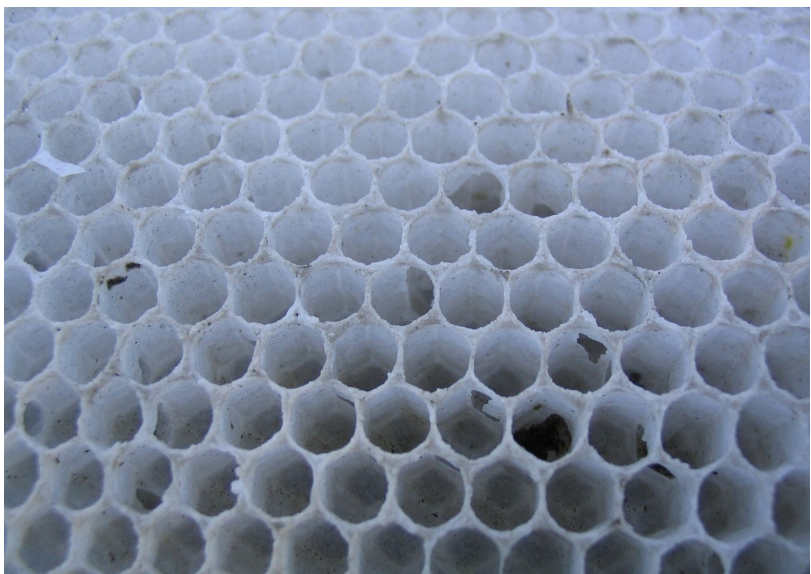
$$x + 2x + 90 = 180 \Rightarrow 3x = 90 \Rightarrow x = 30^\circ.$$

Quadrilaterals

A **quadrilateral** 四边形 has four sides and angles that add up to 360° .

Shape	Property
square 正方形	four equal sides, four right angles
rectangle 矩形	opposite sides equal, four right angles
parallelogram 平行四边形	opposite sides parallel and equal
rhombus 菱形	four equal sides, opposite sides parallel
kite 鸢形	two pairs of equal sides next to each other
trapezium 梯形	one pair of parallel sides

Polygons



A honeycomb tessellates the plane with regular hexagons.

A **polygon** 多边形 is a shape with straight sides. A **regular polygon** 正多边形 has all sides and all angles equal.

Sides	Name
5	pentagon 五边形
6	hexagon 六边形
8	octagon 八边形
10	decagon 十边形

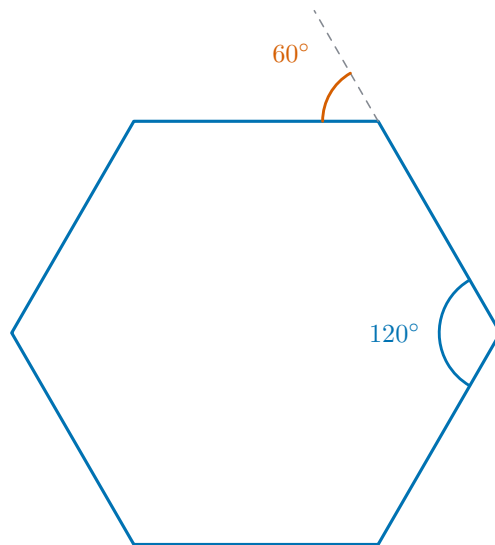
For a polygon with n sides:

$$\text{sum of interior angles} = (n - 2) \times 180^\circ, \quad \text{sum of exterior angles} = 360^\circ.$$

The **interior angle** 内角 and the **exterior angle** 外角 at each corner add up to 180° .

Worked example. Find each interior angle of a regular hexagon.

The exterior angle is $\frac{360^\circ}{6} = 60^\circ$, so each interior angle is $180^\circ - 60^\circ = 120^\circ$.



$$\text{interior} + \text{exterior} = 180^\circ$$

At each corner the interior and exterior angles add to 180° ; a regular hexagon has 60° exterior and 120° interior angles.

Symmetry



The Taj Mahal has a clear line of symmetry down its centre.

- A shape has **line symmetry** 轴对称 if a mirror line splits it into two matching

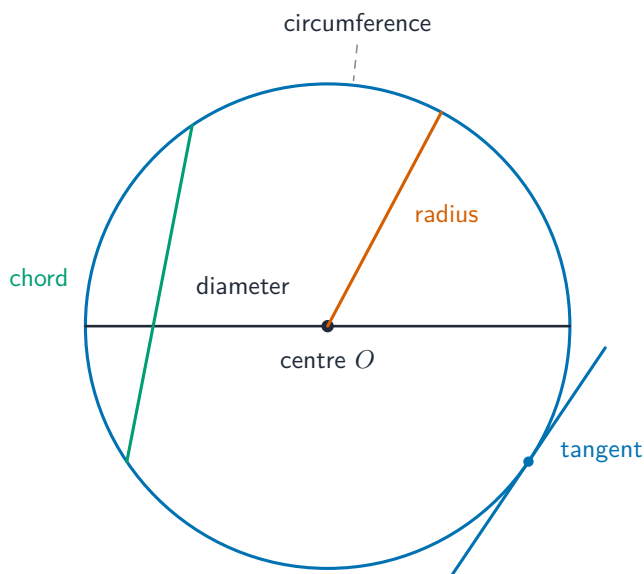
halves.

- A shape has **rotational symmetry** 旋转对称 if it fits onto itself as you turn it. The order is how many times it fits in one full turn.

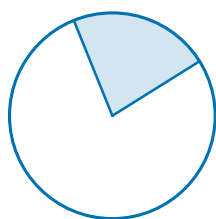
For solids (**Extended**), a flat slice that splits the solid into mirror halves is a **plane of symmetry** 对称面, and a line you can spin it around is an **axis of symmetry** 对称轴.

Circles: the parts

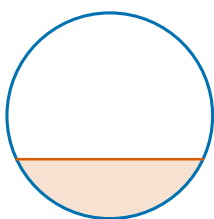
Term	Meaning
circle 圆	all points the same distance from a centre
centre 圆心	the middle point
radius 半径	centre to edge (plural <i>radii</i>)
diameter 直径	right across through the centre ($= 2 \times \text{radius}$)
circumference 圆周	the distance all the way round
chord 弦	a straight line joining two points on the circle
arc 弧	part of the circumference
sector 扇形	a "pizza slice" between two radii
segment 弓形	the region cut off by a chord
semicircle 半圆	half a circle
tangent 切线	a line that touches the circle at one point



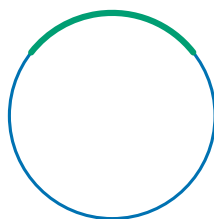
The main parts of a circle: a chord joins two points, while a tangent touches at just one.



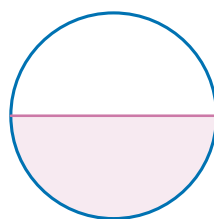
sector



segment



arc



semicircle

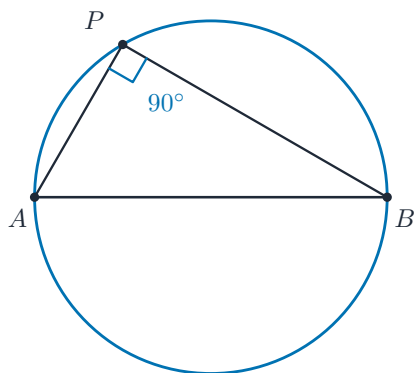
A sector is a slice between two radii, a segment is cut off by a chord, an arc is part of the circumference, and a semicircle is half the circle.

Circle theorems

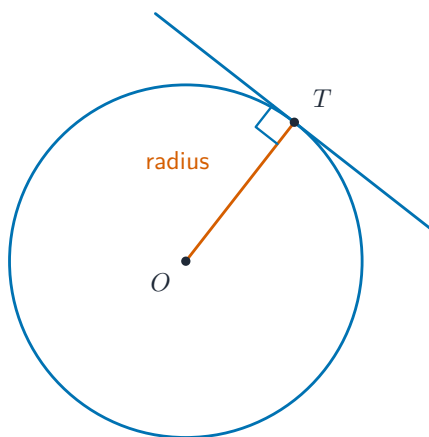
Use these to find unknown angles, always giving the reason.

For both levels:

- The angle in a **semicircle** is 90° .
- The angle between a **tangent** and a **radius** is 90° .



angle in a
semicircle = 90°



tangent meets
radius = 90°

Two theorems for both levels: the angle in a semicircle is 90° , and a tangent meets a radius at 90° .

Extended (theorems I):

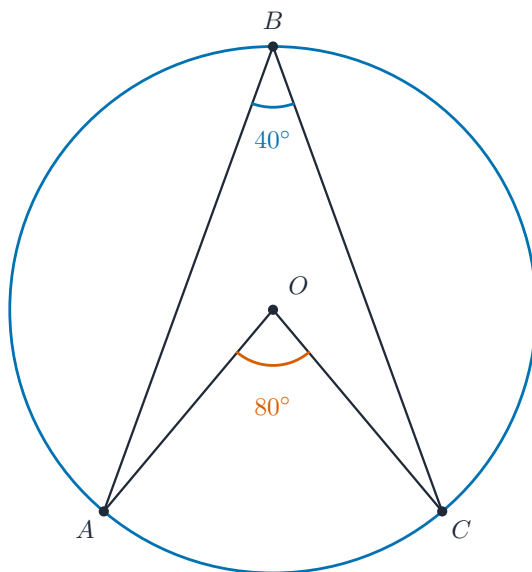
- The angle at the **centre** is twice the angle at the **circumference** (standing on the same arc).

- Angles in the same **segment** are equal.
- Opposite angles of a **cyclic quadrilateral** 圆内接四边形 add up to 180° .
- The **alternate segment theorem** 弦切角定理: the angle between a tangent and a chord equals the angle in the other segment.

Extended (theorems II): equal chords are the same distance from the centre; the perpendicular bisector of a chord passes through the centre; two tangents from the same outside point are equal in length.

Worked example. A, B, C are on a circle. The angle at the circumference ABC is 40° . Find the angle AOC at the centre O .

The angle at the centre is twice the angle at the circumference: $2 \times 40^\circ = 80^\circ$.



angle at centre = $2 \times$ angle at circumference

The angle at the centre (80°) is twice the angle at the circumference (40°) standing on the same arc.

Similar shapes

Two shapes are **similar** 相似 if one is an enlargement of the other: same angles, and all sides multiplied by the same **scale factor** 比例因子 k . (Shapes that are exactly the same size and shape are **congruent** 全等.)



Nesting dolls are similar shapes: each one is the same shape as the others, just multiplied by a scale factor

For similar shapes and solids:

$$\frac{\text{area of } A}{\text{area of } B} = k^2, \quad \frac{\text{volume of } A}{\text{volume of } B} = k^3.$$

Worked example. Two similar solids have lengths in the ratio $2 : 3$. The smaller has volume 40 cm^3 . Find the volume of the larger.

The volume ratio is $2^3 : 3^3 = 8 : 27$. So the larger volume is $40 \times \frac{27}{8} = 135 \text{ cm}^3$.

Bearings

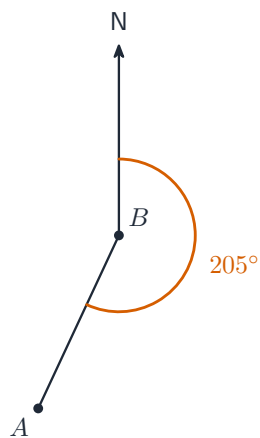
A **bearing** 方位角 gives a direction as a three-figure angle, measured **clockwise** 顺时针 from north, from 000° to 360° . So due east is 090° and due south is 180° .

Worked example. The bearing of B from A is 025° . Find the bearing of A from B .

The return (back) bearing differs by 180° : $025^\circ + 180^\circ = 205^\circ$.



Bearing of B from A

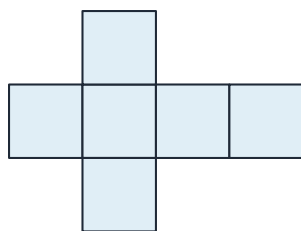


Bearing of A from B

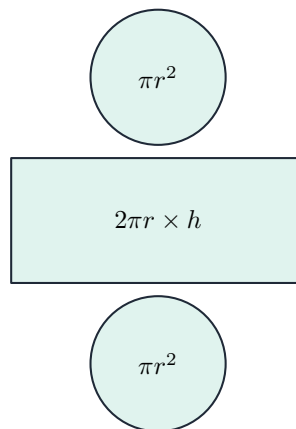
A bearing is measured clockwise from north as three figures; the back bearing differs by 180° .

Constructions, nets and solids

- To **construct** 作图 a triangle from three given sides, draw the base with a ruler, then use a pair of **compasses** 圆规 to mark each other side as an arc. Leave the construction arcs showing.
- A **net** 展开图 is a flat shape that folds up into a solid. You can use a net to work out surface areas.
- A **scale drawing** 比例图 shows a real object smaller (or larger) by a fixed scale, such as 1 cm to 5 m.



Cube net
6 equal square faces



Cylinder net
2 circles + a rectangle

Unfolding a solid gives its net. Add the areas of the pieces to get the surface area: a cylinder is $2\pi r^2 + 2\pi rh$.

Common **solids** 几何体 and their parts:

Solid	Note
cube 立方体 / cuboid 长方体	box shapes
prism 棱柱	the same shape all along its length
cylinder 圆柱	a circular prism
pyramid 棱锥 / cone 圆锥	come to a point
sphere 球 / hemisphere 半球	a ball / half a ball
frustum 平截头体	a cone or pyramid with the top cut off

A flat side of a solid is a **face** 面, two faces meet at an **edge** 棱, and the whole outside is its **surface** 表面.

Exam tips

- Angles on a straight line add to 180° , around a point to 360° , and in a triangle to 180° . Give a **reason** for every step of an angle question.
- Learn the circle theorems: the angle at the centre is *twice* the angle at the circumference; the angle in a semicircle is 90° ; angles in the same segment are equal; opposite angles of a cyclic quadrilateral add to 180° .
- The **exterior angles** of any polygon add to 360° , and each interior angle + its exterior angle = 180° .
- **Bearings** are measured clockwise from north and always written with three figures (e.g. 072°).

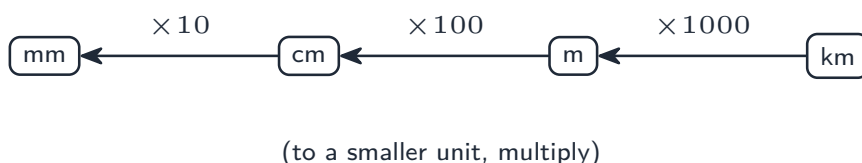
Mensuration

IGCSE Mathematics

This handout covers Topic 5, Mensuration (measuring length, area and volume). The Core and Extended content here is almost the same. In the exam, some formulas are given in the List of formulas, but you should still learn them all.

Units of measure

We use **metric** 公制 units for **mass** 质量 (g, kg), **length** 长度 (mm, cm, m, km), **area** 面积, **volume** 体积 and **capacity** 容量 (ml, litres —the space inside a container).



Changing length units: multiply going down, divide going up

To change units, be careful with squares and cubes:

- Length: $1 \text{ m} = 100 \text{ cm}$.
- Area: $1 \text{ m}^2 = 100^2 = 10\,000 \text{ cm}^2$.
- Volume: $1 \text{ m}^3 = 100^3 = 1\,000\,000 \text{ cm}^3$.
- Capacity: $1 \text{ litre} = 1000 \text{ cm}^3$, so $1 \text{ m}^3 = 1000 \text{ litres}$.

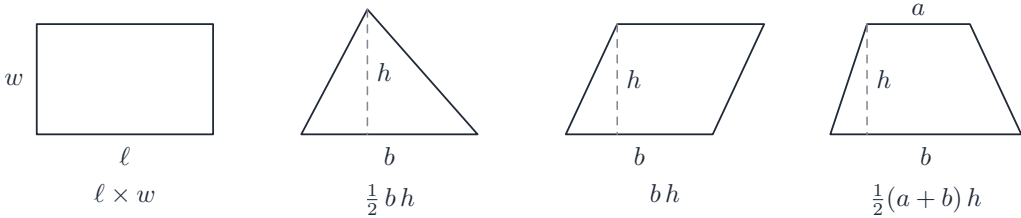
Worked example. Convert 3 m^2 to cm^2 .

$$3 \times 10\,000 = 30\,000 \text{ cm}^2.$$

Perimeter and area of basic shapes

The **perimeter** 周长 is the distance all the way round a shape. The **area** is the amount of flat space inside it. Here b is the **base** 底 and h is the perpendicular **height** 高.

Shape	Area
rectangle 矩形	length $\times$ width
triangle 三角形	$\frac{1}{2} \times b \times h$
parallelogram 平行四边形	$b \times h$
trapezium 梯形	$\frac{1}{2}(a + b)h$, where a and b are the two parallel sides



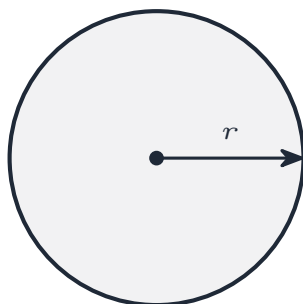
Area of the basic shapes; b is the base, h the perpendicular height, and a, b the two parallel sides of a trapezium.

Worked example. A trapezium has parallel sides 6 cm and 10 cm, and height 4 cm. Find its area.

$$\frac{1}{2}(6 + 10) \times 4 = \frac{1}{2} \times 16 \times 4 = 32 \text{ cm}^2.$$

Circles

For a **circle** 圆 with **radius** 半径 r (and **diameter** 直径 $d = 2r$):



$$\text{circumference} = \pi d$$

$$\text{area} = \pi r^2$$

$$(\text{diameter } d = 2r)$$

A circle: circumference = πd and area = πr^2

$$\text{circumference} = 2\pi r = \pi d, \quad \text{area} = \pi r^2.$$

The **circumference** 圆周 is the distance round the circle.

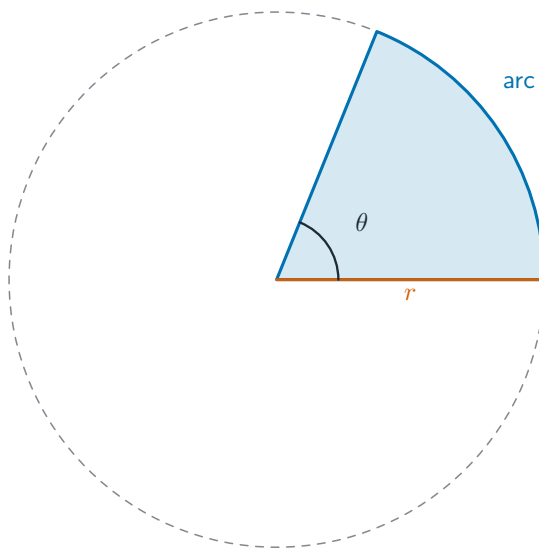
Worked example. A circle has radius 7 cm. Find its circumference and area (leave π in the answer).

$$\text{circumference} = 2\pi \times 7 = 14\pi \text{ cm}, \quad \text{area} = \pi \times 7^2 = 49\pi \text{ cm}^2.$$

Arcs and sectors

An **arc** 弧 is part of the circumference. A **sector** 扇形 is a "pizza slice" between two radii. If the sector angle is θ , the arc and sector are that fraction $\frac{\theta}{360}$ of the whole circle:

$$\text{arc length} = \frac{\theta}{360} \times 2\pi r, \quad \text{sector area} = \frac{\theta}{360} \times \pi r^2.$$



A sector is the fraction $\frac{\theta}{360}$ of the whole circle, so its arc and area are that fraction of the circumference and the area.

A small slice is a **minor sector** 小扇形; the large rest is a **major sector** 大扇形.

Worked example. Find the **arc length** 弧长 and area of a sector with angle 90° and radius 8 cm.

The fraction is $\frac{90}{360} = \frac{1}{4}$, so

$$\text{arc} = \frac{1}{4} \times 2\pi \times 8 = 4\pi \text{ cm}, \quad \text{area} = \frac{1}{4} \times \pi \times 8^2 = 16\pi \text{ cm}^2.$$

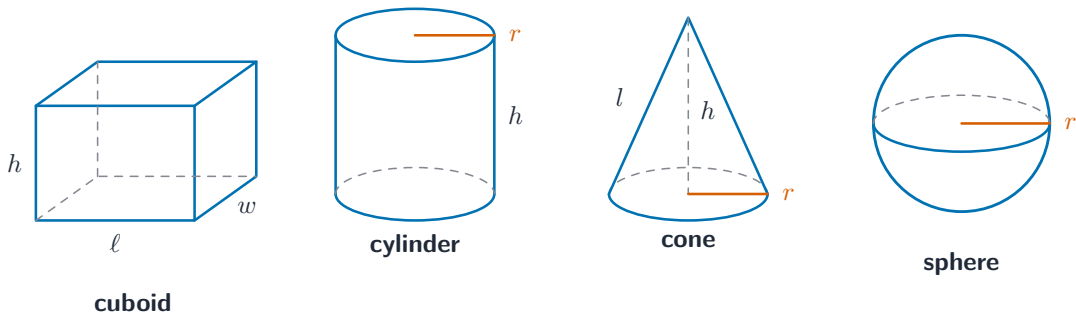
Surface area and volume of solids



The pyramids of Giza are square-based pyramids — a 3-D solid.

The **surface area** 表面积 is the total area of all the outside faces. The volume is the space inside. For these solids (r = radius, h = height):

Solid	Volume	Surface area
cuboid 长方体	length $\times$ width $\times$ height	add the six faces
prism 棱柱	(cross-section 横截面 area) $\times$ length	—
cylinder 圆柱	$\pi r^2 h$	$2\pi r h$ (curved surface area 侧面积) $+ 2\pi r^2$
pyramid 棱锥	$\frac{1}{3} \times$ base area $\times h$	—
cone 圆锥	$\frac{1}{3}\pi r^2 h$	$\pi r l$ (curved) $+ \pi r^2$, where l is the slant height 斜高
sphere 球	$\frac{4}{3}\pi r^3$	$4\pi r^2$



Common solids and the dimensions (r , h , ℓ , slant l) used in their volume and surface-area formulas.

Worked example. A cylinder has radius 5 cm and height 10 cm. Find its volume

and total surface area (in terms of π).

$$\text{volume} = \pi \times 5^2 \times 10 = 250\pi \text{ cm}^3.$$

$$\text{surface area} = 2\pi(5)(10) + 2\pi(5)^2 = 100\pi + 50\pi = 150\pi \text{ cm}^2.$$

Compound shapes and parts of shapes



Stacked containers are cuboids; volume is length times width times height.

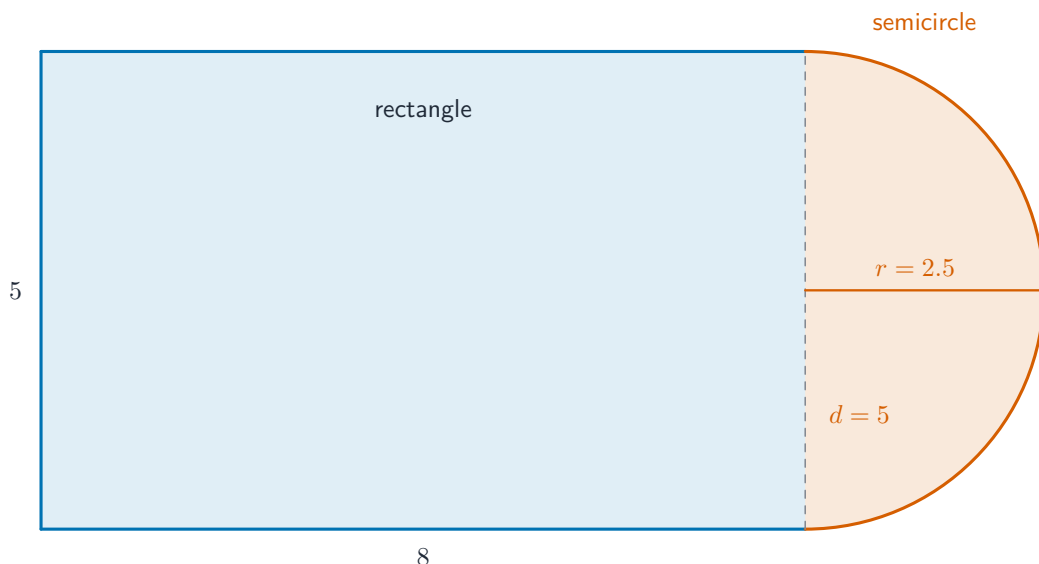
A **compound shape** 组合图形 is made by joining or cutting simple shapes. Split it into parts you know, then add or subtract.

For "parts" of a circle or solid, take the right fraction. For example, a **hemisphere** 半球 (half a sphere) has volume

$$\frac{1}{2} \times \frac{4}{3}\pi r^3 = \frac{2}{3}\pi r^3.$$

A **frustum** 平截头体 is a cone or pyramid with its top cut off; find its volume by subtracting the small top cone from the whole cone.

Worked example. Find the area of a shape made of a rectangle $8 \text{ cm} \times 5 \text{ cm}$ with a semicircle of diameter 5 cm on one end.



Split a compound shape into parts you know — here a rectangle plus a semicircle — then add the areas.

The semicircle has radius 2.5 cm:

$$\text{area} = 8 \times 5 + \frac{1}{2}\pi(2.5)^2 = 40 + 3.125\pi \approx 49.8 \text{ cm}^2.$$

Exam tips

- Match the formula to the shape: the **area** of a circle is πr^2 , the **circumference** is πd (or $2\pi r$) —do not mix them up.
- Keep units consistent, and remember area units are squared and volume units cubed (e.g. $1 \text{ m}^2 = 10\,000 \text{ cm}^2$).
- For an **arc** or **sector**, take the fraction $\frac{\theta}{360}$ of the whole circumference or area.
- Surface area is the total of *all* the faces —draw the net if you are unsure. Leave an answer in terms of π only if the question allows it.

Trigonometry

IGCSE Mathematics

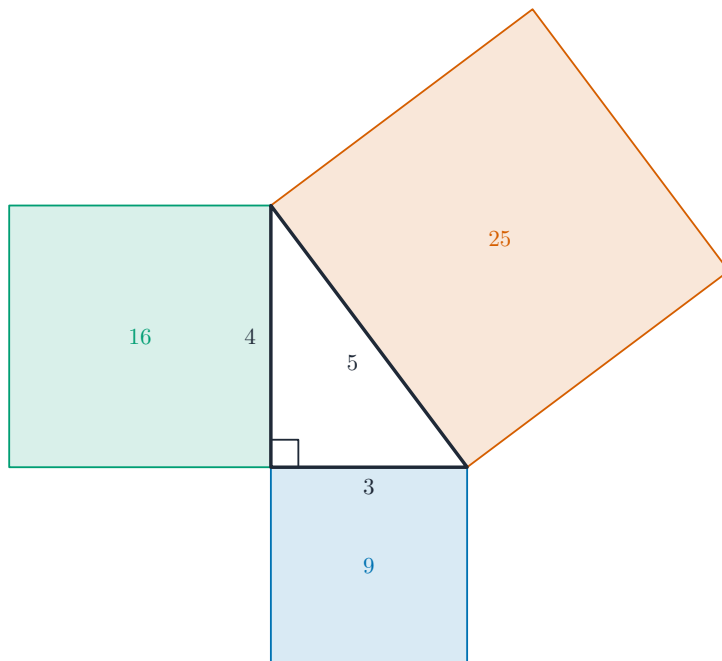
This handout covers Topic 6, Trigonometry. Parts marked **(Extended)** are only tested on the Extended papers; everything else is for both levels. Give angle answers to one decimal place.

Pythagoras' theorem

Pythagoras' theorem 勾股定理 links the three sides of a **right-angled triangle** 直角三角形 (a **triangle** 三角形 with one 90° angle). If the longest side (the **hypotenuse** 斜边, opposite the right angle) is c , then

$$a^2 + b^2 = c^2.$$

Use it to find a missing side.



Pythagoras as areas: the squares on the two shorter sides ($9 + 16$) add up to the square on the hypotenuse (25).

Worked example. A right-angled triangle has a hypotenuse of 13 cm and one short side of 5 cm. Find the other short side.

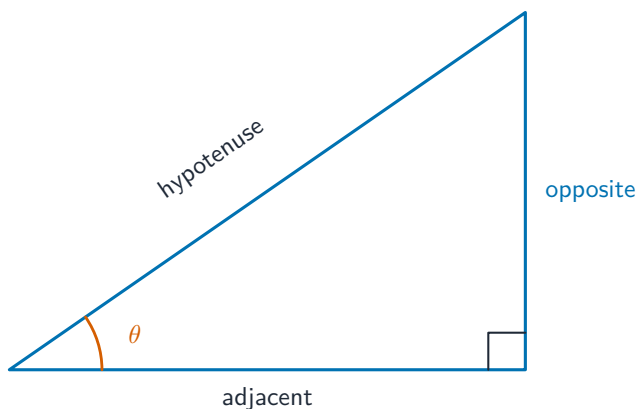
$$b^2 = 13^2 - 5^2 = 169 - 25 = 144, \quad b = \sqrt{144} = 12 \text{ cm.}$$

Trigonometry in right-angled triangles

Label the sides from the angle you are using: the **opposite** 对边 (across from the angle), the **adjacent** 邻边 (next to the angle), and the hypotenuse. The three ratios are the **sine** 正弦, **cosine** 余弦 and **tangent ratio** 正切 (sin, cos, tan):

$$\sin \theta = \frac{\text{opp}}{\text{hyp}}, \quad \cos \theta = \frac{\text{adj}}{\text{hyp}}, \quad \tan \theta = \frac{\text{opp}}{\text{adj}}.$$

Remember them as **SOH-CAH-TOA**. To find an angle, use the inverse ($\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$).



Name the sides from the angle θ : the opposite is across from it, the adjacent next to it, and the hypotenuse opposite the right angle (SOH-CAH-TOA).

Worked example (find a side). In a right-angled triangle the hypotenuse is 10 cm and the angle is 30° . Find the opposite side.

$$\text{opp} = 10 \times \sin 30^\circ = 10 \times 0.5 = 5 \text{ cm.}$$

Worked example (find an angle). The opposite side is 4 cm and the adjacent side is 3 cm.

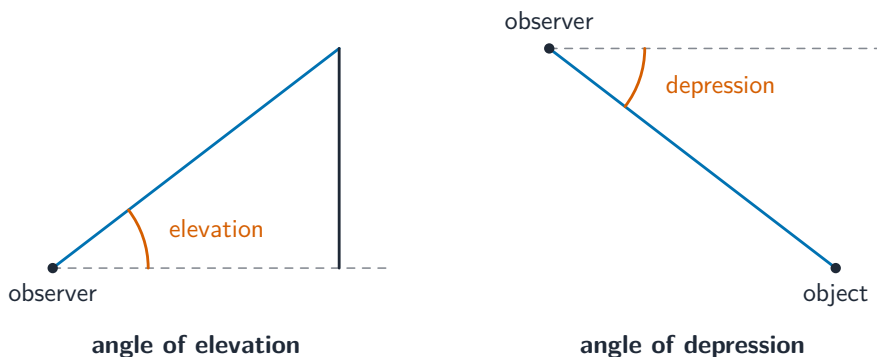
$$\tan \theta = \frac{4}{3}, \quad \theta = \tan^{-1}\left(\frac{4}{3}\right) = 53.1^\circ.$$

Angles of elevation and depression (Extended)



Looking up at a tower involves an angle of elevation.

The **angle of elevation** 仰角 is the angle up from the horizontal to an object above you. The **angle of depression** 俯角 is the angle down from the horizontal to an object below you. The shortest distance from a point to a line is the **perpendicular** 垂直 distance.



The angle of elevation looks up from the horizontal; the angle of depression looks down.

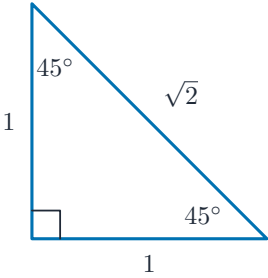
Worked example. From a point 50 m from the foot of a tower, the angle of elevation to the top is 40° . Find the height of the tower.

$$\text{height} = 50 \times \tan 40^\circ = 50 \times 0.839 = 42.0 \text{ m.}$$

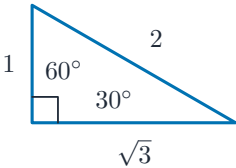
Exact trigonometric values (Extended)

You must know these exact values without a calculator.

x	0°	30°	45°	60°	90°
$\sin x$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan x$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	—



45° triangle



30°–60° triangle

These two special triangles are where the exact values come from —worth memorising.

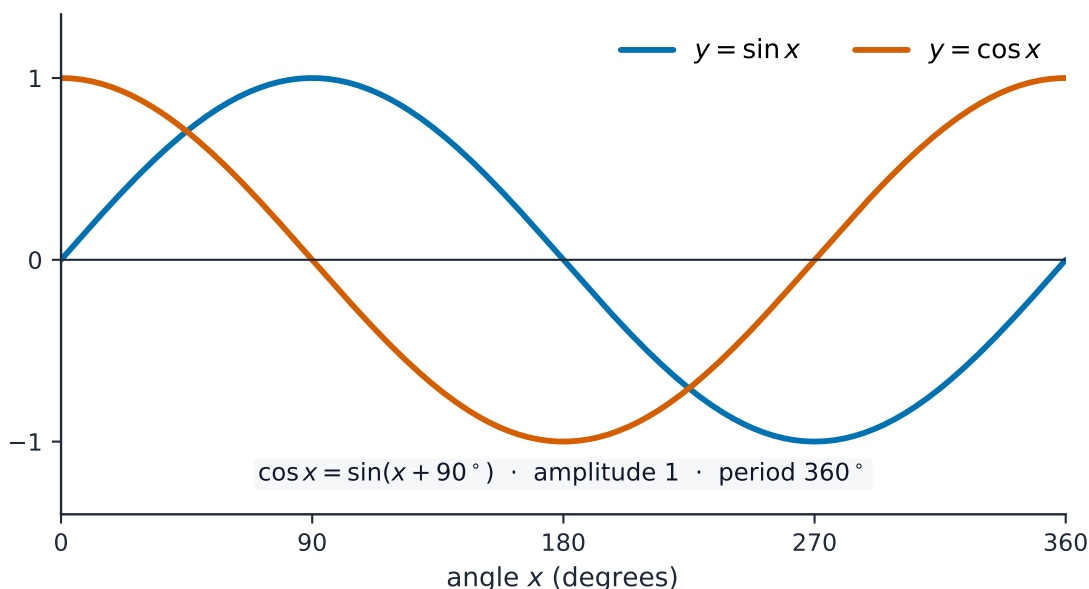
Graphs and trigonometric equations (Extended)



A Ferris wheel: a point on the rim traces a sine curve as it turns.

For $0^\circ \leq x \leq 360^\circ$:

- $y = \sin x$ is a wave starting at 0, peaking at 90° , back to 0 at 180° , down to -1 at 270° .
- $y = \cos x$ is the same wave but starting at 1.
- $y = \tan x$ rises steeply and repeats every 180° .



$y = \sin x$ and $y = \cos x$ are smooth waves between -1 and 1 ; cosine is sine shifted left by 90° .

A **trigonometric equation** 三角方程 often has more than one answer in this range. Use the graph (or the symmetry of the wave) to find them all.

Worked example. Solve $\sin x = \frac{\sqrt{3}}{2}$ for $0^\circ \leq x \leq 360^\circ$.

One answer is $x = 60^\circ$. The sine wave is also $\frac{\sqrt{3}}{2}$ at $180^\circ - 60^\circ = 120^\circ$. So $x = 60^\circ$ or 120° .

Worked example. Solve $2 \cos x + 1 = 0$ for $0^\circ \leq x \leq 360^\circ$.

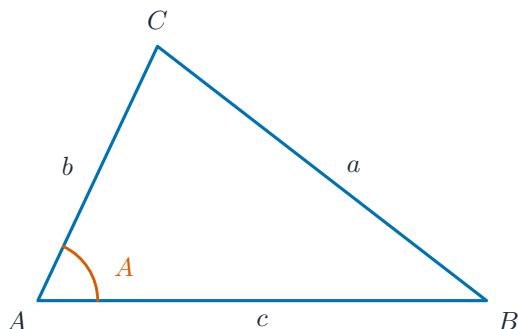
$$\cos x = -\frac{1}{2} \Rightarrow x = 120^\circ \text{ or } 240^\circ.$$

The sine and cosine rules (Extended)

For **any** triangle (not just right-angled), with sides a, b, c opposite angles A, B, C :

$$\text{sine rule: } \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C},$$

$$\text{cosine rule: } a^2 = b^2 + c^2 - 2bc \cos A.$$



In any triangle, sides a, b, c lie opposite the angles A, B, C of the same letter.

Use the **sine rule** 正弦定理 when you have a side and its opposite angle. Use the **cosine rule** 余弦定理 when you have two sides and the angle between them, or all three sides. With the sine rule, watch for the **ambiguous case** 两解情况, where an angle could be acute or obtuse.

The area of any triangle is

$$\text{area} = \frac{1}{2}ab \sin C.$$

Worked example. A triangle has $b = 7$ cm, $c = 8$ cm and the angle between them $A = 40^\circ$. Find side a .

$$a^2 = 7^2 + 8^2 - 2(7)(8) \cos 40^\circ = 113 - 112 \times 0.766 = 27.2,$$

$$a = \sqrt{27.2} = 5.2 \text{ cm.}$$

Pythagoras and trigonometry in 3D (Extended)

In three dimensions, find a right-angled triangle inside the solid, then use Pythagoras or trigonometry on it. A common task is the angle between a line and a flat surface (a **plane** 平面).

Worked example. A box has a base 6 cm by 8 cm and height 5 cm. Find the angle between a space diagonal and the base.

First the base diagonal: $\sqrt{6^2 + 8^2} = \sqrt{100} = 10$ cm. This diagonal and the height form a right-angled triangle, so the angle θ with the base is

$$\tan \theta = \frac{5}{10} = 0.5, \quad \theta = \tan^{-1}(0.5) = 26.6^\circ.$$

Exam tips

- Use **Pythagoras** ($a^2 + b^2 = c^2$) when there is no angle; use **SOH-CAH-TOA** when an angle is involved.
- The **hypotenuse** is always opposite the right angle. Label the sides (opposite, adjacent, hypotenuse) relative to the angle you are using.
- To find an angle, use the inverse ($\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$), and check your calculator is set to **degrees**.
- For a triangle with no right angle, use the **sine rule** or the **cosine rule** —the cosine rule when you know two sides and the angle between them.

Transformations and vectors

IGCSE Mathematics

This handout covers Topic 7, Transformations and vectors. Parts marked (**Extended**) are only tested on the Extended papers; everything else is for both levels. Vectors as a whole topic are Extended.

Transformations



Geometric tiles are built by reflecting, rotating and translating shapes.

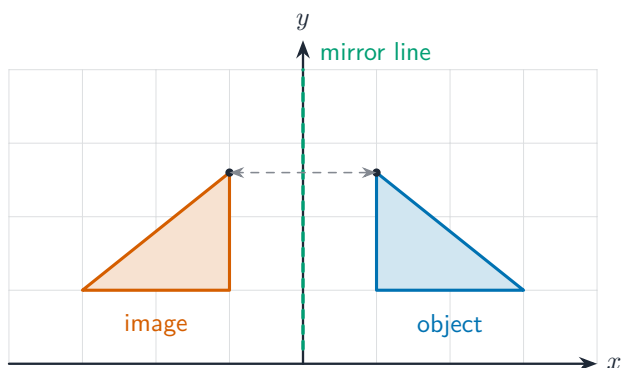
A **transformation** 变换 changes the position or size of a shape. The original is the *object* and the result is the *image*. There are four types. When asked to **describe** one, you must name the type and give all the details below.

Reflection

A **reflection** 反射 flips the shape over a **mirror line** 对称轴. Each image point is the same distance from the line as the object point, on the other side.

- To describe it, give the **equation of the mirror line** (for Core, a horizontal or vertical line; Extended allows any line such as $y = x$).

Worked example. Reflect the point $(3, 2)$ in the y -axis. Only the sign of x changes: the image is $(-3, 2)$. (In the x -axis it would be $(3, -2)$.)



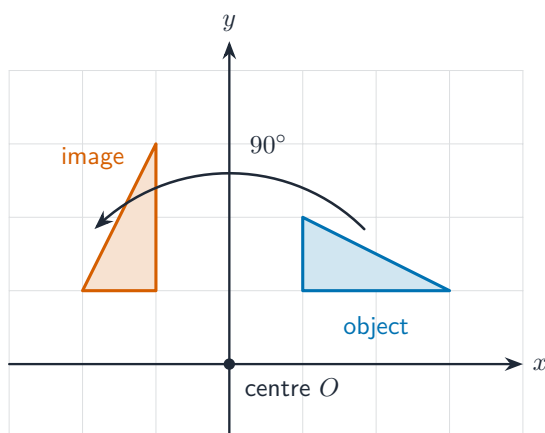
A reflection flips the shape over a mirror line (here the y -axis); each point and its image are the same distance from the line.

Rotation

A **rotation** 旋转 turns the shape about a fixed point, the **centre** 中心 of rotation, through multiples of 90° .

- To describe it, give the centre, the angle, and the direction (clockwise or anticlockwise).

Worked example. Rotate $(3, 1)$ by 90° anticlockwise about the origin. The rule is $(x, y) \rightarrow (-y, x)$, so the image is $(-1, 3)$.



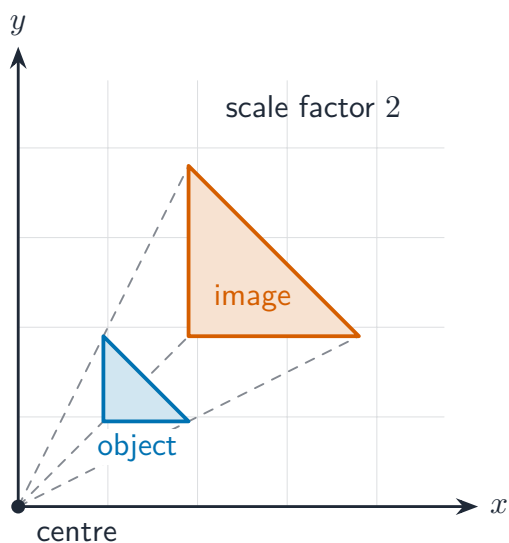
A rotation turns the shape about a fixed centre —here 90° anticlockwise about the origin.

Enlargement

An **enlargement** 放大 changes the size by a **scale factor** 比例因子 k , measured from a fixed centre. Each distance from the centre is multiplied by k .

- To describe it, give the centre and the scale factor. A **fractional** scale factor (between 0 and 1) makes the shape smaller. For Extended, k may also be **negative** (the image appears on the other side of the centre).

Worked example. Enlarge $(1, 2)$ from the origin by scale factor 2. Multiply both coordinates: the image is $(2, 4)$.

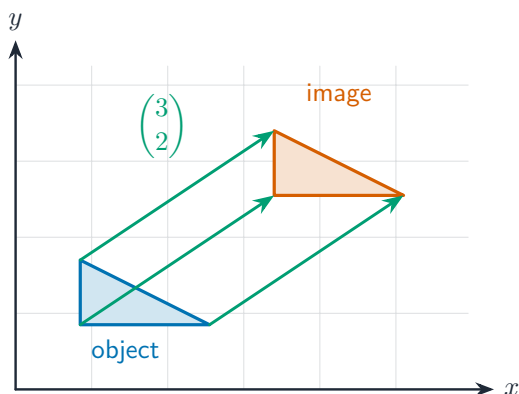


An enlargement multiplies every distance from the centre by the scale factor (here 2).

Translation

A **translation** 平移 slides the shape with no turning, by a **vector** 向量 written as a **column vector** 列向量 $\begin{pmatrix} x \\ y \end{pmatrix}$ (x across, y up).

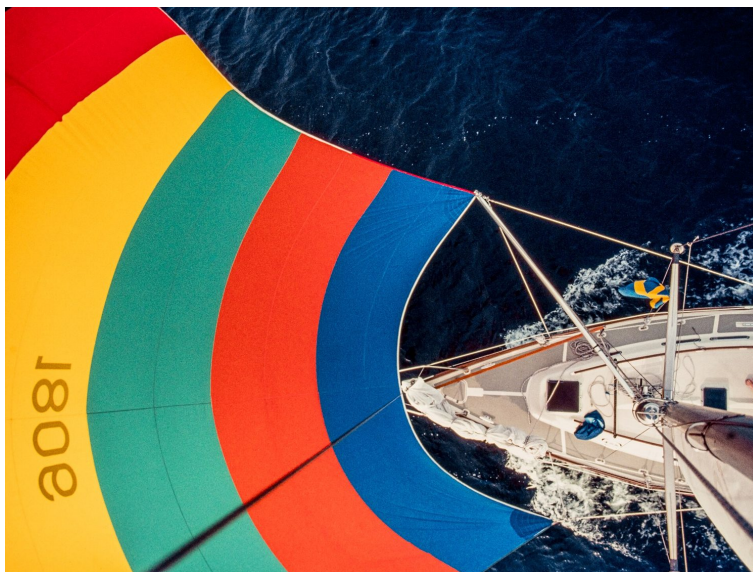
Worked example. Translate $(5, 3)$ by $\begin{pmatrix} -2 \\ 4 \end{pmatrix}$: move 2 left and 4 up to get $(3, 7)$.



A translation slides every point by the same column vector, with no turning.

(**Extended:** a question may ask you to combine two transformations and describe the single transformation that has the same effect.)

Vectors in two dimensions (Extended)



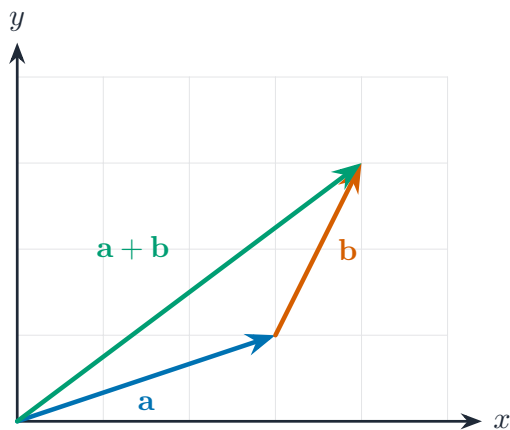
Forces like wind are vectors, with both size and direction.

A vector has both size and direction. It can be written as a column vector, as $\overrightarrow{AB}$ (from A to B), or in bold as $\mathbf{a}$.

- **Add or subtract** by working on the top and bottom numbers separately.
- **Multiply by a scalar** 标量 (an ordinary number) by multiplying both numbers.

Worked example. If $\mathbf{a} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$, then

$$\mathbf{a} + \mathbf{b} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}, \quad 3\mathbf{a} = \begin{pmatrix} 9 \\ 3 \end{pmatrix}.$$



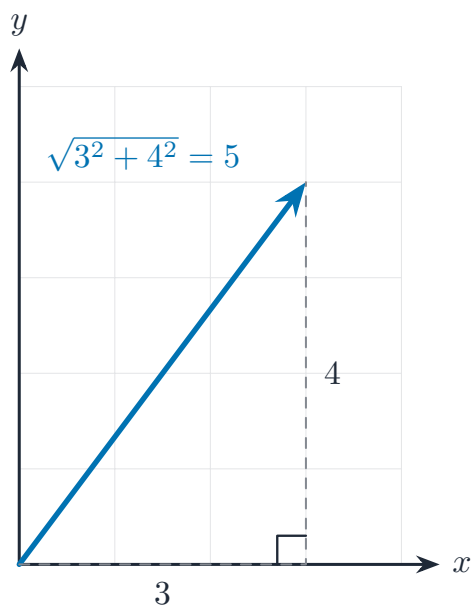
*Adding by the triangle law: draw **b** from the tip of **a**, and **a + b** runs from start to finish.*

Magnitude of a vector (Extended)

The **magnitude** 模 (length) of a vector is found with Pythagoras. For $\begin{pmatrix} x \\ y \end{pmatrix}$,

$$\left| \begin{pmatrix} x \\ y \end{pmatrix} \right| = \sqrt{x^2 + y^2}.$$

Worked example. The magnitude of $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ is $\sqrt{3^2 + 4^2} = \sqrt{25} = 5$.



The magnitude of $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ comes from Pythagoras on its horizontal and vertical parts.

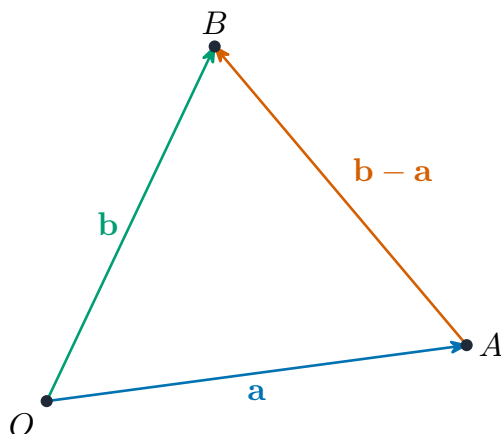
Vector geometry (Extended)

A vector can be drawn as a **directed line segment** 有向线段 (an arrow). The **position vector** 位置向量 of a point is the vector from the origin O to that point.

A key idea: the vector from A to B is

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a},$$

where $\mathbf{a}$ and $\mathbf{b}$ are the position vectors of A and B .



To go from A to B , travel back along $\mathbf{a}$ to O then forward along $\mathbf{b}$: so $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$.

Two vectors are **parallel** 平行 if one is a scalar multiple of the other (for example $\overrightarrow{AB} = 2\overrightarrow{CD}$). Three points are **collinear** 共线 (in a straight line) if the vectors between them are parallel and share a point. You can express any vector in terms of two **coplanar** 共面 vectors.

Worked example. O is the origin, with $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. M is the midpoint of AB . Find $\overrightarrow{OM}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

$$\overrightarrow{OM} = \mathbf{a} + \frac{1}{2}\overrightarrow{AB} = \mathbf{a} + \frac{1}{2}(\mathbf{b} - \mathbf{a}) = \frac{1}{2}(\mathbf{a} + \mathbf{b}).$$

Exam tips

- Fully **describe** each transformation: a translation (a vector), a reflection (the mirror line), a rotation (centre, angle *and* direction), an enlargement (centre and scale factor).

- A **negative** scale factor turns the image upside down through the centre; a fractional one (between 0 and 1) makes it smaller.
- Add vectors tip-to-tail —add the top numbers, then the bottom numbers. The **magnitude** of a vector comes from Pythagoras on its components.
- $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$ (end minus start). Two vectors are **parallel** if one is a scalar multiple of the other.

Probability

IGCSE Mathematics

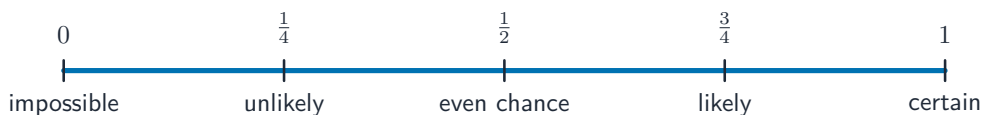
This handout covers Topic 8, Probability. Parts marked (**Extended**) are only tested on the Extended papers; everything else is for both levels.

The probability scale



Dice: the probability scale runs from impossible (0) to certain (1).

Probability 概率 measures how likely an **event** 事件 is. It runs on a scale from 0 (impossible) to 1 (certain), and can be written as a fraction, decimal or percentage. We write $P(A)$ for the probability of event A .



Probability runs from 0 (impossible) to 1 (certain), with $\frac{1}{2}$ an even chance.

For equally likely **outcomes** 结果,

$$P(\text{event}) = \frac{\text{number of favourable outcomes}}{\text{total number of outcomes}}.$$

The probability that an event does **not** happen is

$$P(A') = 1 - P(A).$$

Worked example. A bag has 3 red and 5 blue counters. Find the probability of *not* drawing red.

$$P(\text{red}) = \frac{3}{8}, \quad P(\text{not red}) = 1 - \frac{3}{8} = \frac{5}{8}.$$

Relative frequency and expected frequency

When outcomes are not equally likely, do an experiment. The **relative frequency** 相对频率 estimates the probability:

$$\text{relative frequency} = \frac{\text{number of times it happened}}{\text{total number of trials}}.$$

The more trials you do, the better the estimate. A **fair** 公平 object gives equal chances; one with **bias** 偏倚 does not; **random** 随机 means each outcome happens by chance.

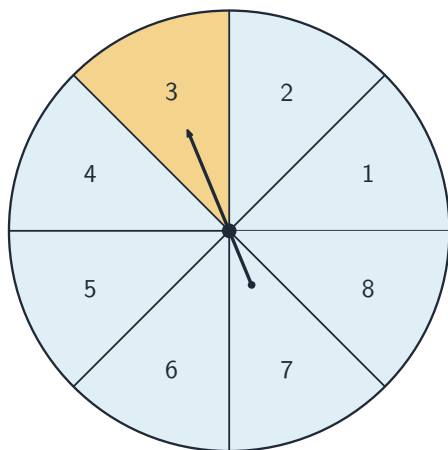
The **expected frequency** 期望频数 is how many times you expect an event in n trials:

$$\text{expected frequency} = P(\text{event}) \times n.$$

Worked example. The probability of rolling a six is $\frac{1}{6}$. How many sixes are expected in 300 rolls?

$$\frac{1}{6} \times 300 = 50.$$

Combined events



Eight identical sectors, so every number is **equally likely**.

$$P(3) = \frac{\text{favourable outcomes}}{\text{total outcomes}} = \frac{1}{8}$$

A fair spinner: eight identical sectors, so each number is an equally likely outcome.

For two or more events together (**combined events** 组合事件), two rules help:

- **AND** (both happen): **multiply** the probabilities —when the events are **independent** 独立事件 (one does not affect the other).
- **OR** (either happens): **add** the probabilities —when the events are **mutually exclusive** 互斥 (they cannot both happen).

Three pictures help you organise the work.

Sample space diagrams

A **sample space diagram** 样本空间图 is a table or grid listing every possible outcome —useful for two dice or two spinners. Count the outcomes you want out of the total.

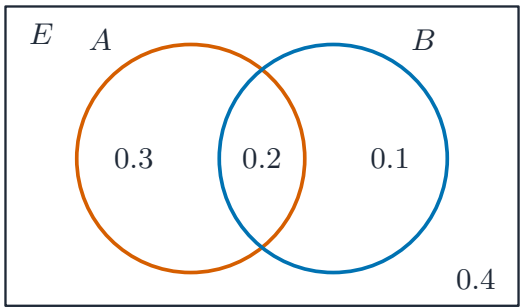
		die 2					
+		1	2	3	4	5	6
die 1	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

the six highlighted cells all give a total of 7

A sample space diagram lists every outcome; for the total of two dice there are 36 equally likely cells.

Venn diagrams

A **Venn diagram** 维恩图 sorts outcomes into overlapping sets. From it you can read $P(A \cap B)$ (in both) and $P(A \cup B)$ (in either).



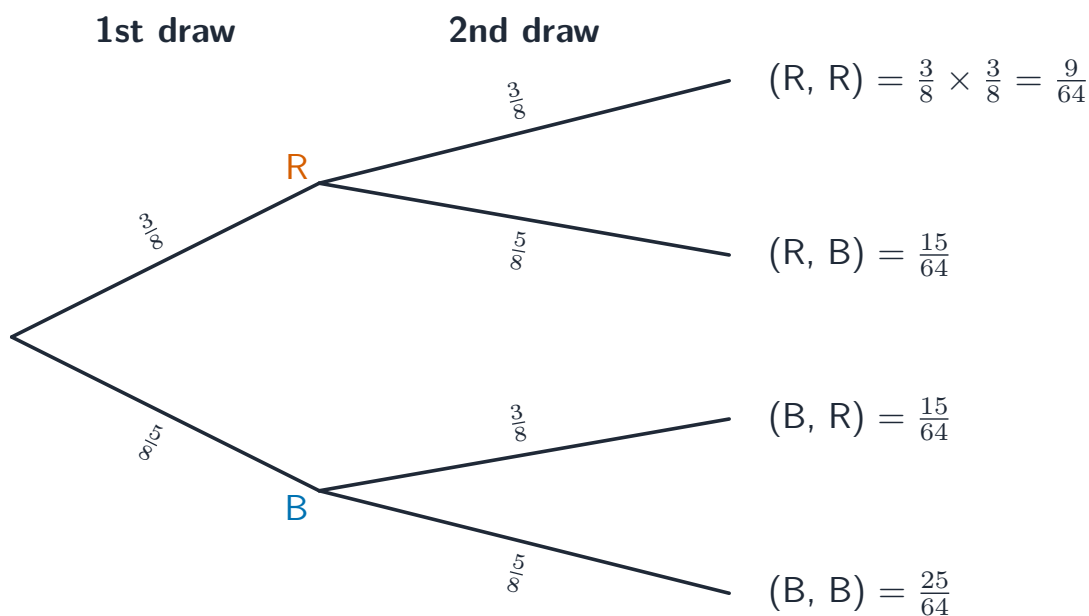
The overlap is $P(A \cap B) = 0.2$; everything inside either circle is $P(A \cup B) = 0.6$; outside both is 0.4.

Tree diagrams

A **tree diagram** 树状图 shows each stage as a set of branches. Write the probability on each branch and the outcome at the end. **Multiply along** the branches, then **add** the paths you want.

Worked example (with replacement). From the bag (3 red, 5 blue), a counter is drawn, replaced, then a second is drawn. This is **with replacement** 有放回, so the chances do not change. Find the probability of two reds.

$$P(\text{red, red}) = \frac{3}{8} \times \frac{3}{8} = \frac{9}{64}.$$



On a tree diagram, multiply the probabilities along the branches; the four outcomes' probabilities add to 1.

Worked example (without replacement, Extended). Now the first counter is *not* replaced —drawing **without replacement** 无放回. After one red is taken, 2 reds remain out of 7:

$$P(\text{red, red}) = \frac{3}{8} \times \frac{2}{7} = \frac{6}{56} = \frac{3}{28}.$$

Conditional probability (Extended)

Conditional probability 条件概率 is the probability of one event *given that* another has already happened. Read it from a Venn diagram, two-way table or tree diagram by looking only at the part that matches the condition.

Worked example. In a class of 30, 18 study French and, of those, 7 also study German. A French student is picked. Find the probability they also study German.

Look only at the 18 French students:

$$P(\text{German} \mid \text{French}) = \frac{7}{18}.$$

Exam tips

- Every probability lies between 0 and 1, and the probabilities of all the outcomes add to 1.
- For "**and**" (both events) *multiply* the probabilities; for "**or**" (either event) *add* them. On a tree diagram, multiply along the branches.
- Watch for **without replacement**: the second probability changes because one item has already been removed.
- Expected frequency = probability $\times$ number of trials.

Statistics

IGCSE Mathematics

This handout covers Topic 9, Statistics. Parts marked **(Extended)** are only tested on the Extended papers; everything else is for both levels.

Collecting and showing data



Data is collected from people — a sample drawn from a population.

To organise **statistical** 统计 **data** 数据, use:

- a **tally table** 计数表—make a mark for each value, then count.
- a **two-way table** 双向表—sorts data by two features at once (for example, boys/girls against walk/bus).

When you read a diagram, only draw conclusions the data really supports.

Averages and range

Three averages describe a typical value of the data, and the range shows how spread out it is. Each is used for a different purpose:



median = 3

$$\text{mean} = \frac{2+3+3+5+7}{5} = 4$$

mode = 3 (most common)

$$\text{range} = 7 - 2 = 5$$

Mean, median, mode and range of a small data set

- **mean** 平均数 = $\frac{\text{sum of all values}}{\text{how many values}}$.
- **median** 中位数 = the middle value when the data is put in order.
- **mode** 众数 = the value that appears most often.
- **range** 极差 = largest value – smallest value (it shows how spread out the data is).

Worked example. Find the mean, median, mode and range of 4, 7, 7, 2, 5.

Order the data: 2, 4, 5, 7, 7.

$$\text{mean} = \frac{4 + 7 + 7 + 2 + 5}{5} = \frac{25}{5} = 5, \quad \text{median} = 5, \quad \text{mode} = 7, \quad \text{range} = 7 - 2 = 5.$$

Averages from a frequency table

When data is listed with its **frequency** 频数 (how many times each value occurs), the mean is

$$\text{mean} = \frac{\sum(\text{value} \times \text{frequency})}{\sum \text{frequency}}.$$

Worked example. Values 1, 2, 3 occur with frequencies 4, 5, 1. Find the mean.

$$\text{mean} = \frac{1(4) + 2(5) + 3(1)}{4 + 5 + 1} = \frac{17}{10} = 1.7.$$

Grouped data (Extended)

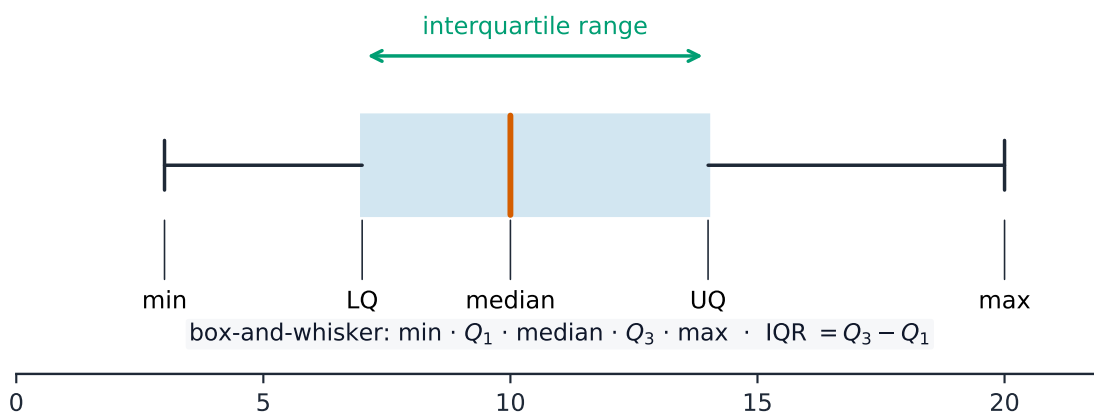
For **grouped data** 分组数据, you cannot find the exact mean, so estimate it using the **midpoint** of each group as the value. The **modal class** 众数组 is simply the group with the highest frequency.

Measures of spread (Extended)

When data is in order, the **quartiles** 四分位数 cut it into four equal parts. The lower quartile (LQ) is one quarter of the way up; the upper quartile (UQ) is three quarters of the way up. The **interquartile range** 四分位距 measures the spread of the middle half:

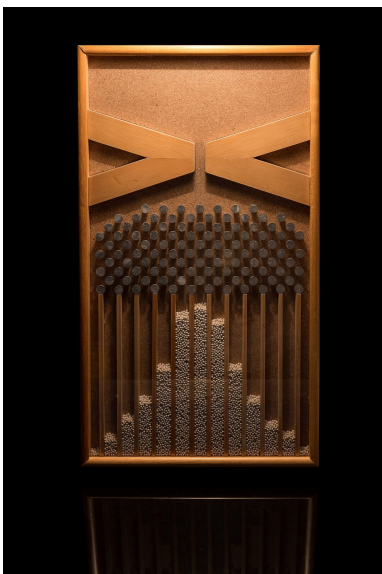
$$\text{interquartile range} = \text{UQ} - \text{LQ}.$$

The interquartile range is useful because, unlike the range, it ignores extreme values.

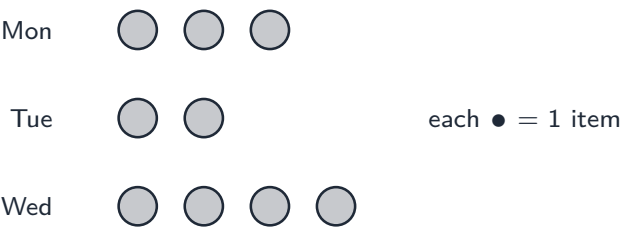


A box-and-whisker plot shows the five-number summary; the box length is the interquartile range.

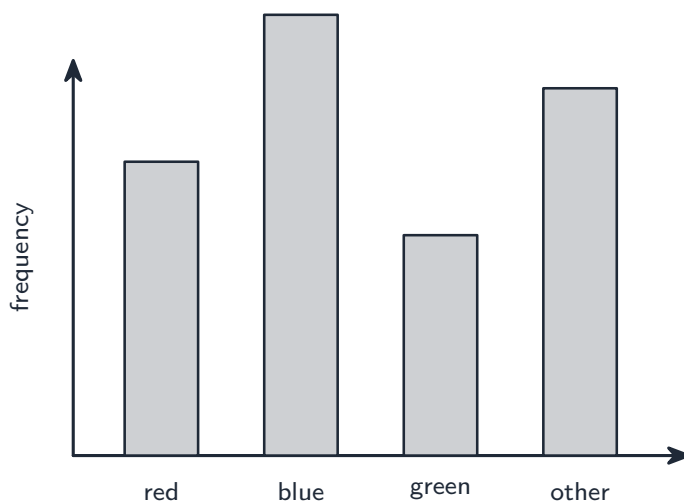
Charts and diagrams



A Galton board shows how data piles up into a distribution.



A pictogram uses a symbol to stand for a number of items

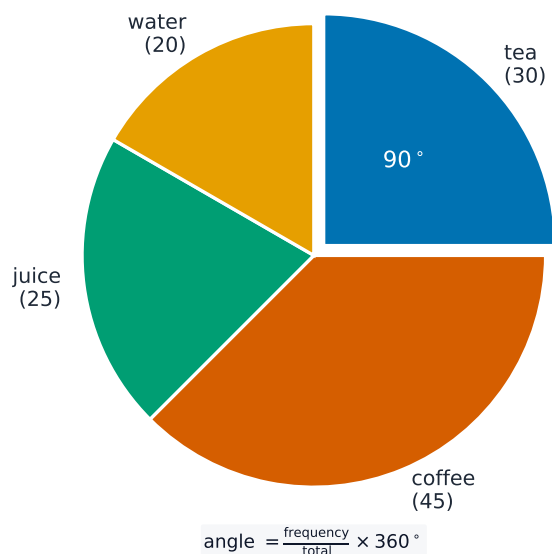


A bar chart shows the frequency of each group

Chart	What it shows
bar chart 条形图	a bar for each category; height is the frequency
pie chart 饼图	a circle split into slices, each a fraction of 360°
pictogram 象形图	uses a symbol to stand for a number of items
stem-and-leaf diagram 茎叶图	keeps the digits of ordered data, with a key
frequency distribution 频数分布	a table of values and their frequencies

Worked example (pie chart). Out of 120 people, 30 chose tea. Find the angle of the tea slice.

$$\frac{30}{120} \times 360^\circ = 90^\circ.$$

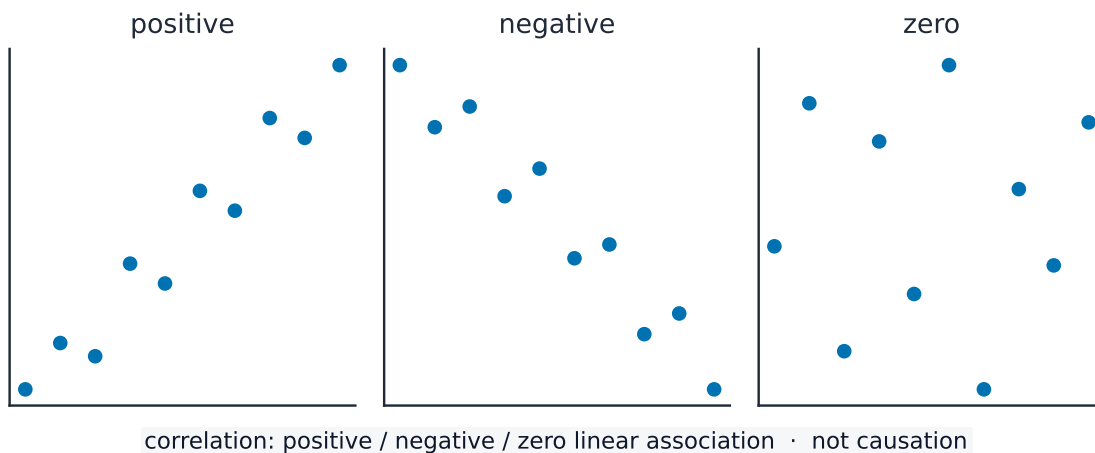


Each slice is its fraction of 360° ; tea is $\frac{30}{120} \times 360^\circ = 90^\circ$.

Scatter diagrams and correlation

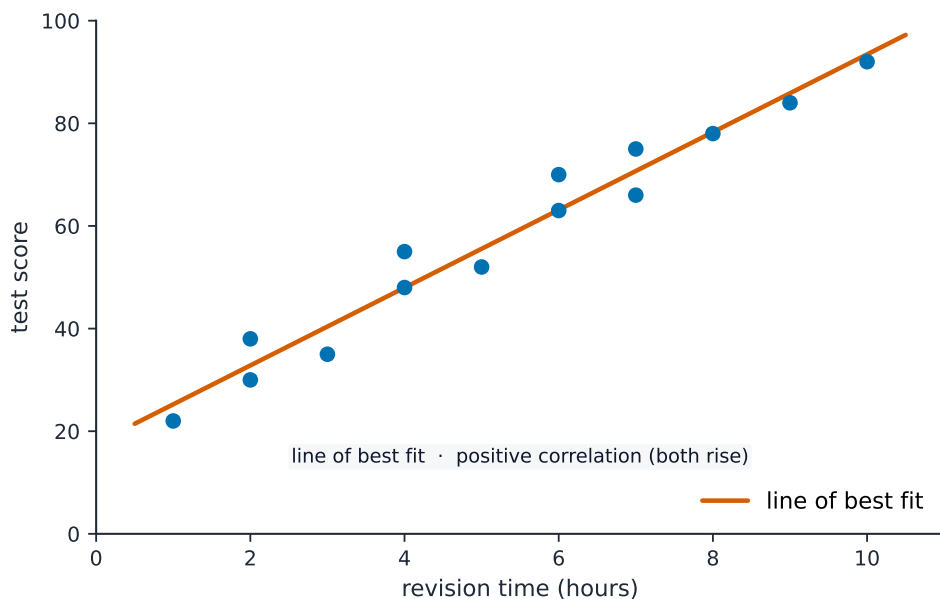
A **scatter diagram** 散点图 plots pairs of values as points to show whether two things are linked. The link is the **correlation** 相关性:

- **positive correlation** 正相关—as one goes up, the other goes up.
- **negative correlation** 负相关—as one goes up, the other goes down.
- **zero correlation** 零相关—no clear link.



Positive correlation rises together; negative falls as the other rises; zero shows no clear link.

If there is correlation, draw a **line of best fit** 最佳拟合线: one straight ruled line through the middle of the points, with about the same number of points on each side. Use it to predict values.

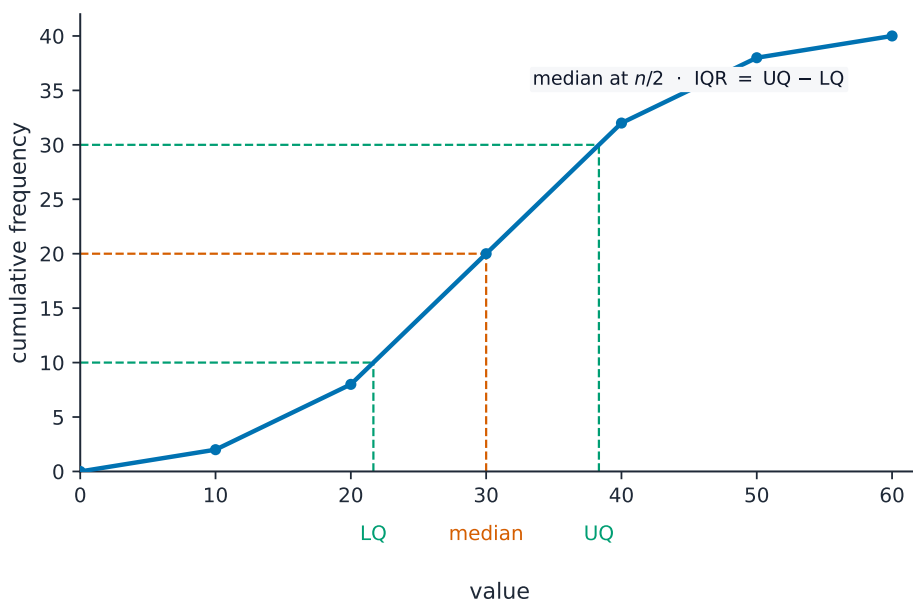


A line of best fit is one straight ruled line through the middle of the points; use it to predict.

Cumulative frequency diagrams (Extended)

The **cumulative frequency** 累积频数 is a running total of the frequencies. Plot it against the upper end of each class and join the points with a smooth curve.

From the curve you can read the median (at half the total), the quartiles (at one quarter and three quarters), and any **percentile** 百分位数 (for example, the 90th percentile is at 90% of the total).

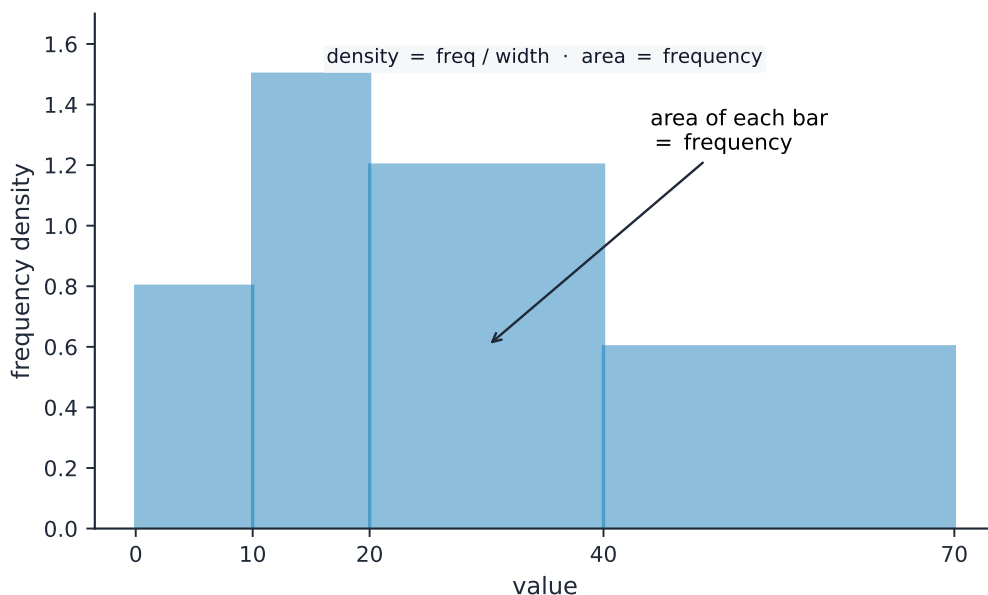


Read the median at half the total, and the quartiles at one quarter and three quarters: across to the curve, then down.

Histograms (Extended)

A **histogram** 直方图 looks like a bar chart, but the bars can have different widths and the area of each bar (not its height) shows the frequency. The vertical axis is the **frequency density** 频数密度:

$$\text{frequency density} = \frac{\text{frequency}}{\text{class width}}.$$



With unequal class widths the bar AREA (not its height) is the frequency, so the axis is frequency density.

Worked example. A class has **class width** 组距 10 and frequency 25. Find the frequency density.

$$\text{frequency density} = \frac{25}{10} = 2.5.$$

Exam tips

- Know the three averages: **mean** (add up ÷ how many), **median** (the middle value when in order), **mode** (the most common). The **range** is highest – lowest.
- From a frequency table the mean is $\frac{\sum fx}{\sum f}$ —divide by the *total frequency*, not the number of rows.

- On a **scatter diagram**, describe the correlation as positive, negative or none, and draw the line of best fit through the mean point.
- For a **histogram** with unequal class widths, the height is the *frequency density* (frequency $\div$ class width), not the frequency.