

Algebra & Graphs

IGCSE Mathematics ■ Topic 2

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What we will cover

- 1 Simplify, expand & factorise
- 2 Solving equations
- 3 Quadratics & inequalities
- 4 Sequences
- 5 Graph shapes
- 6 Travel graphs



Simplify and expand

$$3 \begin{array}{|c|c|} \hline & \\ \hline x & 2 \\ \hline 3x & 6 \\ \hline \end{array}$$

$$3(x + 2) = 3x + 6$$

In **algebra** we use letters to stand for numbers.

A letter whose value can change is a **variable** .

The words an algebra question uses

Expression and constant

An **expression** has no equals sign; a **constant** is a fixed number in it, as against a variable.

Substitute

To **substitute** is to replace each letter by its value and evaluate – brackets first, and mind the signs.

Algebraic fraction

An **algebraic fraction** (a **rational expression**) is one polynomial over another; simplify by factorising both and cancelling common factors.

Fractional equation

A **fractional equation** has the unknown in a denominator: multiply through by the common denominator, then check no solution makes it zero.

A **perfect square** factorises as $(x + a)^2$ – spotting one saves the whole method of completing the square.

Factorising

common factor

$$\begin{aligned} &9x^2 + 15xy \\ &= 3x(3x + 5y) \end{aligned}$$

**difference
of squares**

$$\begin{aligned} &a^2 - b^2 \\ &= (a+b)(a-b) \end{aligned}$$

quadratic

split the middle
term, then group

Factorise is the reverse of expanding: take out the common factor first.

Solving equations

$$2x + 3 = 11 \text{ (start)}$$

$$2x = 8 \quad \text{subtract 3 from both sides}$$

$$x = 4 \quad \text{divide both sides by 2}$$

Do the **same** to both sides until the **unknown** is alone.

Simultaneous

$$2x + y = 7, \quad 3x - y = 8$$

$$\text{add: } 5x = 15, \text{ so } x = 3, y = 1$$

Changing the subject

To **change the subject** of a formula means to rearrange it so a chosen letter is alone on one side.

Worked example. Make r the subject of $A = \pi r^2$ (**Extended**, because of the power).



Quadratic equations

factorise

$$(x + 2)(x + 3) = 0$$

**complete
the square**

$$(x + 3)^2 = 8$$

formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Three routes to a quadratic's roots

Factorising

Fastest when it works: write as a product of two brackets and set each to zero.

Completing the square

Gives the **maximum** or minimum point directly, and the **axis of symmetry** with it.

By the quadratic formula

By the quadratic formula : $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ – always available, and the discriminant tells you how many roots first.

Reading the graph

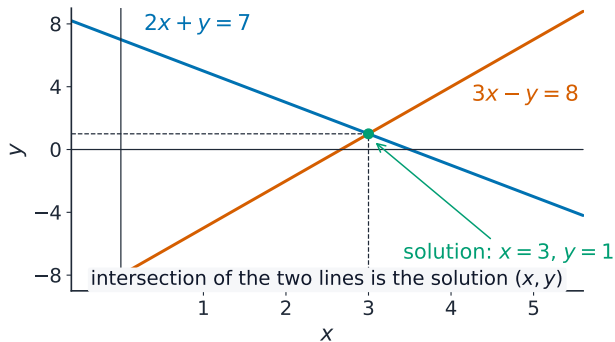
The roots are where the curve meets the x-axis; the **intercept** on the y-axis is the constant term.

An **intersection point** of two graphs is the solution of their simultaneous equations – and vice versa.

Simultaneous equations

Simultaneous equations are two equations solved **together**. For two linear equations, add or subtract to remove one letter.

The solution is where the two graphs **cross** – one point that satisfies both equations at once.



Inequalities

Solve like an equation, but **reverse** the sign if you $\times$ or $\div$ by a negative. Number line: **closed** circle includes the value, **open** excludes it.

$-\frac{1}{3}$ included (closed circle) 3 excluded (open circle)



An **inequality** uses $<$, $>$, $\leq$ or $\geq$.

Solve it like an equation, but **reverse the sign** if you multiply or divide by a negative number.

Sequences



Find the n th term from how the terms change.

- linear $2, 5, 8, 11 \dots \rightarrow 3n - 1$

- quadratic: $2 \times$ the 2nd difference $= n^2$ part

$2, 5, 10, 17 \dots \rightarrow n^2 + 1.$

Sequences, indices in algebra, and the derivative

a sequence

a list of numbers following a rule

the term-to-term rule

tells you how to get the **next** term from the one before

the n th-term rule

gives **any** term directly from its position – far more useful

indices in algebra

$$a^m \times a^n = a^{m+n}, \quad a^m \div a^n = a^{m-n}, \quad (a^m)^n = a^{mn}$$

(Extended) the derivative

if $y = ax^n$ then $\frac{dy}{dx} = anx^{n-1}$

what it gives you

the gradient at any point, so stationary points come from $\frac{dy}{dx} = 0$

“Multiply the power by the coefficient, then reduce the power by one.” Say it aloud once and the rule stops needing to be looked up.

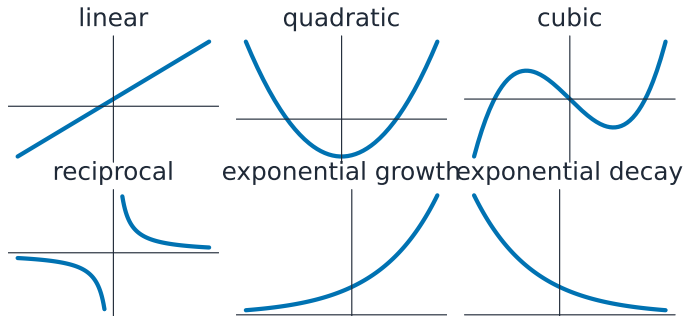


Graph shapes

Know each shape: **linear** , **quadratic** (parabola), cubic, **reciprocal** & **exponential** .

Roots are where the graph crosses the x-axis ($y = 0$).

Graph shapes, in detail



recognise $y = x^2$, $y = 1/x$, $y = 2^x$ from shape

Know each shape: **linear** , **quadratic** (parabola), cubic, **reciprocal** & **exponential** .

Growth, decay and the line a curve never meets

Exponential growth

Exponential growth : $y = ab^x$ with $b > 1$ – a fixed *multiplier* each step, not a fixed increase.

Exponential decay

Exponential decay : the same form with $0 < b < 1$. An **exponential sequence** is its discrete version, with a common ratio.

Asymptote

An **asymptote** is a line the curve approaches but never reaches – the x-axis for an exponential, both axes for $y = 1/x$.

Direct proportion

Direct proportion means $y = kx$: a straight line through the origin. Inverse proportion means $y = k/x$, a hyperbola.

The **base** b decides everything: above one it grows, below one it decays, and at one it is a horizontal line.

Drawing graphs, and reading a region

a table of values

choose values of x , work out y , then plot the **coordinates**

how many points

a straight line needs three (two plus a check); a curve needs five or more

a conversion graph

a straight line used to change between two units – miles and kilometres

a travel graph

distance–time: the **gradient is speed**. Speed–time: the gradient is acceleration and the **area is distance**

(Extended) a region

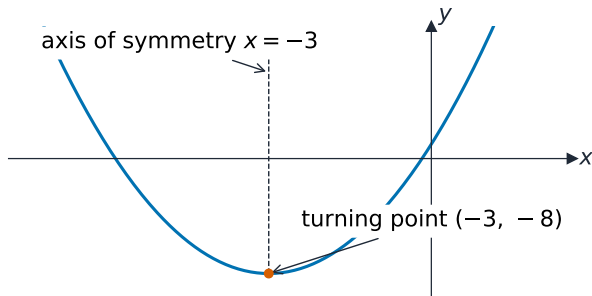
an inequality in two letters describes a region

drawing it

draw the boundary line – **broken** for $<$ or $>$, **solid** for $\leq$ or $\geq$ – then shade

Test the origin in the inequality to decide which side to shade. If $(0,0)$ satisfies it, shade the side containing the origin.

Completing the square



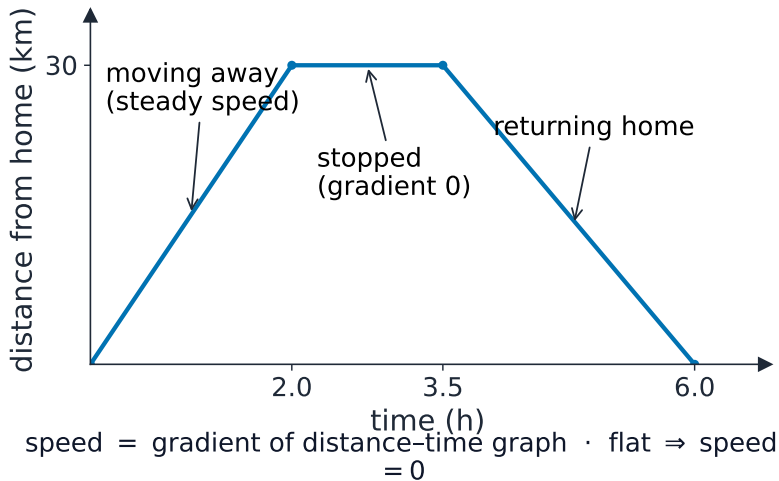
$$y = (x + 3)^2 - 8$$

Writing $y = (x + p)^2 + q$ shows the **turning point** at $(-p, q)$.

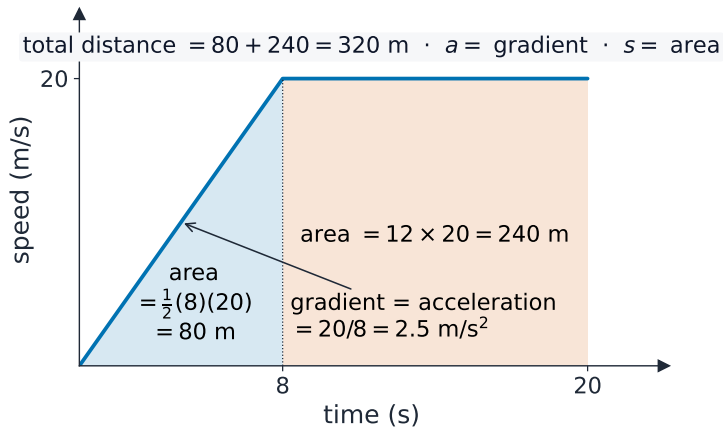
$$x^2 + 6x + 1 = (x + 3)^2 - 8$$

So the lowest point is $(-3, -8)$.

Travel graphs – one



Travel graphs – two



Sketching curves

A quick sketch should show the right shape and the key features: where it crosses the axes, any **symmetry** , and any line the curve gets close to.

For a parabola, completing the square gives the **turning point** (the lowest or highest point).

Differentiation (Extended)

Differentiation finds the gradient of a curve. If $y = ax^n$:

$$\frac{dy}{dx} = a n x^{n-1}$$

differentiate a sum term by term.

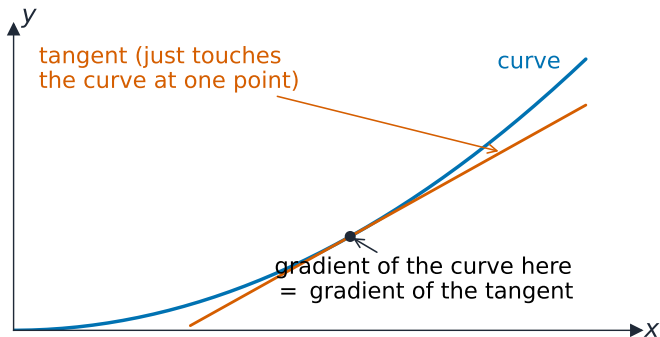
A **turning point** is where the gradient is **zero** – set $dy/dx = 0$ and solve.

Turning point of $y = x^2 - 6x + 5$

$$\frac{dy}{dx} = 2x - 6 = 0 \Rightarrow x = 3$$

$$y = 9 - 18 + 5 = -4, \text{ so } (3, -4)$$

Differentiation (Extended)



$$\frac{dy}{dx} = \text{gradient of the tangent}$$

The gradient of a curve at a point equals the gradient of the tangent there – what differentiation finds.

Functions (Extended)

A **function** turns each input into one output, $f(x)$.

- **inverse** f^{-1} **undoes** it: swap x and y , solve
- **composite** $gf(x)$ means do f **first**, then g

$$f(x) = 2x, g(x) = x + 3$$

$$gf(x) = g(2x) = 2x + 3$$

$$fg(x) = f(x + 3) = 2x + 6$$

Order matters: $gf \neq fg$.

Direct and inverse proportion (Extended)

- **direct** : $y = kx$ – one is a fixed multiple of the other
- **inverse** : $y = k/x$ – one rises as the other falls

Always **find k first** from the given pair.

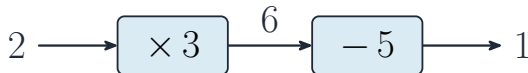
$y \propto x$; $y = 12$ when $x = 3$. Find y at $x = 7$.

$$12 = k \times 3 \Rightarrow k = 4, \text{ so } y = 4x$$

$$y = 4 \times 7 = 28$$

Functions (Extended)

$f(x) = 3x - 5$: do $\times 3$, then -5



$f^{-1}(x) = \frac{x+5}{3}$:



reverse the order, undo each step

the inverse takes the
output back to the
input: $f(2) = 1$,
so $f^{-1}(1) = 2$

A function as a machine; the inverse runs it backwards - reverse the order, undo each step

Function words, and rate of change

Domain

The **domain** is the set of inputs allowed – restricted wherever a denominator vanishes or a square root would go negative.

Composite function

A **composite function** $fg(x)$ applies g first, then f . The order changes the answer.

Inverse function

An **inverse function** undoes f : swap x and y and make y the subject. Its graph is the reflection in $y = x$.

Rate of change

The gradient of a travel graph is a **rate of change** – speed on a distance–time graph, and a negative gradient there is a **deceleration**.

An inverse exists only if each output came from one input – which is why a domain is so often restricted first.

Simplifying and expanding

$$3a + 2a + 4b - b$$

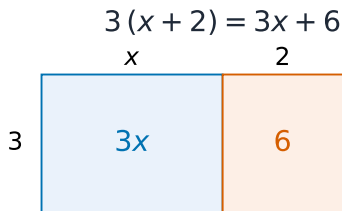


collect like terms

$$5a + 3b$$

Collecting like terms: add terms with the same letter

Expand by distributing: $a(b+c)=ab+ac$ (area model)



$a(b+c) = ab + ac$ · area model of expansion

$$2a^2 + 3ab - 1 + 5a^2 - 9ab + 4$$

↓ group like terms

$$7a^2 - 6ab + 3$$

terms that share a colour are like terms

like terms: same letters/powers · add coefficients only

Like terms share the same letters and powers – add coefficients only

Expand by distributing: $a(b+c)=ab+ac$ (area model)

Completing the square (Extended)



A suspension bridge cable hangs in a parabola – the graph of a quadratic.

Exam tips

- When you **expand brackets**, multiply every term and watch the signs, especially with a minus in front: $-(x - 3) = -x + 3$.
- To solve an equation, do the same thing to both sides. When you multiply or divide an **inequality** by a negative number, *flip* the sign.
- **Factorise fully**: take out the highest common factor first, then look for a difference of two squares or a quadratic pattern.
- A **quadratic** usually has two solutions – give both. Check whether the question wants factorising, the formula, or completing the square.
- When substituting into a formula, put each value in brackets first, so signs and powers come out right.



Topic 2 – key takeaways

1

Expand

multiply out; factorise reverses it

2

Quadratics

factorise, complete the square, or formula

3

Inequalities

flip the sign when dividing by a negative

4

Graphs

gradient = rate; area under speed–time = distance

Check yourself

- 1 Expand $(x + 2)(x - 4)$.
- 2 Factorise $9x^2 - 16$.
- 3 When must you reverse an inequality sign?
- 4 What does the area under a speed–time graph give?



Answers 1. $x^2 - 2x - 8$ 2. $(3x + 4)(3x - 4)$ 3. when $\times$ or $\div$ by a negative 4. the distance travelled