# Simple harmonic motion

C.1 · SL and AHL

# Why does a tall building sway at its own rhythm?

SHM approximates small oscillations when the **restoring force** is proportional to displacement.

- Swings, guitar strings, car springs and atoms in crystals all oscillate.
- Each repeats its motion about a fixed **equilibrium position**.
- This unit: conditions $\rightarrow$ a $= -\omega^2$x $\rightarrow$ graphs $\rightarrow$ springs and pendulums $\rightarrow$ energy.

# Conditions for SHM

## SHM

Ideal linear spring: resultant $F_{\text{net}} = -kx$.

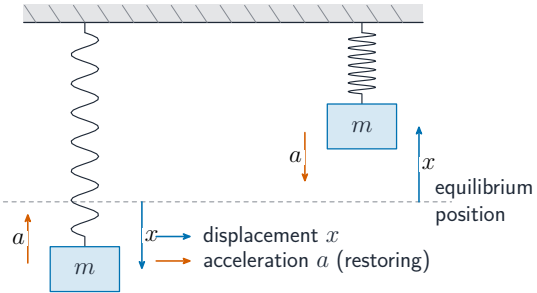
Pendulum at small angles.

## Not SHM

A bouncing ball.

In flight its acceleration is g, independent of height.

# Acceleration points back to equilibrium




$$a = -\omega^2 x$$

# The defining equation

- Acceleration is proportional to displacement.
- The minus sign means acceleration always points back to equilibrium.

$$a = -\omega^2 x$$

$\omega = 12.6$ rad s$^{-1}$ at $x = +0.020$ m gives $a = -3.2$ m s$^{-2}$.

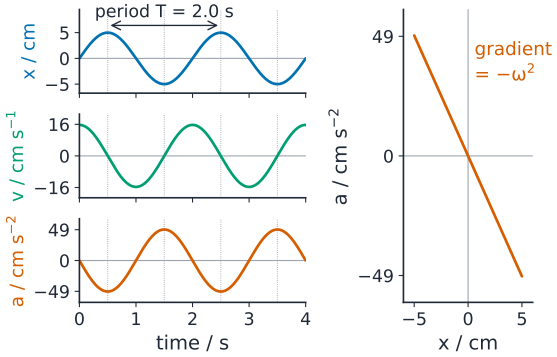
# Period, frequency and angular frequency

- **Amplitude** $x_0$ is measured from equilibrium, not end to end.
- A 440 Hz string has $T = 2.3$ ms and $\omega = 2.8 \times 10^3$ rad s$^{-1}$.

$$T = \frac{1}{f} = \frac{2\pi}{\omega}$$

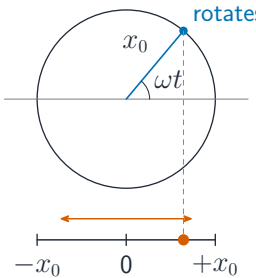
The period of SHM does not depend on the amplitude.

# Graphs of SHM




Speed peaks at equilibrium; acceleration magnitude peaks at the ends.

# SHM is the shadow of circular motion




One turn of $2\pi$ is one **oscillation**

# The mass–spring system

- 0.40 kg on 40 N m$^{-1}$ gives $T = 0.63$ s.
- A four times larger mass doubles the period.

$$T = 2\pi\sqrt{\frac{m}{k}}$$

Ideal constant-k model; x is from equilibrium. Gravity shifts that equilibrium.

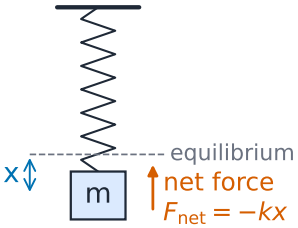
# Worked example: a hanging spring

1. **Given.** Mass 0.40 kg; stiffness 40 N/m; g = 9.8 m/s$^2$; x = +0.025 m below equilibrium. Down is positive.
2. **Equilibrium.** $s_{\text{eq}} = mg/k = 0.098$ m; actual extension $s = 0.123$ m.
3. **Forces.** Weight +3.92 N; spring force −4.92 N; resultant −1.00 N.
4. **Acceleration.** $a = -1.00/0.40 = -2.5$ m/s$^2$, upward.

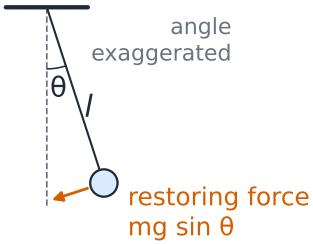
## Decision

Check: $-kx = -40(0.025) = -1.00$ N uses x from equilibrium.

# Springs and pendulums




$$T = 2\pi\sqrt{\frac{m}{k}}$$




$$T = 2\pi\sqrt{\frac{l}{g}}$$

Spring: net **restoring force**. Pendulum: tangential weight.

# The simple pendulum

- A 1.0 m pendulum has $T = 2.0$ s on Earth and 5.0 s on the Moon.
- Plot $T^2$ against $l$; the gradient is $4\pi^2/g$.

$$T = 2\pi\sqrt{\frac{l}{g}}$$

Small-angle tangential motion; the bob also has radial acceleration.

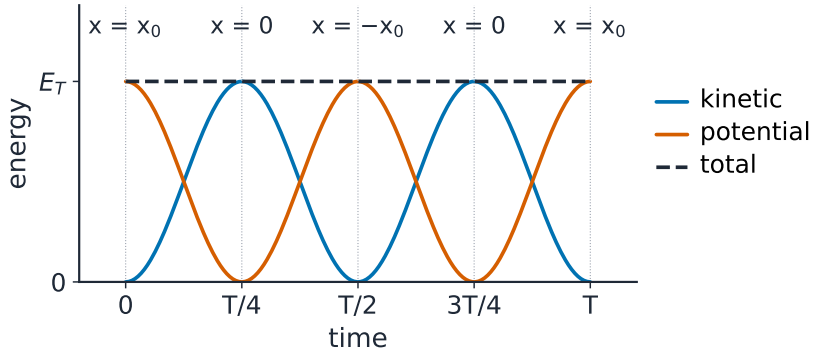
# Check: a pendulum

**Your turn**

An ideal small-angle pendulum has period 2.0 s. Its length is multiplied by four at unchanged g. What is the new period?

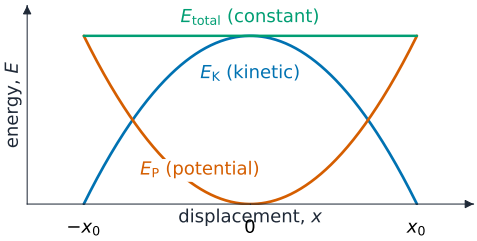
A. 1.0 s
B. 2.0 s
C. 4.0 s
D. 8.0 s

# Energy in SHM




Undriven, no dissipation: kinetic + potential energy stays constant.

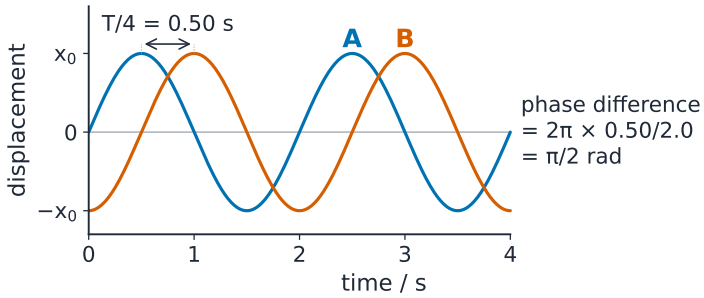
# Energy in SHM against displacement




SHM: $E_k + E_p =$ const · max $E_k$ at centre

Their sum is constant

# Phase angle




- Phase is measured in radians; one cycle is $2\pi$ rad.
- In phase means 0 rad apart; **antiphase** means $\pi$ rad apart.

# The equations of SHM

- Set the calculator to radians.
- $v_{\max} = \omega x_0$ at equilibrium; $a_{\max} = \omega^2 x_0$ at the ends.

$$x = x_0 \sin(\omega t + \phi) \qquad v = \pm\omega\sqrt{x_0^2 - x^2}$$

$v = \omega x_0 \cos(\omega t + \phi)$ gives the velocity at a given time.

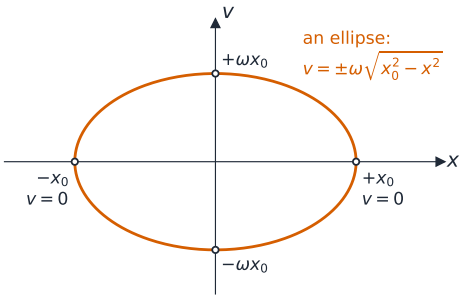
# Worked example: a mass on a spring

1. **Known.** **Angular frequency** 10 rad/s, **amplitude** 0.050 m; passes equilibrium in the positive direction at t = 0. Find x and v at 0.10 s.
2. **Displacement.** $x = 0.050 \sin(1.0) = 0.042$ m.
3. **Velocity.** $v = 10 \times 0.050 \cos(1.0) = 0.27$ m s$^{-1}$.

## Decision

$10\sqrt{0.050^2 - 0.042^2} = 0.27$ m s$^{-1}$ agrees

# Velocity against displacement




$$v = \pm\omega\sqrt{x_0^2 - x^2}$$

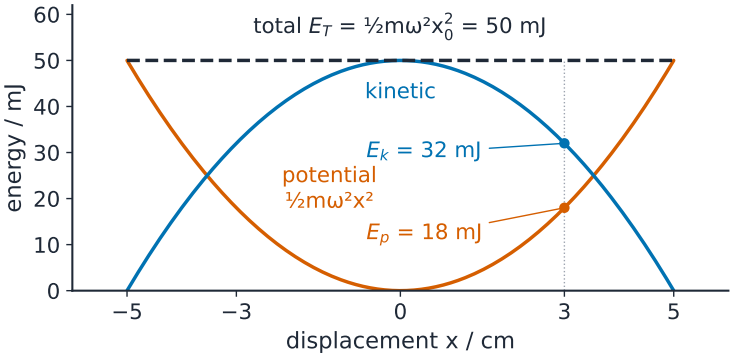
# Energy and displacement

- $E_T = 0.050$ J; at $x = 0.030$ m, $E_p = 0.018$ J and $E_k = 0.032$ J.
- Doubling the **amplitude** gives four times the energy.

$$E_T = \tfrac{1}{2} m\omega^2 x_0^2 \qquad E_p = \tfrac{1}{2} m\omega^2 x^2$$

Potential energy relative to equilibrium; for a vertical spring include elastic and gravitational contributions.

# The energy parabolas




The kinetic and potential curves sum to the same mechanical total.

# Exam tips

> To show SHM, show that $a \propto -x$.

- Measure **amplitude** from equilibrium.
- Ideal linear spring or small-angle pendulum: period is independent of amplitude.
- Speed peaks at equilibrium; acceleration magnitude peaks at the ends.

# Check yourself

## Try these

Find the period of a 0.50 kg mass on a spring of 200 N $m^{-1}$.

Find the length of a pendulum with a period of 1.0 s.

Describe the energy changes of a pendulum during one oscillation.

AHL Find the speed at $x = 0.040$ m when $x_0 = 0.080$ m and $\omega = 5.0$ rad $s^{-1}$.