Rigid body mechanics

A.4 · AHL

Why is a door handle far from the hinge?

> For turning, where and in which direction a force acts matters as much as its size.

- The same push turns a door easily at the handle, hardly at the hinge.
- The whole unit is additional higher level.
- This unit: torque $\rightarrow$ angular motion $\rightarrow \tau = I\alpha \rightarrow L = I\omega \rightarrow$ rolling.

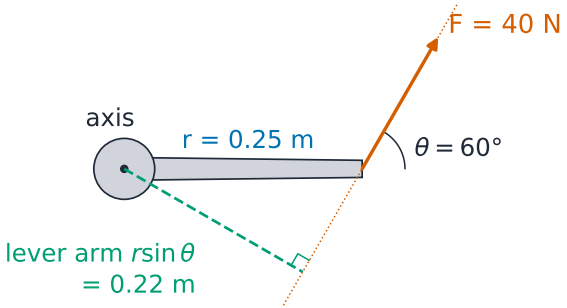
Torque

- $r\sin\theta$ is the perpendicular distance to the line of action.
- State the sense, clockwise or anticlockwise.

$$\tau = Fr\sin\theta$$

A force through the axis has no turning effect.

The lever arm



- 40 N at 60° on a 0.25 m spanner.
- $\tau = 40 \times 0.25 \times \sin 60° = 8.7$ N m.

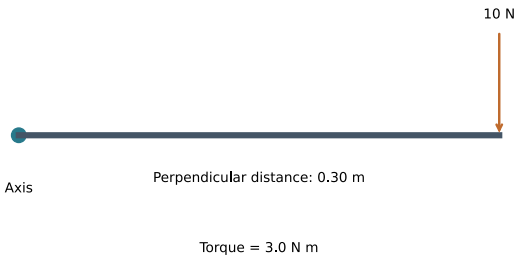
Rotational equilibrium

1. **Choose.** A uniform 4.0 m, 20 kg plank rests on end supports; a 60 kg person stands 1.0 m from the left. Use $g = 9.8$ m s^{-2}; take moments about the left end.
2. **Balance.** $196(2.0) + 588(1.0) = 4.0 R_{\text{right}}$.
3. **Forces.** $R_{\text{left}} = 784 - 245 = 539$ N.

Decision

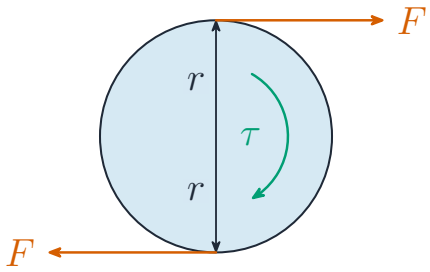
$R_{\text{right}} = 245$ N; the support nearer the person carries more

Torque from a perpendicular force



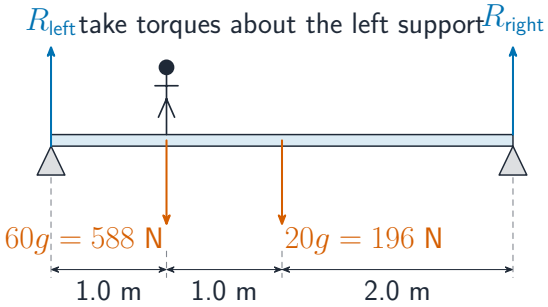
A perpendicular force gives the largest torque

A couple turns without moving



A **couple** rotates a body but does not accelerate its centre

A person on a plank



The support nearer the person carries more

Linear and angular quantities

Linear	**Angular**	**Link**
s	$\Delta\theta$ / rad	$s = r\Delta\theta$
v	ω / rad s^{-1}	$v = \omega r$
a_t	α / rad s^{-2}	$a_t = \alpha r$

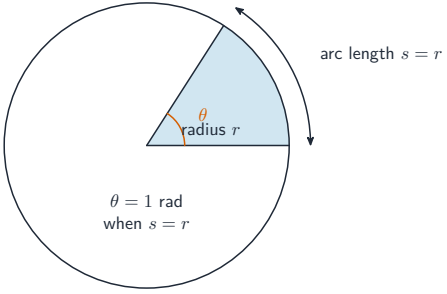
Worked example: a fan slows down

1. **Given.** $\omega_i = 30$ rad s^{-1}, $\omega_f = 0$, displacement 40 revolutions; assume uniform **angular acceleration**.
2. **Convert.** $\Delta\theta = 40 \times 2\pi = 251$ rad.
3. **Choose.** $\omega_f^2 = \omega_i^2 + 2\alpha\Delta\theta$.

Decision

$\alpha = -900/502 = -1.8$ rad s^{-2}

One radian



A full turn is 2π radians

Moment of inertia

- Mass far from the axis counts most, because r is squared.
- Ring MR^2, disc $\frac{1}{2}MR^2$, sphere $\frac{2}{5}MR^2$: given when needed.

$$I = \sum mr^2$$

Two 0.50 kg masses at 0.40 m give 0.16 kg m^2; at 0.20 m only 0.040.

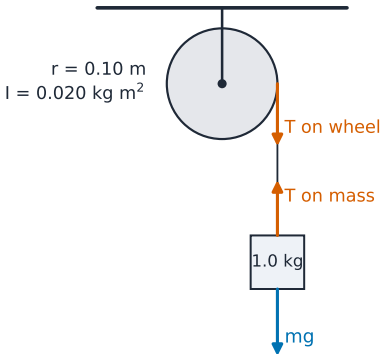
Newton's second law for rotation

$\tau = I\alpha$, the rotational twin of $F = ma$.

- τ is the resultant torque; constant inertia about a fixed axis.
- 3.0 N m on $I = 0.60$ kg m^2 gives $\alpha = 5.0$ rad s^{-2}.

In experiments, guard rotating parts, clamp the axle and secure hanging loads.

Coupled bodies



wheel: $Tr = I\alpha$

mass: $mg - T = ma$

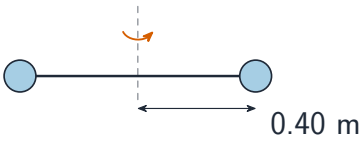
link: $a = \alpha r$

$a = 3.3$ m s^{-2}, $T = 6.5$ N

- One equation per body, linked by $a = \alpha r$; the wheel takes energy, so $a < g$.

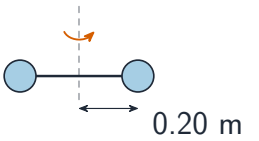
Moment of inertia depends on distance

$m = 0.50$ kg each



$I = 2mr^2 = 0.16$ kg m^2

masses slid in



$I = 2mr^2 = 0.040$ kg m^2

Halve the distance: a quarter of the inertia

Angular momentum is conserved

- Skater: I 3.0 $\to$ 1.0 kg m^2, so ω 2.0 $\to$ 6.0 rad s^{-1}.
- Kinetic energy rises from 6.0 J to 18 J; the muscles do the work.

Angular momentum $L = I\omega$, constant unless a resultant external torque acts

Worked example: coupled discs

1. **Given.** Coaxial discs: $I_1 = 0.20$ kg m^2, $\omega_1 = 30$ rad s^{-1}; $I_2 = 0.10$ kg m^2, initially stationary. They stick; neglect external **angular impulse**.
2. **Conserve** L. $0.20 \times 30 = 0.30\,\omega$, so $\omega = 20$ rad s^{-1}.
3. **Energy.** E_k falls from 90 J to 60 J.

Decision

Like a perfectly inelastic collision

Angular impulse

- With initial rotation positive, a brake torque of -15 N m for 4.0 s gives $\Delta L = -60$ kg m^2 s^{-1}.
- A 2.0 kg m^2 flywheel slows from 50 to 20 rad s^{-1}.

$$\Delta L = \overline{\tau}\Delta t = \Delta(I\omega)$$

Use average resultant external torque; signed torque–time area gives **angular impulse**.

Check: a turntable

Your turn

A freely rotating turntable has $I = 0.050$ kg m^2 and angular velocity 3.0 rad s^{-1}. A 0.10 kg lump of clay drops vertically onto it 0.15 m from the axis and sticks. Neglect external angular impulse. What is their common final angular velocity?

A. 2.0 rad s^{-1}

B. 2.9 rad s^{-1}

C. 3.0 rad s^{-1}

D. 3.1 rad s^{-1}

Angular impulse from a changing torque

1. **Given.** Invented triangular net-torque pulse: zero $\to +4.0$ N m $\to$ zero in 3.0 s. Fixed $I = 0.50$ kg m^2; $\omega_i = +8.0$ rad s^{-1}.
2. **Area.** $\Delta L = \frac{1}{2}(3.0)(4.0) = +6.0$ N m s; $\overline{\tau} = 6.0/3.0 = +2.0$ N m.
3. **Change.** $\Delta\omega = 6.0/0.50 = +12$ rad s^{-1}; $\omega_f = 8.0 + 12 = +20$ rad s^{-1}.

Decision

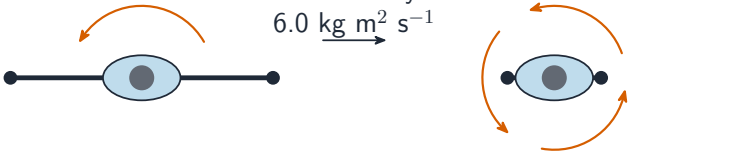
Use the triangle area; peak torque times duration doubles the impulse.

A skater pulls the arms in

arms out

arms pulled in

$L = I\omega$ stays
6.0 kg m^2 s^{-1}

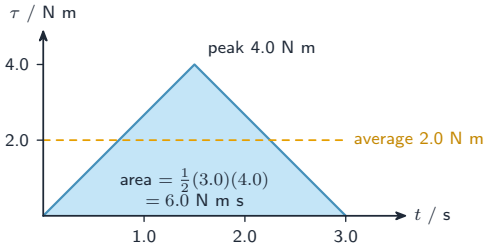


$I = 3.0$ kg m^2, $\omega = 2.0$ rad s^{-1}

$I = 1.0$ kg m^2, $\omega = 6.0$ rad s^{-1}

$L = I\omega = 6.0$ both times

Angular impulse is an area



The same area as an average 2.0 N m for 3.0 s

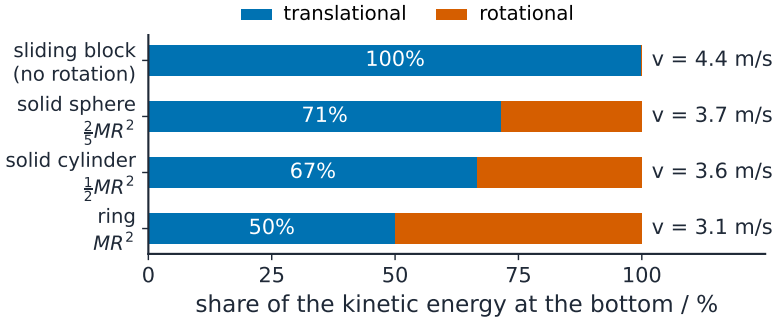
Rotational kinetic energy

- Use $L^2/2I$ when L is conserved.
- Rolling without slipping: $v = \omega r$.

$$E_k = \tfrac{1}{2} I\omega^2 = \frac{L^2}{2I}$$

Rolling: $E_k = \tfrac{1}{2} Mv^2 + \tfrac{1}{2} I\omega^2$; v is centre-of-mass speed, I is about the **centre of mass**.

The race down a slope



- Solid cylinder after a 1.0 m drop: $v = \sqrt{\frac{4}{3}gh} = 3.6$ m s^{-1}.
- Same slope, rest start, no slip or dissipation: smaller $I/(MR^2)$ wins.

Exam tips

> Torque uses the perpendicular distance to the line of action.

- Take torques about the point with the most unknowns.
- Convert revolutions with 2π and rpm with $2\pi/60$.
- When I changes with no external torque, conserve L, not E_k.

Check yourself

Try these

A uniform metre rule is pivoted at 50 cm. Balance 2.0 N at 20 cm with 3.0 N on the other side: where?

A constant net torque of 2.0 N m acts for 6.0 s on a disc initially at rest, $I = 0.40$ kg m^2. Find ω, L, E_k.

A solid disc ($I = MR^2/2$) and a ring ($I = MR^2$) roll without slip from rest down the same 0.50 m drop. Neglect losses; use $g = 9.8$ m s^{-2}. Find both speeds.