Course Companion

# International Baccalaureate · IB Diploma · Mathematics: Applications and Interpretation · SL

Every topic handout in one volume

全部专题讲义合集

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# Number and algebra

International Baccalaureate · IB Diploma · Mathematics: Applications and Interpretation · SL

## Scope and prerequisites

This handout teaches topic 1, Number and algebra, of Mathematics: Applications and Interpretation, the course first assessed in 2021: arithmetic and geometric sequences and series and their applications, compound interest, depreciation, loans and annuities with technology, standard form, approximation, bounds and percentage error, exponents and logarithms, and solving equations with technology, and, for HL, the laws of logarithms, rational exponents, complex numbers, matrices, and eigenvalues and eigenvectors.

Prerequisites: percentages, negative numbers, substitution into formulae, and solving linear equations. You need a graphic display calculator (GDC) with a finance (TVM) solver and an equation solver.

## Sequences and growth

### ■ Franklin's thousand pounds

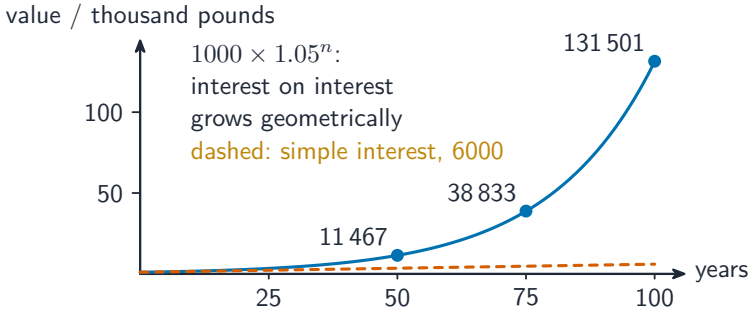
When Benjamin Franklin died in 1790, he left £1000 each to the cities of Boston and Philadelphia. The money was to be lent to young workers at 5 % interest, and the cities were not to touch it for 100 years. Franklin did the calculation himself: at 5 % a year, £1000 would grow to about £131 000 in 100 years. He was right about the arithmetic, since $1000 \times 1.05^{100} \approx 131\,501$. In reality the funds grew more slowly, because not every loan was repaid, but 200 years later they were still worth millions of dollars. Money that earns interest on its interest grows geometrically. This topic is about sequences like this one and the financial tools built from them.



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*A population growing by the same percentage each year is a geometric sequence*




*Franklin's £1000 at 5% compound interest*

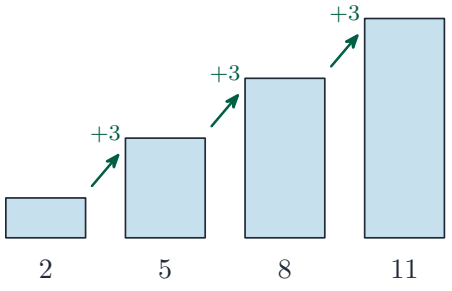
## ■ Arithmetic sequences

A **sequence** 数列 is an ordered list of numbers, its terms $u_1, u_2, u_3, \ldots$ An **arithmetic sequence** 等差数列 adds the same **common difference** 公差 $d$ each time. The formula booklet gives



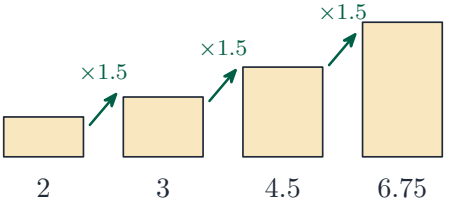
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**arithmetic**: add $d$ each step




$u_n = a + (n-1)d$: equal steps

**geometric**: multiply by $r$ each step




$u_n = ar^{n-1}$: growing steps

*An arithmetic sequence climbs in equal steps; a geometric one multiplies by a ratio*

$$u_n = u_1 + (n-1)d, \qquad S_n = \frac{n}{2}\big(2u_1 + (n-1)d\big) = \frac{n}{2}(u_1 + u_n)$$

where $S_n$ is the **series** 级数, the sum of the first $n$ terms.

> **Worked example.** A theatre has 20 seats in the first row and 2 more in each row behind, for 25 rows. The last row has $u_{25} = 20 + 24(2) = 68$ seats, and the theatre has $S_{25} = \frac{25}{2}(20 + 68) = 1100$ seats.

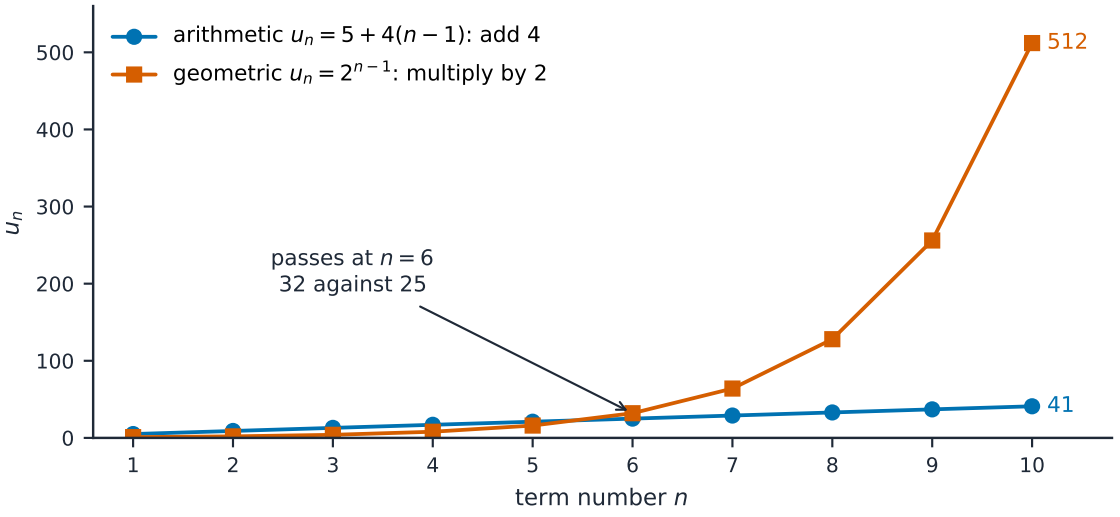
Arithmetic sequences model anything that changes by a fixed amount: simple interest, a salary with a fixed yearly rise, the cost of a taxi ride per kilometre. Their graph is a set of points on a straight line.

## Geometric sequences

A **geometric sequence** 等比数列 multiplies by the same **common ratio** 公比 $r$ each time:

$$u_n = u_1 r^{n-1}, \qquad S_n = \frac{u_1(r^n - 1)}{r - 1} = \frac{u_1(1 - r^n)}{1 - r}, \quad r \neq 1$$

A growth of $p$ % a year means multiplying by $r = 1 + \frac{p}{100}$; a decrease of $p$ % means $r = 1 - \frac{p}{100}$.




*Adding gives a straight line; multiplying eventually wins*

**Worked example.** A town of 2400 people grows by 3 % a year.

- After 10 years: $2400 \times 1.03^{10} \approx 3225$ people.
- It first exceeds 4000 when $2400 \times 1.03^{n} > 4000$. Use the GDC: graph $y = 2400 \times 1.03^{x}$ and $y = 4000$, or use a table, to find $n > 17.3$. So it takes 18 years.

Population models like this rarely stay geometric for long, because space and resources run out. A model is only as good as its **assumptions** 假设: here, a constant growth rate.

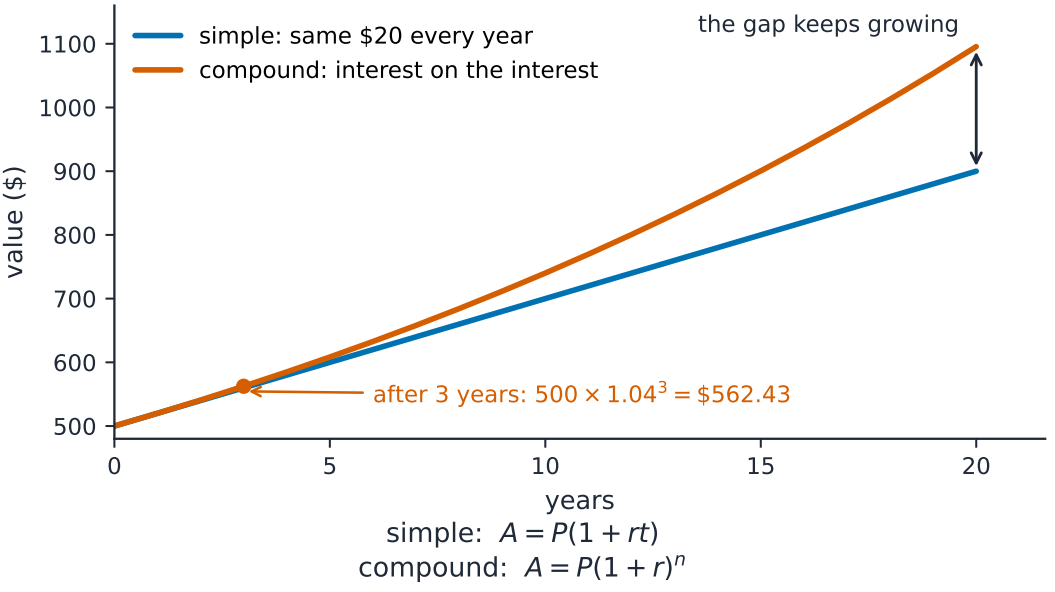
## Financial mathematics

### ■ Compound interest and depreciation

With **compound interest** 复利 the interest is added to the balance and then earns interest itself. A present value $PV$ invested at $r$ % a year, compounded $k$ times a year, for $n$ years, becomes




*Shop discounts and interest rates are both percentage changes: multiply by a factor, again and again*




*Simple interest grows in a straight line; compound interest grows faster every year*

$$FV = PV\left(1 + \frac{r}{100k}\right)^{kn}$$

**Depreciation** 折旧 is the opposite: a car or machine loses $r$ % of its value each year, so $V = V_0\left(1 - \frac{r}{100}\right)^n$.

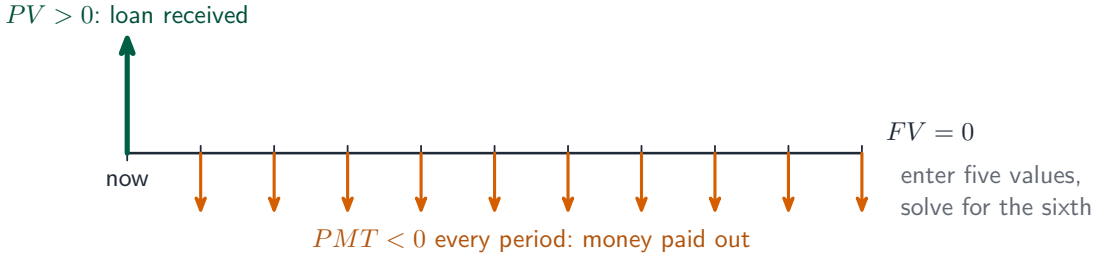
**Worked example.** ¥10 000 is invested at 4 % a year for 5 years. Compounded annually it becomes $10\,000 \times 1.04^5 \approx$ ¥12 166.53; compounded monthly, $10\,000\left(1+\frac{4}{1200}\right)^{60} \approx$ ¥12 209.97.

**Worked example.** A car bought for ¥200 000 loses 15 % of its value a year. After 4 years it is worth $200\,000 \times 0.85^4 \approx$ ¥104 401.

**Inflation** 通货膨胀 means prices rise, so money buys less. If savings earn 3 % while prices rise 2 %, the **real** value of the savings rises by only about 1 % a year.

## The finance solver

Most financial questions in this course are answered with the GDC's finance solver, usually called the TVM (time value of money) solver. It links six values:




*The finance solver's sign convention*

| Variable | Meaning |
|---|---|
| $N$ | total number of payments or compounding periods |
| $I\%$ | interest rate per year, as a percentage |
| $PV$ | present value |
| $PMT$ | payment each period |
| $FV$ | future value |
| $P/Y$, $C/Y$ | payments and compounding periods per year |

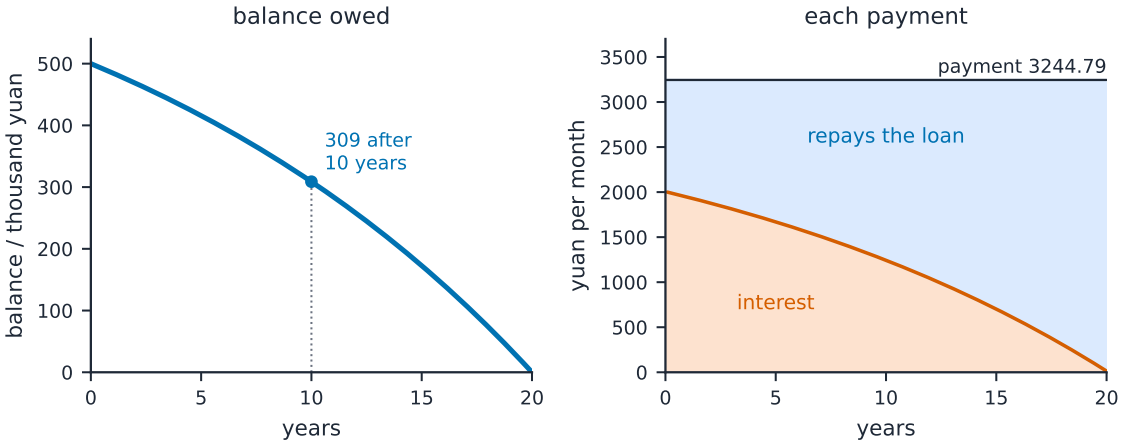
Enter any five and solve for the sixth. **Sign convention:** money you pay out is negative, and money you receive is positive. Always write down the values you entered: they earn method marks, and a wrong sign is the most common error.

## Loans and annuities

A loan repaid by equal regular payments is **amortised** 分期偿还: each payment covers the interest due and repays part of the loan.

**Worked example.** A family borrows ¥500 000 to buy a flat at 4.8 % a year, compounded monthly, repaid monthly over 20 years.

- TVM: $N = 240$, $I\% = 4.8$, $PV = 500\,000$, $FV = 0$, $P/Y = C/Y = 12$. Solve for $PMT \approx -3244.79$: a payment of ¥3244.79 a month.
- Total paid: $240 \times 3244.79 \approx$ ¥778 749, so the interest is about ¥278 749.
- The first month's interest is $500\,000 \times 0.004 =$ ¥2000, so only ¥1244.79 of the first payment repays the loan.




*Early payments are mostly interest*

An **annuity** 年金 is a series of equal regular payments. Saving ¥2000 a month at 3 % a year, compounded monthly, for 10 years gives (TVM with $PV = 0$, $PMT = -2000$, $N = 120$) a future value of about ¥279 483, of which ¥240 000 was paid in.

## Approximation and error

### Standard form and rounding

A number in **standard form** 科学记数法 is written $a \times 10^k$, where $1 \le a < 10$ and $k$ is an integer: $3\,600\,000 = 3.6 \times 10^6$ and $0.000\,307 = 3.07 \times 10^{-4}$. Your GDC may show 3.07E−4, but you must write $3.07 \times 10^{-4}$.



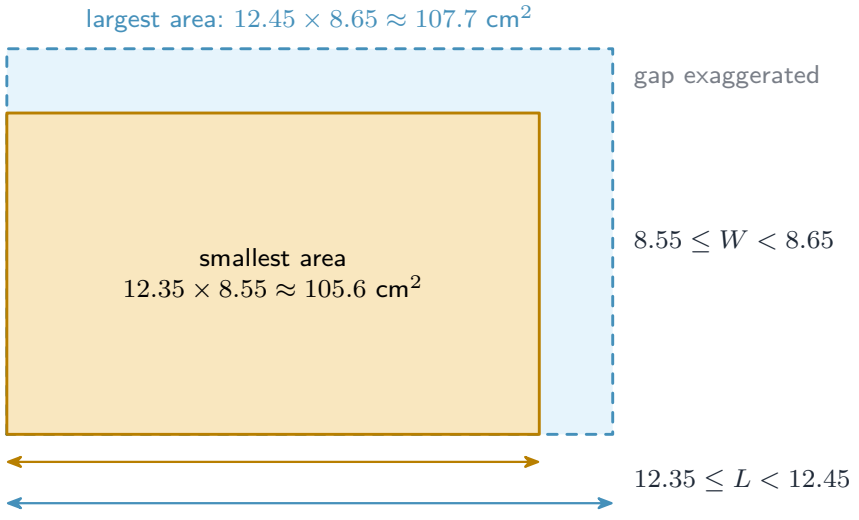

*Vernier calipers read to 0.02 mm: every measurement carries an upper and lower bound*


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Round to **decimal places** (digits after the point) or to **significant figures** 有效数字 (digits from the first non-zero digit). In IB exams, give final answers exactly or to 3 significant figures unless the question says otherwise. Keep full calculator values in the working: rounding too early is a common way to lose accuracy marks.

## Bounds

A measurement rounded to a given accuracy lies between a **lower bound** 下界 and an **upper bound** 上界. A length of 12.4 cm, to 1 decimal place, lies in $12.35 \leq L < 12.45$.




*Lower bounds give the smallest area, upper bounds the largest.*

**Worked example.** A rectangle is 12.4 cm by 8.6 cm, both to 1 decimal place. The smallest possible area is $12.35 \times 8.55 \approx 105.6$ cm$^2$, and the largest is $12.45 \times 8.65 \approx 107.7$ cm$^2$.

For a difference or a quotient, think about which bounds make it largest: the largest value of $a - b$ uses the upper bound of $a$ and the lower bound of $b$.

## Percentage error

When an approximate value $v_A$ is used for an exact value $v_E$,

$$\varepsilon = \left| \frac{v_A - v_E}{v_E} \right| \times 100\%$$

**Worked example.** Using $\frac{22}{7}$ for $\pi$ gives $\varepsilon = \left| \dfrac{3.142857 - 3.141593}{3.141593} \right| \times 100\% \approx 0.0402\%$.

Divide by the **exact** value, not the approximate one.


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# Exponents and logarithms

## ■ Laws of exponents

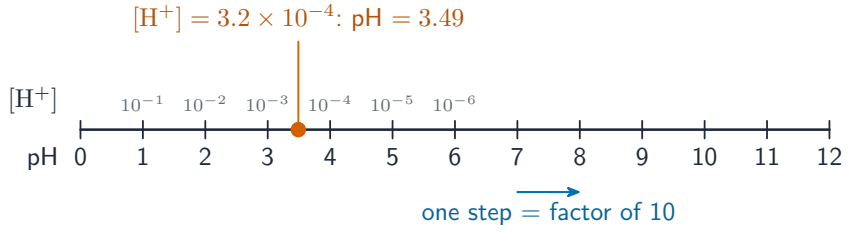
For integer exponents:

$$a^m \times a^n = a^{m+n}, \qquad \frac{a^m}{a^n} = a^{m-n}, \qquad (a^m)^n = a^{mn}, \qquad a^0 = 1, \qquad a^{-n} = \frac{1}{a^n}$$

For example, $\dfrac{(3x^2y)^3}{9x^4y} = \dfrac{27x^6y^3}{9x^4y} = 3x^2y^2$.

## ■ Logarithms

A **logarithm** 对数 answers the question "what power?":




*A logarithmic scale: pH*

$$a^x = b \iff x = \log_a b$$

Your GDC has two bases: $\log x$ means $\log_{10} x$, and the **natural logarithm** 自然对数 $\ln x$ means $\log_e x$, where $e \approx 2.718$. So $\log 1000 = 3$ and $\ln e^2 = 2$.

> **Worked example.** Solve $10^x = 250$: $x = \log 250 \approx 2.40$. Solve $e^{0.2t} = 5$: $0.2t = \ln 5$, so $t = 5 \ln 5 \approx 8.05$.

Logarithmic scales compress huge ranges of numbers. The acidity of a solution is $\text{pH} = -\log[\text{H}^+]$: when $[\text{H}^+] = 3.2 \times 10^{-4}$, the pH is $-\log(3.2 \times 10^{-4}) \approx 3.49$. A change of 1 in pH means a factor of 10 in $[\text{H}^+]$.

# Equations with technology

## ■ Systems of linear equations

The GDC's equation solver solves systems of up to three linear equations in three unknowns. Write the equations in the standard form $ax + by + cz = d$ first, and write down the system you entered.

| order | equation | check with $t=8$, $c=14$, $j=15$ |
|---|---|---|
| 2 tea, 3 coffee, 1 juice | $2t+3c+j=73$ | $16+42+15=73$ ✓ |
| 1 tea, 2 coffee, 2 juice | $t+2c+2j=66$ | $8+28+30=66$ ✓ |
| 3 tea, 1 coffee, 1 juice | $3t+c+j=53$ | $24+14+15=53$ ✓ |

*Write the system, solve, then check every equation*

**Worked example.** A café sells tea, coffee and juice. Order 1 (2 teas, 3 coffees, 1 juice) costs ¥73; order 2 (1 tea, 2 coffees, 2 juices) costs ¥66; order 3 (3 teas, 1 coffee, 1 juice) costs ¥53. With $t$, $c$ and $j$ the prices in yuan:

$$2t+3c+j=73, \qquad t+2c+2j=66, \qquad 3t+c+j=53$$

The solver gives $t=8$, $c=14$, $j=15$. Check in every equation: $16+42+15=73$, $8+28+30=66$ and $24+14+15=53$. Substituting back catches entry errors, which are the most common mistake with the solver.

## Polynomial equations

The polynomial solver finds the real (and, at HL, complex) roots of polynomial equations.

**Worked example.** $x^3-2x^2-5x+6=0$ has roots $x=-2$, 1 and 3. Check: $(x+2)(x-1)(x-3)$ expands to the original polynomial.

For other equations, such as $2^x=x+3$, graph both sides and use the intersection tool.

**Common mistakes.**

- Taking the measured value as an upper bound: 3.5 cm to the nearest 0.1 cm lies between 3.45 cm and 3.55 cm.
- Dividing by the approximate value in a percentage error. Use $\left|\dfrac{v_A-v_E}{v_E}\right|\times 100\%$.
- Writing $34\times10^3$ as standard form. The number in front must satisfy $1\le a<10$: $3.4\times10^4$.
- Using $u_n=u_1+nd$ for an arithmetic sequence. It is $u_n=u_1+(n-1)d$.
- Giving PV and FV the same sign in the finance solver. Money paid out and money received have opposite signs.
- Setting $N$ to the number of years when interest is paid monthly. $N$ counts periods, so 5 years is $N=60$.
- Rounding in the middle of a calculation. Keep full calculator values and round only the final answer.


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## Practice

1. Write 0.000 307 in standard form, and find $(2.4 \times 10^6) \times (5 \times 10^{-4})$ in standard form.
2. The side of a square is 6.2 cm, correct to 1 decimal place. Find the lower and upper bounds of its area.
3. A student measures $g$ as 9.6 m/s$^2$. The accepted value is 9.81 m/s$^2$. Find the percentage error.
4. Kai saves ¥50 in week 1, ¥55 in week 2, ¥60 in week 3, and so on. Find the amount saved in week 20 and the total after 20 weeks.
5. A town of 2400 people grows by 3 % a year. Find the population after 10 years, and the number of complete years until it first exceeds 4000.
6. Find the sum of the first 10 terms of $6 + 9 + 13.5 + \ldots$
7. ¥8000 is invested at 2.5 % a year, compounded monthly. Find its value after 6 years.
8. A machine costs ¥60 000 and depreciates by 12 % a year. Find its value after 5 years.
9. A car loan of ¥120 000 is repaid monthly over 5 years at 6 % a year, compounded monthly. Find the monthly payment and the total interest paid.
10. Wei saves ¥1500 a month for 15 years at 3.6 % a year, compounded monthly. Find the final value, and how much of it is interest.
11. Use technology to solve $2x + 3y - z = 5$, $x - y + 2z = 5$, $3x + y + z = 9$.
12. Use technology to solve $x^3 - 2x^2 - 5x + 6 = 0$.
13. Solve $10^x = 250$ and $e^{0.2t} = 5$.
14. Find the pH of a solution with $[\mathrm{H}^+] = 3.2 \times 10^{-4}$, using $\mathrm{pH} = -\log[\mathrm{H}^+]$.

**Answers.**

1. $3.07 \times 10^{-4}$; $1.2 \times 10^3$.
2. $6.15^2 = 37.8225$ cm$^2$ and $6.25^2 = 39.0625$ cm$^2$, so $37.8225 \le A < 39.0625$ cm$^2$.
3. $\varepsilon = \dfrac{|9.6 - 9.81|}{9.81} \times 100\% \approx 2.14\%$.
4. $u_{20} = 50 + 19(5) = ¥145$; $S_{20} = 10(50 + 145) = ¥1950$.
5. $2400 \times 1.03^{10} \approx 3225$; $1.03^n > \frac{5}{3}$ gives $n > 17.3$, so 18 years.
6. $r = 1.5$: $S_{10} = \dfrac{6(1.5^{10} - 1)}{0.5} \approx 680$.
7. $8000\left(1 + \frac{2.5}{1200}\right)^{72} \approx ¥9293.22$.
8. $60\,000 \times 0.88^5 \approx ¥31\ 663.92$.
9. TVM: $N = 60$, $I\% = 6$, $PV = 120\,000$, $FV = 0$, $P/Y = C/Y = 12$: $PMT \approx ¥2319.94$. Interest $= 60 \times 2319.94 - 120\,000 \approx ¥19\ 196$.
10. TVM: $N = 180$, $I\% = 3.6$, $PV = 0$, $PMT = -1500$, $P/Y = C/Y = 12$: $FV \approx ¥357\ 310$. Paid in $180 \times 1500 = ¥270\ 000$, so the interest is about ¥87 310.


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11. $x = 2$, $y = 1$, $z = 2$.
12. $x = -2$, $1$, $3$.
13. $x = \log 250 \approx 2.40$; $t = 5 \ln 5 \approx 8.05$.
14. $\text{pH} = -\log(3.2 \times 10^{-4}) \approx 3.49$.

## Exam tips

- Write down every value you enter in the TVM solver, and check the sign of each payment.
- Give answers to 3 significant figures unless told otherwise, but money to 2 decimal places.
- For "how many years" questions, give a whole number and check the values either side.
- In percentage error, divide by the exact value.
- Write $3.07 \times 10^{-4}$, never the GDC's 3.07E−4.
- Substitute solutions back into every equation of a system to catch entry errors.


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# Functions

International Baccalaureate · IB Diploma · Mathematics: Applications and Interpretation · SL

## Scope and prerequisites

This handout teaches topic 2, Functions, of Mathematics: Applications and Interpretation, the course first assessed in 2021: equations of straight lines, parallel and perpendicular lines, functions, domain, range and inverse, key features of graphs found with technology, and modelling with linear, quadratic, exponential, direct and inverse variation, cubic and sinusoidal functions, and, for HL, composite and inverse functions, transformations of graphs, models with limits such as Newton's law of cooling and logistic growth, and straightening data with logarithmic scales.

Prerequisites: substitution into formulae, solving linear equations, plotting points, and topic 1 (sequences, exponents and logarithms).

## Straight lines and linear models

### A law that held for fifty years

In 1965 the engineer Gordon Moore noticed that the number of components on a computer chip had been doubling every year, and he predicted that it would carry on. In 1975 he revised the rate to doubling about every two years. "Moore's law" is not a law of nature, only a **model** 模型: a function chosen to describe data and make predictions. It held remarkably well for decades and guided the planning of the whole chip industry, until the components became so small that physical limits began to slow it. This topic is about choosing such functions, using them, and judging how far to trust them.

### Gradient and the equation of a line

The **gradient** 斜率 of the line through $(x_1, y_1)$ and $(x_2, y_2)$ is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{rise}}{\text{run}}$$

A line can be written in three forms:

| Form | Equation | Use it when |
|---|---|---|
| Gradient-intercept | $y = mx + c$ | you know $m$ and the $y$-intercept $c$ |
| Point-gradient | $y - y_1 = m(x - x_1)$ | you know $m$ and one point |
| General | $ax + by + d = 0$ | the question asks for integer coefficients |

**Parallel** 平行 lines have equal gradients. **Perpendicular** 垂直 lines have gradients

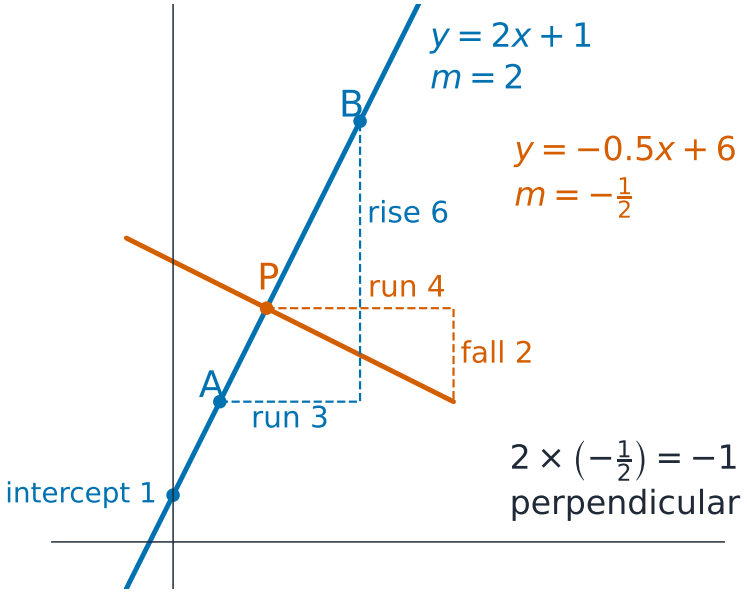


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that multiply to $-1$: if one gradient is $m$, the other is $-\frac{1}{m}$.

> **Worked example.** Find the line through $A(1,3)$ and $B(4,9)$, and the line through $P(2,5)$ perpendicular to it.

- $m = \frac{9-3}{4-1} = 2$, so $y - 3 = 2(x-1)$, that is $y = 2x + 1$, or $2x - y + 1 = 0$.
- The perpendicular gradient is $-\frac{1}{2}$, so $y - 5 = -\frac{1}{2}(x-2)$, that is $y = -0.5x + 6$.

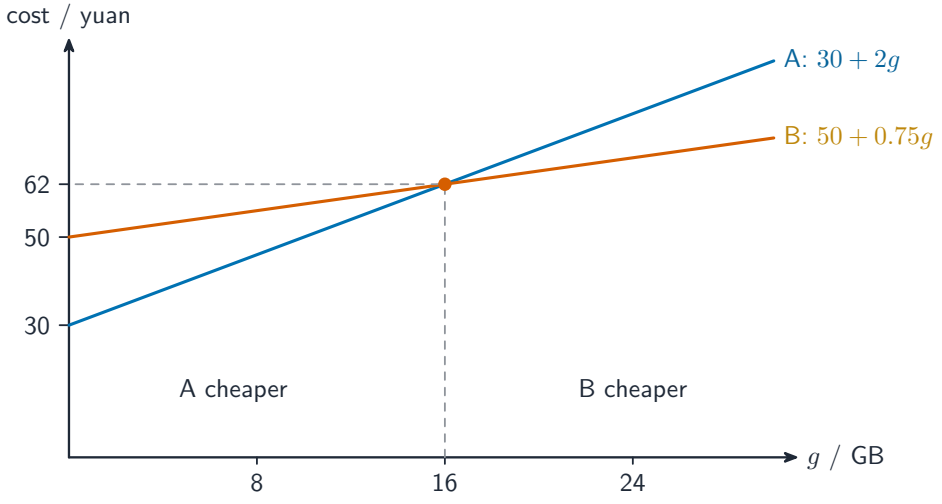



*Perpendicular gradients multiply to $-1$*

## Linear models

A **linear model** 线性模型 $y = mx + c$ describes a quantity that changes at a constant rate. The gradient is the rate of change, and the intercept is the value when $x = 0$.


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*Two linear models cross at 16 GB*

**Worked example.** A plumber charges $C = 120 + 85h$ yuan for a job lasting $h$ hours. The gradient 85 is the charge per hour, and the intercept 120 is a fixed call-out charge. A 3.5-hour job costs $120 + 85(3.5) = ¥417.50$.

**Worked example.** Phone plan A costs $30 + 2g$ yuan a month for $g$ GB of data, and plan B costs $50 + 0.75g$. They cost the same when $30 + 2g = 50 + 0.75g$, so $g = 16$ GB. Plan A is cheaper below 16 GB and plan B above it. This is the same problem as finding where two lines intersect, which a GDC does directly.

## Functions and their graphs

### ■ Domain, range and notation

A **function** 函数 gives exactly one output for each input. In $f(x) = 2x - 5$, $x$ is the input and $f(x)$ the output.

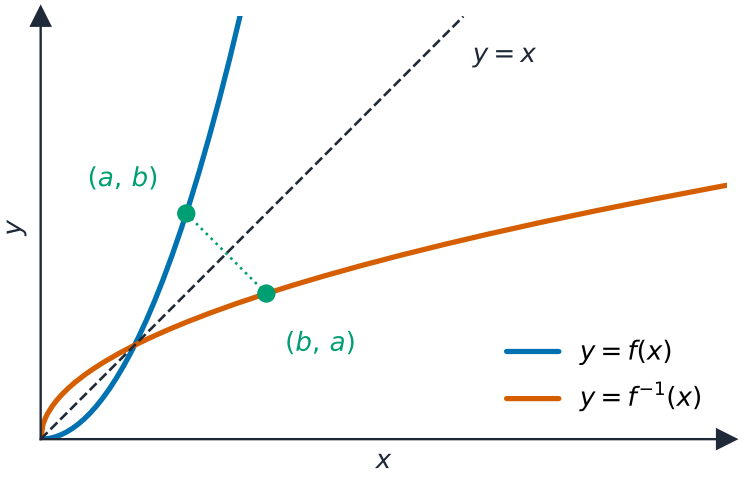
- The **domain** 定义域 is the set of allowed inputs. In a model it is usually limited by the context: a time cannot be negative.
- The **range** 值域 is the set of outputs that the function actually produces.

**Worked example.** $f(x) = 2x - 5$ with domain $0 \le x \le 6$ has range $-5 \le f(x) \le 7$.

### ■ Inverse functions

An **inverse function** 反函数 $f^{-1}$ undoes $f$: if $f(a) = b$, then $f^{-1}(b) = a$. The graph of $y = f^{-1}(x)$ is the reflection of $y = f(x)$ in the line $y = x$, and the domain and range swap.


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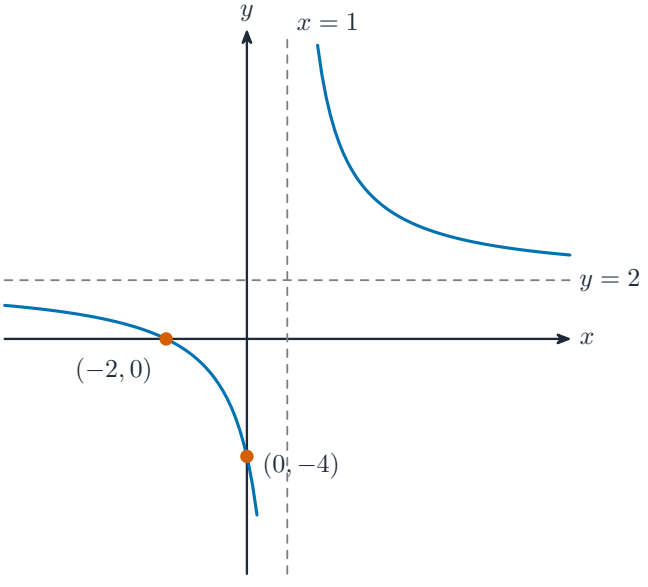

inverse = reflection in $y = x$ · domain of $f^{-1}$ = range of $f$

*The inverse is the reflection in $y = x$: $(a, b)$ becomes $(b, a)$*

For $f(x) = 2x - 5$: $f(4) = 3$, so $f^{-1}(3) = 4$. To undo "multiply by 2, then subtract 5", add 5 and then divide by 2: $f^{-1}(x) = \frac{x+5}{2}$.

## ■ Key features with technology

Use the GDC to sketch a graph and find its key features. Copy the window you used, and label the features on your sketch.




*Key features of f(x) = 6/(x − 1) + 2*



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- **Intercepts:** where the graph meets the axes. The $x$-intercepts are the **zeros** 零点 of the function.
- **Maximum and minimum points**, also called turning points.
- **Asymptotes** 渐近线: lines that the graph approaches as an input tends to a finite value or infinity. A graph may cross a horizontal asymptote.
- **Intersections** of two graphs, which solve equations such as $f(x) = g(x)$.

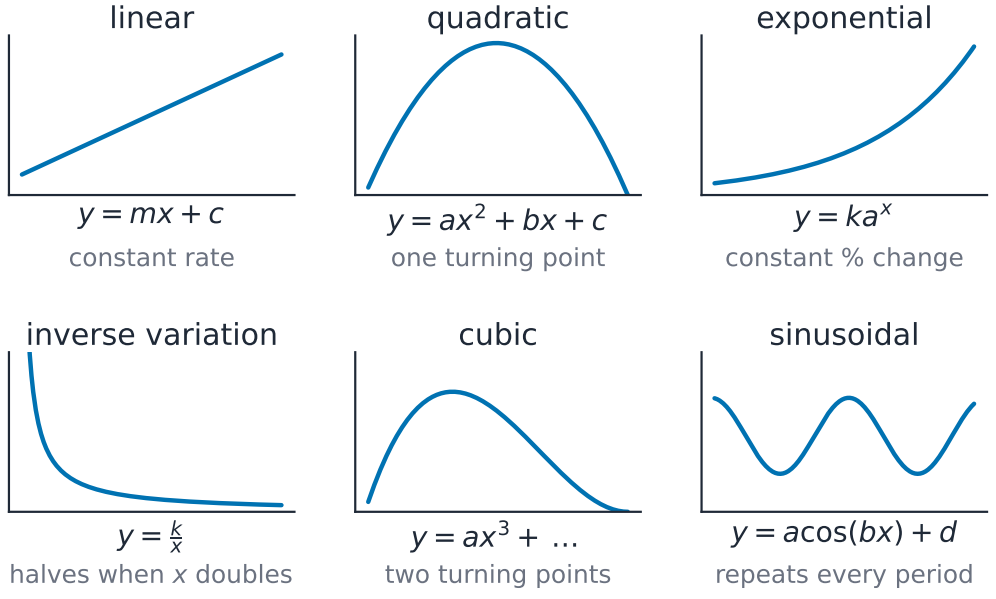
**Worked example.** $f(x) = \dfrac{6}{x-1} + 2$ has a vertical asymptote $x = 1$ (where the denominator is 0) and a horizontal asymptote $y = 2$. Its $y$-intercept is $f(0) = -4$, and its $x$-intercept is where $\frac{6}{x-1} = -2$, that is $x = -2$.

**Worked example.** $y = x^2 - 4$ and $y = 2x$ intersect where $x^2 - 2x - 4 = 0$. The GDC gives $(3.24, 6.47)$ and $(-1.24, -2.47)$.

## A family of models

### Choosing the shape

Each type of function has a typical shape and a typical story. Choose a model by thinking about how the quantity changes, not only by how the points look.




*Match the shape to how the quantity changes*

| Model | Equation | Typical context |
|---|---|---|
| Linear | $f(x) = mx + c$ | fixed charge plus a rate |
| Quadratic | $f(x) = ax^2 + bx + c$ | thrown objects, area, profit |
| Exponential | $f(x) = ka^x + c$ or $ke^{rx} + c$ | growth or decay by a fixed percentage |
| Direct or inverse variation | $f(x) = ax^n$, $n$ positive or negative | $y \propto x^2$, $y \propto \frac{1}{x}$ |
| Cubic | $f(x) = ax^3 + bx^2 + cx + d$ | volumes |
| Sinusoidal | $f(x) = a\sin(bx) + d$ or $a\cos(bx) + d$ | tides, daylight, rotation |

## Quadratic models

A **quadratic model** 二次模型 has one turning point, on its **axis of symmetry** 对称轴 $x = -\frac{b}{2a}$.

> **Worked example.** A ball is thrown so that its height is $h = -4.9t^2 + 14t + 1.5$ metres after $t$ seconds.

- Maximum at $t = -\frac{14}{2(-4.9)} \approx 1.43$ s, height $h \approx 11.5$ m.
- It lands when $h = 0$: the GDC gives $t \approx 2.96$ s. The other root is negative, outside the domain.

## Exponential models

A quantity that changes by a fixed **percentage** each step follows an **exponential model** 指数模型. With $a > 1$ it grows; with $0 < a < 1$ it decays.

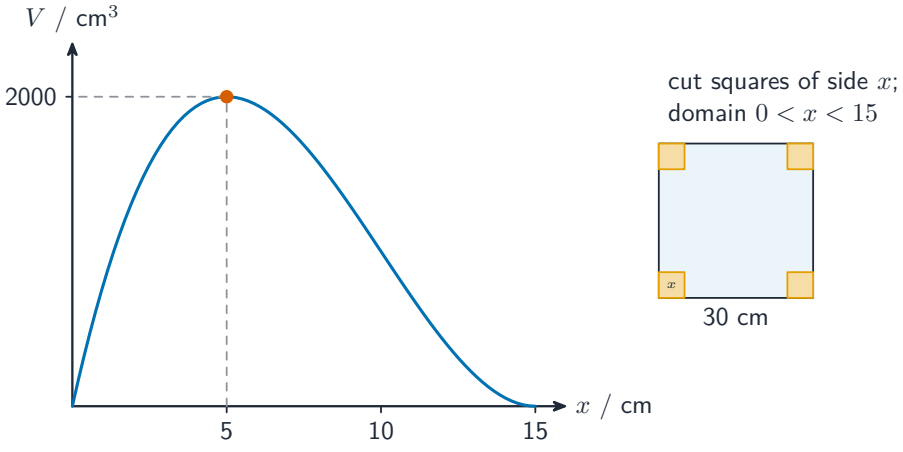
> **Worked example.** 200 mg of a medicine is in the blood, and 15 % is removed each hour, so $A = 200 \times 0.85^t$. After 3 hours $A \approx 122.8$ mg. It falls below 50 mg when $0.85^t < 0.25$: the GDC gives $t > 8.53$ hours.

## Variation and cubic models

If $y$ is **directly proportional** 正比例 to $x^n$, then $y = kx^n$; if $y$ is **inversely proportional** 反比例 to $x^n$, then $y = \dfrac{k}{x^n}$. Find $k$ from one pair of values.



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*The open-box model and its maximum*

**Worked example.** The intensity of light $I$ is inversely proportional to the square of the distance $d$. If $I = 12$ when $d = 2$, then $12 = \frac{k}{4}$, so $k = 48$ and $I = \frac{48}{d^2}$. At $d = 4$, $I = 3$: doubling the distance divides the intensity by 4.

**Worked example.** Squares of side $x$ cm are cut from the corners of a 30 cm square sheet, which is folded into an open box. $V = x(30 - 2x)^2$ with domain $0 < x < 15$. The GDC's maximum is at $x = 5$, giving $V = 2000$ cm$^3$.

### ■ Sinusoidal models

A **sinusoidal model** 正弦模型 $f(x) = a\sin(bx) + d$ or $a\cos(bx) + d$ repeats with **period** 周期 $\dfrac{360°}{b}$ (or $\frac{2\pi}{b}$ in radians). Its **amplitude** 振幅 is $|a|$, and its **principal axis** 主轴 is $y = d$. The maximum is $d + |a|$ and the minimum is $d - |a|$.

**Worked example.** The depth of water in a harbour is $h = 1.8\cos(30t°) + 4.2$ metres, $t$ hours after midnight. The maximum depth is 6 m and the minimum 2.4 m; the period is $\frac{360}{30} = 12$ hours, so high tide comes at midnight and noon.

## Testing and improving models

### ■ Developing a model

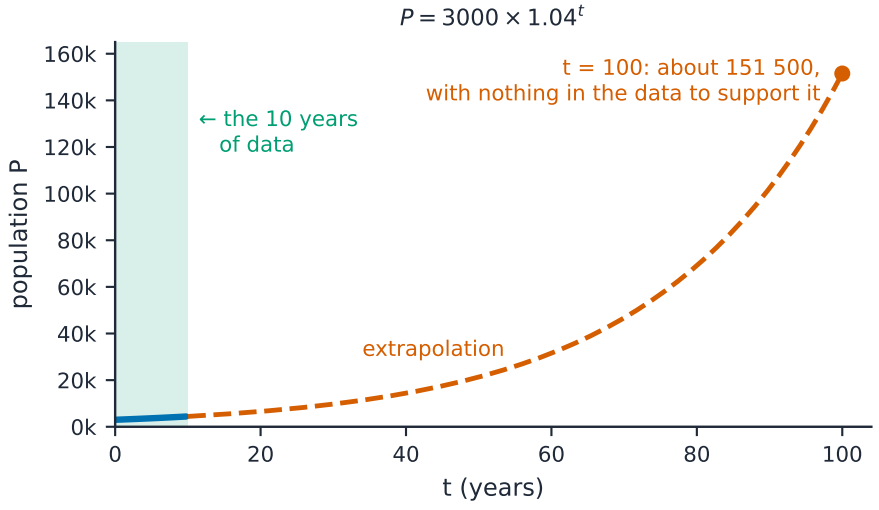
Modelling is a cycle:

1. Identify the variables and make **assumptions** 假设.
2. Choose a function and find its parameters, from given values or by regression with technology.
3. Test it: compare its predictions with data, and check that its values make sense.
4. Reflect: change the assumptions or the function, and test again.

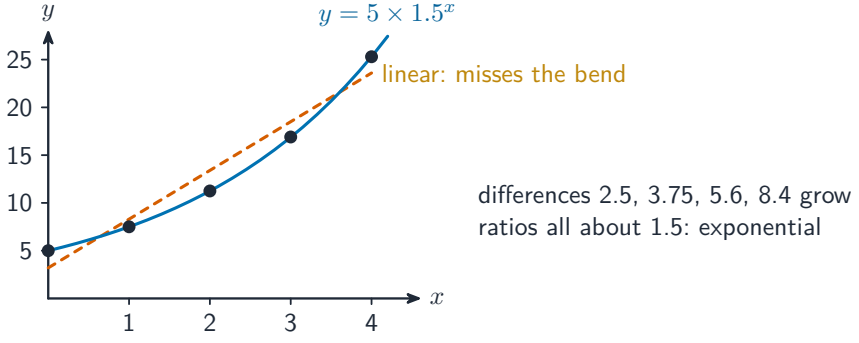

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### Is the model reasonable?

- **Domain.** A population model $P = 3000 \times 1.04^t$ fitted to 10 years of data predicts more than 150 000 people after 100 years. Nothing in the data supports that: real populations are limited by space and resources.
- **Values.** A quadratic model for the height of a ball gives negative heights after it lands; restrict the domain.
- **Fit.** A good model's predictions are close to the data, with no pattern in the errors. Topic 4 measures this with residuals and $R^2$.




*Fitted on 10 years, trusted for 100: the extrapolation runs away*




*Growing differences, constant ratios: exponential*

**Worked example.** Data: $x = 0, 1, 2, 3, 4$ and $y = 5, 7.5, 11.25, 16.9, 25.3$. The differences grow (2.5, 3.75, 5.6, 8.4), so a linear model is wrong. The ratios are all about 1.5, so try $y = 5 \times 1.5^x$: it gives 5, 7.5, 11.25, 16.875 and 25.3, a very close fit.

Predicting within the data (**interpolation** 内插) is safer than predicting beyond it (**extrapolation** 外推).

**Common mistakes.**

- Reading $f^{-1}(x)$ as $\dfrac{1}{f(x)}$. It is the inverse function.
- Mixing up domain (the inputs) and range (the outputs).
- Using a model far outside the data it was built from, such as a linear growth model for 50 years.
- Reading the base of $y = k\,a^x$ as the rate: $a = 1.05$ means 5% growth per unit, and $a = 0.92$ means an 8% fall.
- Taking $b$ in $y = a\sin(bx) + d$ as the period. The period is $\dfrac{360^\circ}{b}$, the amplitude is $|a|$ and the principal axis is $y = d$.
- Reporting a model without checking it against the data and the context.

## Practice

1. Find the equation of the line through $(-2, 7)$ and $(4, -2)$ in the forms $y = mx + c$ and $ax + by + d = 0$.
2. Find the line through $(8, 3)$ perpendicular to $y = 4x - 1$.
3. A plumber charges $C = 120 + 85h$ yuan. Interpret the 120 and the 85, and find the cost of a 3.5-hour job.
4. $f(x) = 2x - 5$ for $0 \le x \le 6$. State the range, find $f^{-1}(x)$ and $f^{-1}(3)$.
5. For $f(x) = \dfrac{6}{x - 1} + 2$, state the asymptotes and find the intercepts.
6. Use technology to find where $y = x^2 - 4$ meets $y = 2x$.
7. A ball's height is $h = -4.9t^2 + 14t + 1.5$ metres. Find the maximum height and when the ball lands.
8. A medicine follows $A = 200 \times 0.85^t$ mg. Find $A$ after 3 hours, and when it first falls below 50 mg.
9. $y$ is inversely proportional to $x^2$, and $y = 12$ when $x = 2$. Find $y$ when $x = 4$.
10. Squares of side $x$ are cut from a 30 cm square sheet to make an open box. Write the volume, state its domain and find the largest volume.
11. A tide follows $h = 1.8\cos(30t^\circ) + 4.2$. Find the maximum and minimum depths and the period.
12. Which model fits $x = 0, 1, 2, 3, 4$ and $y = 5, 7.5, 11.25, 16.9, 25.3$? Give its equation.
13. A town's population is modelled by $P = 3000 \times 1.04^t$, fitted to data for $0 \le t \le 10$. Explain why the model should not be used for $t = 100$.
14. Phone plan A costs $30 + 2g$ and plan B costs $50 + 0.75g$ yuan for $g$ GB. When is plan B cheaper?

**Answers.**



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1. $m = \frac{-2-7}{4-(-2)} = -1.5$, so $y = -1.5x + 4$, or $3x + 2y - 8 = 0$.
2. Gradient $-\frac{1}{4}$: $y - 3 = -\frac{1}{4}(x - 8)$, so $y = -0.25x + 5$.
3. ¥120 is a fixed call-out charge and ¥85 is the charge per hour. Cost ¥417.50.
4. Range $-5 \leq f(x) \leq 7$; $f^{-1}(x) = \frac{x+5}{2}$; $f^{-1}(3) = 4$.
5. Asymptotes $x = 1$ and $y = 2$; $y$-intercept $(0, -4)$; $x$-intercept $(-2, 0)$.
6. $(3.24, 6.47)$ and $(-1.24, -2.47)$.
7. Maximum 11.5 m at $t \approx 1.43$ s; it lands at $t \approx 2.96$ s.
8. $A \approx 122.8$ mg; it falls below 50 mg after about 8.53 hours.
9. $k = 48$, so $y = \frac{48}{16} = 3$.
10. $V = x(30 - 2x)^2$, $0 < x < 15$; maximum $V = 2000$ cm$^3$ at $x = 5$.
11. Maximum 6 m, minimum 2.4 m, period 12 hours.
12. Ratios are about 1.5, so exponential: $y = 5 \times 1.5^x$.
13. It is fitted only to $0 \leq t \leq 10$; at $t = 100$ it predicts over 150 000 people, an extrapolation that ignores limits on space and resources.
14. $30 + 2g > 50 + 0.75g$ when $g > 16$: plan B is cheaper for more than 16 GB.

## Exam tips

- Give a line in the form the question asks for; general form needs integer coefficients.
- For perpendicular lines, use the negative reciprocal of the gradient.
- Sketch GDC graphs with labelled axes, key points and asymptotes, and state the window if it matters.
- State the domain of a model from its context.
- Choose a model from how the quantity changes, and test it against the data.
- Interpret parameters in context, with units: “85 is the charge in yuan per hour”.


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# Geometry and trigonometry

International Baccalaureate · IB Diploma · Mathematics: Applications and Interpretation · SL

## Scope and prerequisites

This handout teaches topic 3, Geometry and trigonometry, of Mathematics: Applications and Interpretation, the course first assessed in 2021: distance, volume, surface area and angles in three dimensions, right-angled and non-right-angled triangles, bearings, arcs and sectors, perpendicular bisectors and Voronoi diagrams, and, for HL, radians and the unit circle, matrix transformations, vectors and motion, and graph theory with its algorithms.

Prerequisites: Pythagoras' theorem, the gradient and equation of a line (topic 2), and, for HL, matrices (topic 1).

## Geometry in three dimensions

### A map that stopped an epidemic

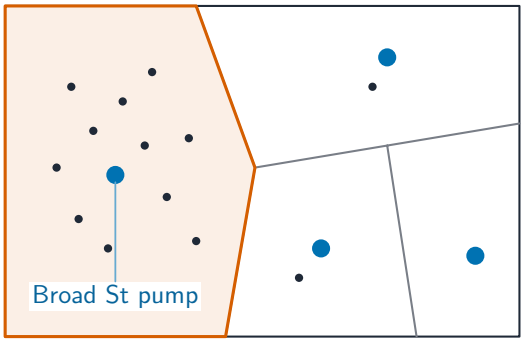
In August 1854 cholera killed more than 500 people in ten days in Soho, London. Most doctors believed the disease spread through bad air. The doctor John Snow marked each death on a street map and noticed that they clustered around one public water pump, on Broad Street. In a later version of his map he drew a line round the area that was closer, by walking distance, to the Broad Street pump than to any other pump, and most of the deaths lay inside it. The pump handle was removed, and the outbreak, already slowing, ended. Snow's line is an early example of a Voronoi diagram, which divides a map into the regions nearest to each of a set of points. This topic is about measuring space: distances, angles, areas and regions.



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*A version of John Snow's 1854 map: each dot is a death and each cross a pump; the deaths cluster around the Broad Street pump*




each cell: points nearest one pump; deaths (dots) cluster in one cell (schematic, not Snow's data)

*Nearest-pump regions (schematic)*

## ■ Distance and midpoint

In three dimensions a point has coordinates $(x, y, z)$. For points $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}, \qquad M = \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right)$$

**Worked example.** For $P(1, 2, 3)$ and $Q(4, 6, 15)$, the differences are 3, 4 and 12, so $PQ = \sqrt{9 + 16 + 144} = 13$. The **midpoint** 中点 is $(2.5, 4, 9)$.



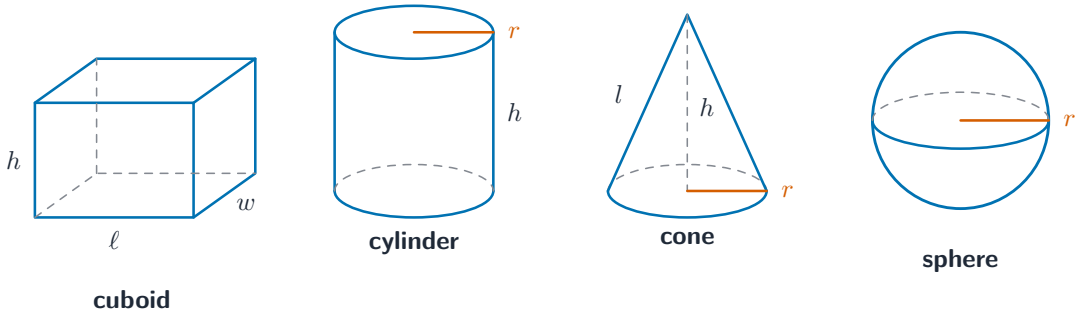
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## Volume and surface area

The formula booklet gives these. Here $r$ is a radius, $h$ a perpendicular height and $l$ the **slant height** 斜高 of a cone.



*The pyramids of Giza: a square-based pyramid has volume one third of base area times height*




*Common solids and the dimensions ($r$, $h$, $\ell$, slant $l$) used in their volume and surface-area formulas.*

| Solid | Volume | Surface area |
|---|---|---|
| Prism, cylinder | area of cross-section × length; cylinder $\pi r^2 h$ | cylinder curved surface $2\pi rh$ |
| **Pyramid** 棱锥 | $\frac{1}{3}\times$ base area $\times h$ | add the areas of the faces |
| **Cone** 圆锥 | $\frac{1}{3}\pi r^2 h$ | curved surface $\pi rl$ |
| **Sphere** 球 | $\frac{4}{3}\pi r^3$ | $4\pi r^2$ |

A **hemisphere** 半球 is half a sphere: volume $\frac{2}{3}\pi r^3$, curved surface $2\pi r^2$.

> **Worked example.** A pyramid has a square base of side 6 cm and a height of 4 cm.

- Volume: $\frac{1}{3}(36)(4) = 48$ cm$^3$.
- Each triangular face has height $\sqrt{4^2 + 3^2} = 5$ cm, from the right-angled triangle between the apex, the centre of the base and the middle of a side.
- Surface area: $36 + 4 \times \frac{1}{2}(6)(5) = 96$ cm$^2$.

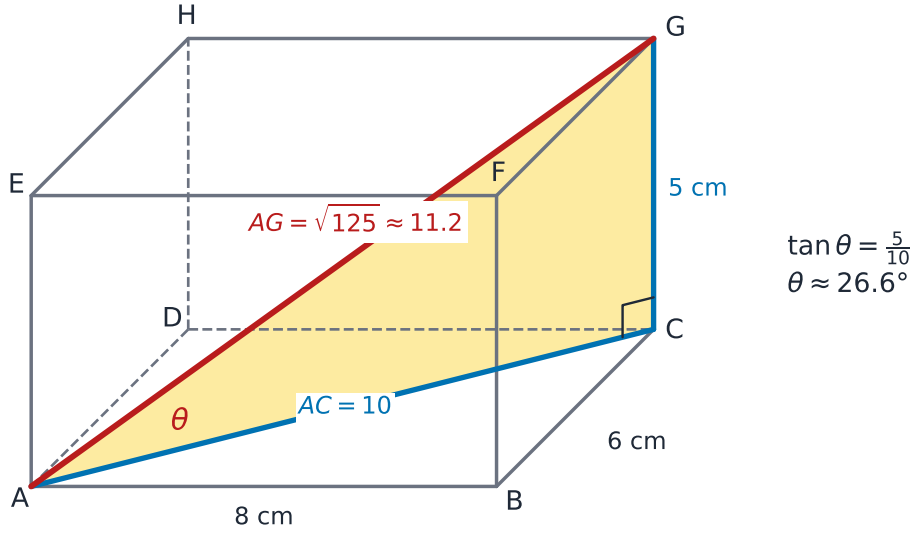
> **Worked example.** A grain silo is a cylinder of radius 3 m and height 10 m with a hemisphere on top. Its volume is $\pi(9)(10) + \frac{2}{3}\pi(27) = 108\pi \approx 339$ m$^3$. The curved surfaces that need painting have area $2\pi(3)(10) + 2\pi(9) = 78\pi \approx 245$ m$^2$. Split a **compound solid** into simple solids, and add only the faces that are on the outside.

## The angle between a line and a plane

To find the angle between a line and a plane, drop a perpendicular from a point on the line to the plane. The angle is between the line and its "shadow" in the plane, and it sits in a right-angled triangle.

> **Worked example.** A cuboid is 8 cm by 6 cm by 5 cm. Find the angle between the **space diagonal** 体对角线 $AG$ and the base.

- Base diagonal: $AC = \sqrt{8^2 + 6^2} = 10$ cm.
- $CG = 5$ cm is vertical, so triangle $ACG$ has a right angle at $C$.
- $\tan\theta = \frac{5}{10}$, so $\theta \approx 26.6°$, and $AG = \sqrt{125} \approx 11.2$ cm.

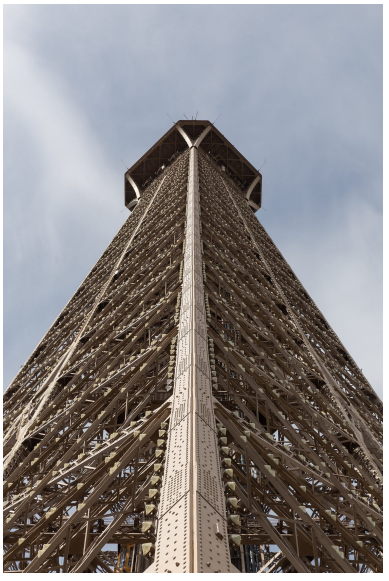



*Find the right-angled triangle that contains the angle*

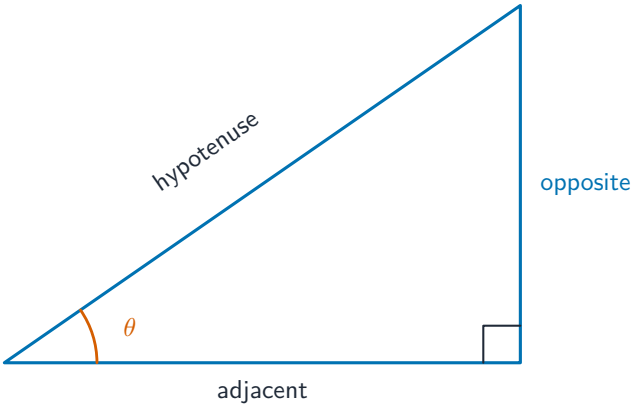
# Triangle trigonometry

## ■ Right-angled triangles

Label the sides from the angle $\theta$ you are using: the **hypotenuse** 斜边 is opposite the right angle, the opposite side faces $\theta$, and the adjacent side is next to it.



*Looking up at a tower: an angle of elevation and a measured distance give its height*




*Name the sides from the angle $\theta$: the opposite is across from it, the adjacent next to it, and the hypotenuse opposite the right angle (SOH-CAH-TOA).*

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}, \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}, \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

- An **angle of elevation** 仰角 is measured up from the horizontal; an **angle of depression** 俯角 is measured down from the horizontal.

- A **bearing** 方位角 is an angle measured clockwise from north, written with three figures: 050°, 160°, 305°.

> **Worked example.** From a point 40 m from the foot of a tower, the angle of elevation of the top is 32°. The height is $40\tan 32° \approx 25.0$ m.

> **Worked example.** From the top of a cliff 60 m high, the angle of depression of a boat is 18°. The angle of elevation of the cliff top from the boat is also 18° (alternate angles), so the boat is $\frac{60}{\tan 18°} \approx 185$ m from the foot of the cliff.

## ■ The sine rule, the cosine rule and area

For any triangle with sides $a$, $b$, $c$ opposite angles $A$, $B$, $C$:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}, \qquad c^2 = a^2 + b^2 - 2ab\cos C, \qquad \text{Area} = \frac{1}{2}ab\sin C$$

- Use the **sine rule** 正弦定理 when you know a side and the angle opposite it.
- Use the **cosine rule** 余弦定理 when you know two sides and the angle between them, or all three sides. To find an angle, rearrange: $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$.

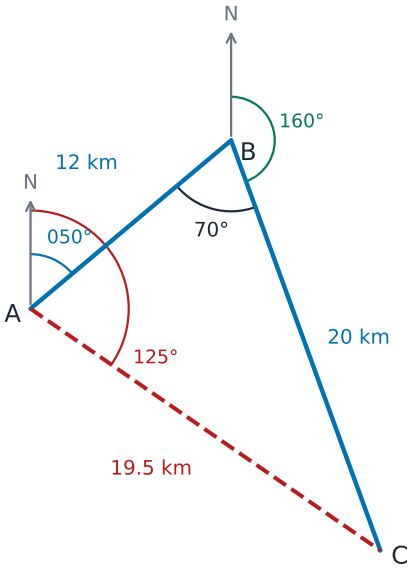
> **Worked example.** In triangle $ABC$, $A = 48°$, $B = 75°$ and $a = 12$ cm. Then $b = \dfrac{12\sin 75°}{\sin 48°} \approx 15.6$ cm, $C = 57°$, and the area is $\frac{1}{2}(12)(15.6)\sin 57° \approx 78.5$ cm$^2$.

> **Worked example.** A triangle has sides 7, 8 and 10 m. The largest angle is opposite the longest side: $\cos C = \dfrac{49 + 64 - 100}{2(7)(8)} = \dfrac{13}{112}$, so $C \approx 83.3°$.

> **Worked example (bearings).** A ship sails 12 km from $A$ on a bearing of 050° to $B$, then 20 km on a bearing of 160° to $C$. Find the distance and bearing of $C$ from $A$, and the area of triangle $ABC$.

- At $B$, the bearing back to $A$ is $050° + 180° = 230°$, so angle $ABC = 230° - 160° = 70°$.
- Cosine rule: $AC^2 = 12^2 + 20^2 - 2(12)(20)\cos 70° \approx 379.8$, so $AC \approx 19.5$ km.
- Angle at $A$: $\cos A = \dfrac{12^2 + 19.49^2 - 20^2}{2(12)(19.49)} \approx 0.265$, so $A \approx 74.6°$.
- Bearing of $C$ from $A$: $050° + 74.6° \approx 125°$.
- Area: $\frac{1}{2}(12)(20)\sin 70° \approx 113$ km$^2$.


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*Draw a north line at every point where a bearing is measured*

At SL you will not meet the ambiguous case of the sine rule: when you find an angle with it, the question will make clear whether the angle is acute or obtuse.

## Arcs, sectors and perpendicular bisectors

### Arc length and sector area in degrees

An **arc** 弧 is part of a circle, and a **sector** 扇形 is the region between an arc and two radii. A sector with angle $\theta°$ is the fraction $\frac{\theta}{360}$ of the whole circle, so

$$\text{arc length } l = \frac{\theta}{360} \times 2\pi r, \qquad \text{sector area } A = \frac{\theta}{360} \times \pi r^2$$

> **Worked example.** A pizza of radius 15 cm is cut into slices with angle 40°. Each slice has arc $\frac{40}{360} \times 30\pi \approx 10.5$ cm, area $\frac{40}{360} \times 225\pi \approx 78.5$ cm$^2$, and perimeter $15 + 15 + 10.5 = 40.5$ cm.

### The perpendicular bisector

The **perpendicular bisector** 垂直平分线 of $AB$ is the line through the midpoint of $AB$ at right angles to it. Every point on it is the same distance from $A$ as from $B$.

> **Worked example.** For $P(2, 1)$ and $Q(6, 5)$: the midpoint is $(4, 3)$, the gradient of $PQ$ is $\frac{5-1}{6-2} = 1$, so the bisector has gradient $-1$ and equation $y - 3 = -(x - 4)$, that is $y = -x + 7$.



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# Voronoi diagrams

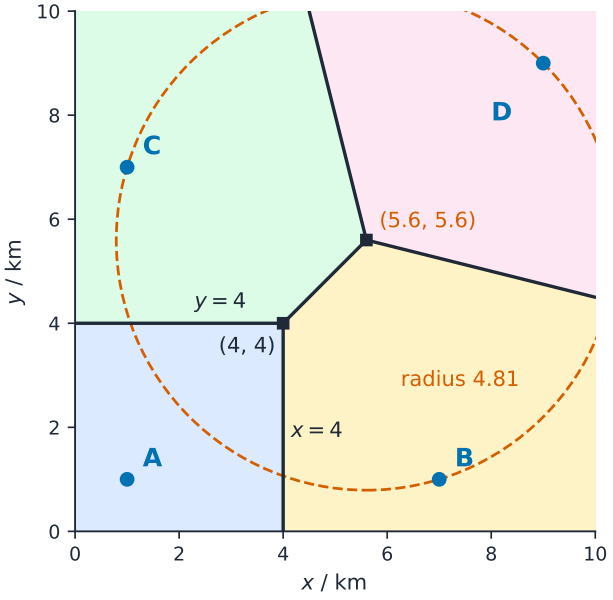
## ■ Sites, cells, edges and vertices

A **Voronoi diagram** 沃罗诺伊图 divides a region into cells, one for each **site** 站点, such as a school, a weather station or a mobile phone mast. Each **cell** 单元 contains the points closer to its site than to any other.

- An **edge** 边 is part of the perpendicular bisector of two neighbouring sites.
- A **vertex** 顶点 is where three edges meet. It is the same distance from three sites.

**Worked example.** Four wells are at $A(1,1)$, $B(7,1)$, $C(1,7)$ and $D(9,9)$, in kilometres, in a farming region $0 \le x \le 10$, $0 \le y \le 10$.

- The bisector of $A$ and $B$ is $x = 4$, and the bisector of $A$ and $C$ is $y = 4$. They meet at the vertex $(4,4)$, which is $\sqrt{3^2+3^2} \approx 4.24$ km from $A$, $B$ and $C$.
- The bisector of $B$ and $C$ passes through their midpoint $(4,4)$ with gradient $1$ (because $BC$ has gradient $-1$): it is $y = x$.
- The bisector of $B$ and $D$ passes through $(8,5)$ with gradient $-\frac{1}{4}$: $y = -0.25x + 7$. It meets $y = x$ at the vertex $(5.6, 5.6)$, which is about $4.81$ km from $B$, $C$ and $D$.




*Each cell holds the points nearest its well; the largest empty circle is centred on a vertex*



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### Using a Voronoi diagram

- **Nearest neighbour interpolation** 最近邻插值: to estimate a value at a point, use the value at the site of the cell that contains it. A farm at $(3, 2)$ is in cell $A$, so its rainfall is estimated by the rainfall at well $A$.
- **Adding a site:** draw the perpendicular bisectors between the new site and its neighbours, then remove the parts of the old edges that now lie inside the new cell.
- **The largest empty circle**, sometimes called the toxic waste dump problem: to place something as far as possible from every site, check the vertices, where edges meet the boundary, and the corners of the region. In the example, the vertex $(5.6, 5.6)$ is 4.81 km from its nearest wells, more than $(4, 4)$ at 4.24 km and more than any boundary point (the best is about 4.61 km), so it is the best place for a new storage tank that must be as far from the wells as possible.

**Common mistakes.**

- Using the slant height of a cone or pyramid where the formula needs the perpendicular height.
- Leaving the calculator in the wrong angle mode.
- Forgetting the fraction of the circle: a sector of $\theta°$ has area $\dfrac{\theta}{360}\pi r^2$.
- Measuring a bearing anticlockwise or from east. Bearings are clockwise from north, with three figures.
- Drawing a Voronoi edge anywhere between two sites. Each edge lies on the perpendicular bisector of the two sites.
- Adding a site without redrawing every cell it takes area from.

## Practice

1. Find the distance between $(1, 2, 3)$ and $(4, 6, 15)$, and their midpoint.
2. A pyramid has a square base of side 6 cm and height 4 cm. Find its volume and total surface area.
3. A silo is a cylinder of radius 3 m and height 10 m with a hemisphere on top. Find its volume.
4. A cuboid measures 8 cm by 6 cm by 5 cm. Find the angle between a space diagonal and the base.
5. The angle of elevation of the top of a tower from 40 m away is 32°. Find its height.
6. From a cliff 60 m high, the angle of depression of a boat is 18°. How far is the boat from the foot of the cliff?
7. In triangle $ABC$, $A = 48°$, $B = 75°$ and $a = 12$ cm. Find $b$ and the area.
8. Find the largest angle of a triangle with sides 7, 8 and 10 m.
9. A ship sails 12 km on a bearing of 050°, then 20 km on 160°. Find its distance and bearing from the start.

10. A slice of pizza of radius 15 cm has angle 40°. Find its arc length, area and perimeter.
11. Find the perpendicular bisector of $P(2, 1)$ and $Q(6, 5)$.
12. Wells are at $A(1, 1)$, $B(7, 1)$ and $C(1, 7)$. Find the edges between them and the Voronoi vertex.
13. A fourth well is at $D(9, 9)$. Find the bisector of $B$ and $D$, and the vertex where it meets $y = x$.
14. Explain why the vertex in question 13 is a good place for a storage tank that must be as far as possible from all four wells, in the region $0 \le x, y \le 10$.

**Answers.**

1. 13; $(2.5, 4, 9)$.
2. 48 cm$^3$; 96 cm$^2$.
3. $108\pi \approx 339$ m$^3$.
4. $\tan\theta = \frac{5}{10}$, $\theta \approx 26.6°$.
5. $40\tan 32° \approx 25.0$ m.
6. $\frac{60}{\tan 18°} \approx 185$ m.
7. $b \approx 15.6$ cm; area $\approx 78.5$ cm$^2$.
8. $\cos C = \frac{13}{112}$, $C \approx 83.3°$.
9. 19.5 km on a bearing of about 125°.
10. 10.5 cm; 78.5 cm$^2$; 40.5 cm.
11. $y = -x + 7$.
12. $x = 4$ for $y \le 4$, $y = 4$ for $x \le 4$, and $y = x$ for $x \ge 4$, meeting at the vertex $(4, 4)$.
13. $y = -0.25x + 7$; vertex $(5.6, 5.6)$.
14. It is about 4.81 km from its three nearest wells, further than the other vertex (4.24 km) and any point on the boundary (at most about 4.61 km).

## Exam tips

- Draw the right-angled triangle you are using on its own, flat on the page.
- Draw a north line at every point where a bearing is measured, and give bearings with three figures.
- Check whether the question wants the curved surface or the total surface area.
- In degrees, a sector is the fraction $\frac{\theta}{360}$ of the circle.
- A Voronoi edge is part of a perpendicular bisector: find the midpoint and the negative reciprocal gradient.
- For the largest empty circle, compare the vertices with the boundary and the corners.


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# Statistics and probability

International Baccalaureate · IB Diploma · Mathematics: Applications and Interpretation · SL

## Scope and prerequisites

This handout teaches topic 4, Statistics and probability, of Mathematics: Applications and Interpretation, the course first assessed in 2021: sampling, presenting and summarising data, correlation and regression, Spearman's rank correlation, probability, discrete random variables, the binomial and normal distributions, and hypothesis testing with the chi-squared and t-tests, and, for HL, reliability and validity, non-linear regression, linear combinations of random variables, the central limit theorem, confidence intervals, the Poisson distribution, critical regions and errors, and Markov chains.

Prerequisites: fractions, decimals and percentages, substitution into formulae, and matrices (topic 1) for the HL Markov chains.

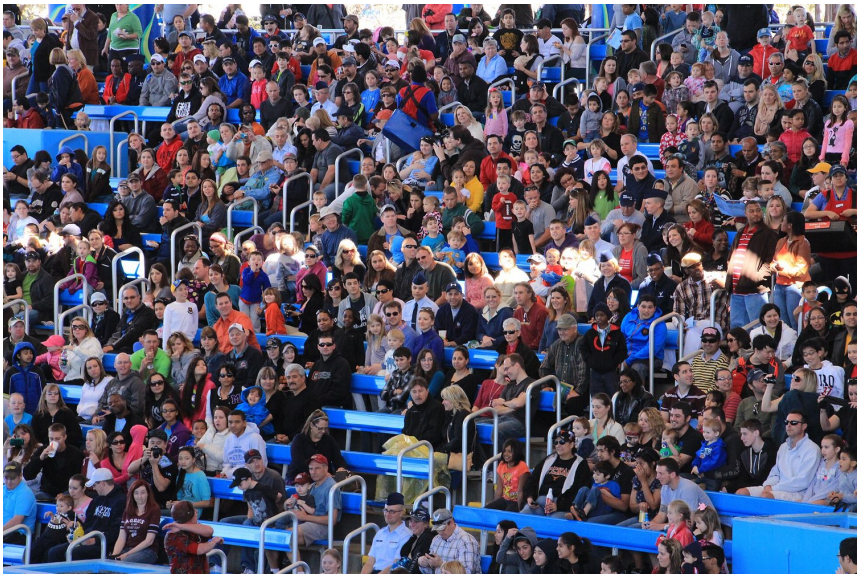
## Sampling and data

### The lady tasting tea

In the 1920s, at an agricultural research station north of London, the biologist Muriel Bristol said she could taste whether the milk or the tea had been poured into a cup first. The statistician Ronald Fisher designed a test: eight cups, four of each kind, in a random order. If she were only guessing, the chance of picking the right four is $\frac{1}{70} \approx 0.014$, so a perfect score would be strong evidence that she was not guessing. According to one of those present, she identified them all. Fisher used the story to open his book on the design of experiments in 1935. Every hypothesis test in this topic asks his question: could the result have happened by chance alone?



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*A crowd is a population: a well-chosen sample tells you about it without asking everyone*




M: milk first, T: tea first; 4 of each in random order
choosing the right 4 by guessing: $1/\binom{8}{4} = 1/70 \approx 0.014$

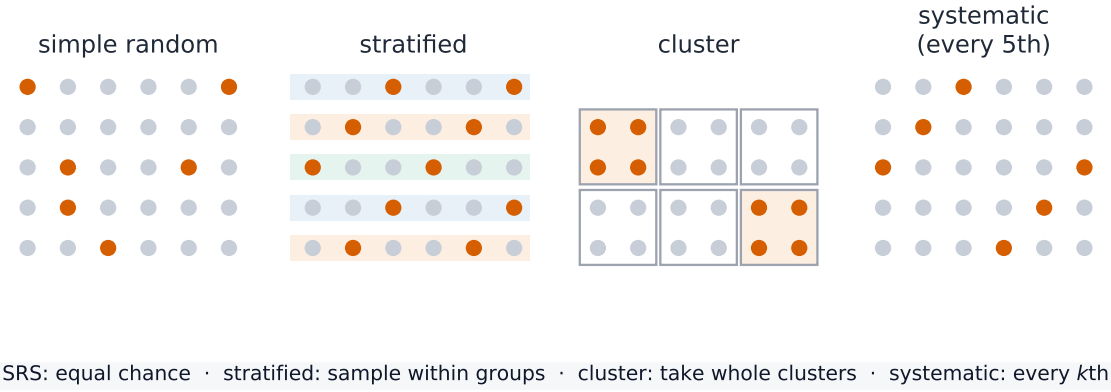
*Fisher's eight cups*

## Populations and samples

- The **population** 总体 is every member of the group being studied; a **sample** 样本 is the part actually measured.
- A sample is **biased** when some members of the population are more likely to be included, so the sample does not represent the population.
- **Discrete data** 离散数据 can take only separate values (number of siblings); **continuous data** 连续数据 can take any value in a range (height, time).

## Sampling techniques

| Technique | How | Strength | Weakness |
|---|---|---|---|
| Simple random | Every member equally likely, such as random numbers | No selection bias | Needs a full list |
| Systematic | Every $k$th member from a list | Quick, spread out | Biased if the list has a pattern |
| Stratified | Random samples from each group in proportion | Represents every group | Needs group sizes |
| Quota | Fixed numbers from each group, not random | Quick, cheap | Interviewer chooses: bias |
| Convenience | Whoever is easy to reach | Easiest | Rarely representative |




*Four random sampling designs: who gets selected, and how*

> **Worked example.** A school has 300 students in Year 10 and 200 in Year 11. A stratified sample of 50 takes $\frac{300}{500} \times 50 = 30$ from Year 10 and 20 from Year 11.

An **outlier** 异常值 is a value far from the rest: more than $1.5 \times$ IQR below $Q_1$ or above $Q_3$. Check an outlier before removing it: a recording error can be corrected, but a genuine extreme value is part of the data.

# Summarising data

## ■ Measures of centre and spread

For the data $3, 5, 7, 7, 8, 10, 12, 20$ ($n = 8$):




median $= 3$

mean $= \frac{2+3+3+5+7}{5} = 4$

mode $= 3$ (most common)

range $= 7 - 2 = 5$

*Mean, median, mode and range of a small data set*

- **mean** 平均数 $\bar{x} = \dfrac{\sum x}{n} = \dfrac{72}{8} = 9$;
- **median** 中位数 $= \dfrac{7+8}{2} = 7.5$, the middle value of the ordered data;
- **mode** 众数 $= 7$, the most frequent value;
- **quartiles** 四分位数 $Q_1 = 6$ and $Q_3 = 11$, so the **interquartile range** 四分位距 is IQR $= 5$. Since $11 + 1.5 \times 5 = 18.5$, the value 20 is an outlier;
- **standard deviation** 标准差 $\sigma \approx 4.90$, from the GDC.

The median and IQR resist outliers; the mean and standard deviation are pulled by them. If every value is multiplied by $a$ and then $b$ is added, the mean and median become $a(\cdot) + b$, while the standard deviation and IQR are multiplied by $|a|$ and not changed by $b$. For $a \geq 0$, quartiles also become $a(\cdot) + b$; for $a < 0$, their order reverses, so the new $Q_1$ is $aQ_3 + b$ and the new $Q_3$ is $aQ_1 + b$.

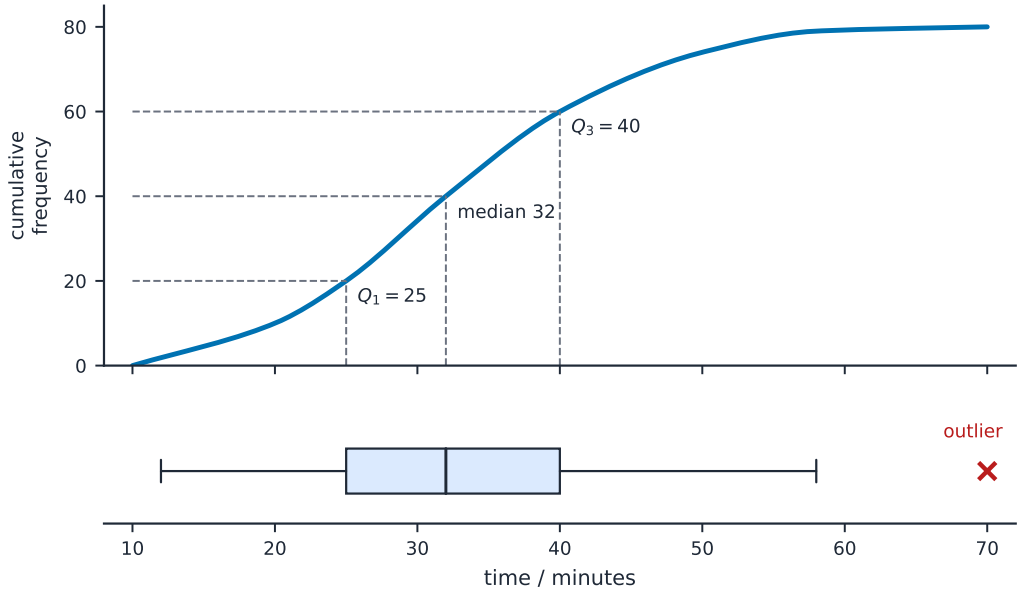
For grouped data, use the mid-interval values: $\bar{x} \approx \dfrac{\sum fx}{\sum f}$. The result is an estimate because the exact data are not known.

## ■ Cumulative frequency and box plots

A **cumulative frequency** 累积频率 graph plots the running total against the upper class boundary. Reading across at $\frac{n}{4}$, $\frac{n}{2}$ and $\frac{3n}{4}$ gives estimates of $Q_1$, the median and $Q_3$. A **box plot** 箱线图 shows the minimum, $Q_1$, median, $Q_3$ and maximum, with outliers marked separately.



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*Quartiles from a cumulative frequency curve, and the matching box plot*

For the 80 students, the upper fence is $40 + 1.5 \times 15 = 62.5$ minutes, so the 70-minute time is an outlier and the whisker stops at the largest value inside the fence, 58 minutes. When comparing two box plots, compare a measure of centre (median) **and** a measure of spread (IQR) in context.

## Correlation and regression

### ■ Pearson's correlation coefficient

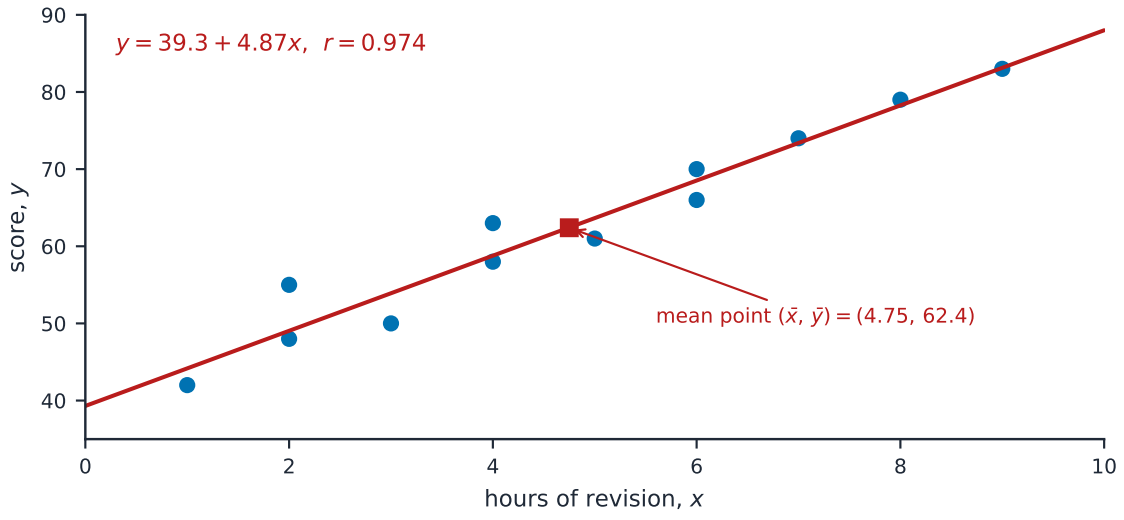
A **scatter diagram** 散点图 shows paired data $(x, y)$. **Pearson's product-moment correlation coefficient** $r$, found with technology, measures the strength of the **linear** relationship: $-1 \le r \le 1$. A common guide to describing it:

| $\lvert r \rvert$ | **Strength of linear correlation** |
|---|---|
| $0 < \lvert r \rvert \le 0.25$ | very weak |
| $0.25 < \lvert r \rvert \le 0.5$ | weak |
| $0.5 < \lvert r \rvert \le 0.75$ | moderate |
| $0.75 < \lvert r \rvert \le 1$ | strong |

Correlation is not causation: ice-cream sales and drownings both rise in hot weather, because a third variable, temperature, drives both.

## The regression line

The **regression line** 回归直线 of $y$ on $x$, found with technology, minimises the squared vertical distances and passes through the mean point $(\bar{x}, \bar{y})$.



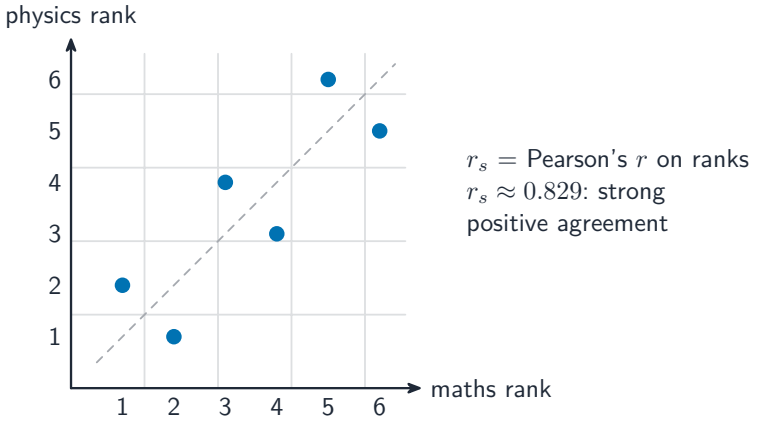

*Regression line of $y$ on $x$ through the mean point*

**Worked example.** For 12 students, hours of revision $x$ and score $y$ give $r = 0.974$ and $y = 39.3 + 4.87x$.

- Prediction for 5.5 hours: $y = 39.3 + 4.87 \times 5.5 \approx 66$. Reliable: $r$ is strong and 5.5 is within the data (**interpolation** 内插).
- The gradient means each extra hour of revision is associated with about 4.9 more marks.
- A prediction for 15 hours gives 112, more than the maximum mark: **extrapolation** 外推 is unreliable.
- Use this line only to predict $y$ from $x$.

## Spearman's rank correlation

**Spearman's rank correlation coefficient** 斯皮尔曼等级相关系数 $r_s$ is Pearson's $r$ applied to the ranks of the data. It measures how well the relationship is **monotonic**: whether $y$ always increases (or always decreases) as $x$ increases, even if not in a straight line. Equal values share the mean of their ranks.




*Two rankings in strong agreement*

**Worked example.** Six students are ranked in mathematics (1 to 6) and in physics as 2, 1, 4, 3, 6, 5. Pearson's $r$ on the ranks, by GDC, gives $r_s \approx 0.829$: a strong positive agreement between the two rankings.

Use $r_s$ when the data are ranks, when there are outliers, or when the pattern is a curve that keeps rising. For data that rise and then fall, neither coefficient describes the relationship.

## Probability

### Events and Venn diagrams

For equally likely outcomes, the **probability** 概率 of event $A$ is $P(A) = \dfrac{n(A)}{n(U)}$, where $U$ is the **sample space** 样本空间. $P(A') = 1 - P(A)$, and the **expected number** of times $A$ happens in $n$ trials is $nP(A)$.



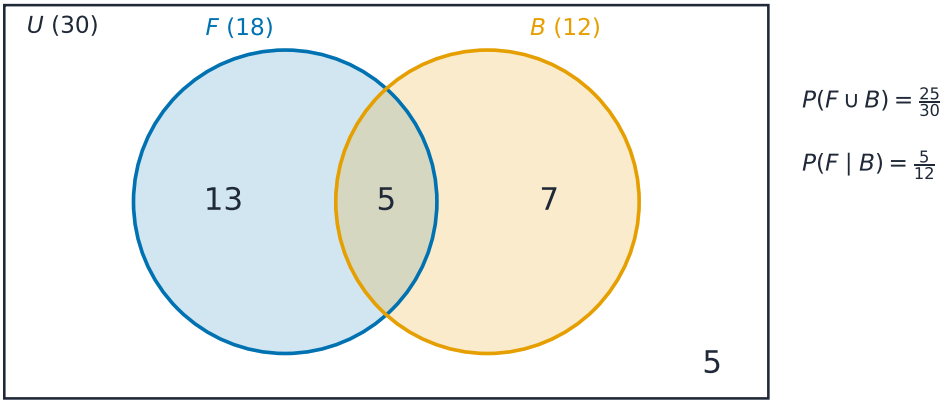
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*Fair dice: each face is equally likely, the simplest probability model*

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

**Mutually exclusive** 互斥 events cannot both happen: $P(A \cap B) = 0$.




*Fill a Venn diagram from the overlap outwards*

**Worked example.** In a class of 30, 18 play football ($F$), 12 basketball ($B$) and 5 both. Then $F$ only $= 13$, $B$ only $= 7$, neither $= 5$, and $P(F \cup B) = \frac{25}{30} = \frac{5}{6}$.

### Conditional probability and independence

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Given that a student plays basketball, the chance they also play football is $P(F \mid B) = \frac{5}{12}$: the sample space shrinks to the 12 basketball players.

Events are **independent** 独立 if $P(A \cap B) = P(A)P(B)$. Here $0.6 \times 0.4 = 0.24 \neq \frac{5}{30}$, so the two sports are not independent.

> **Worked example.** Two counters are taken without replacement from a bag of 3 red and 2 blue. A **tree diagram** 树状图 gives $P(RR) = \frac{3}{5} \times \frac{2}{4} = 0.3$ and $P(BB) = \frac{2}{5} \times \frac{1}{4} = 0.1$, so $P(\text{same colour}) = 0.4$. Multiply along branches, add between them.

## Discrete random variables and the binomial distribution

### Expected value

A **discrete random variable** 离散随机变量 $X$ takes separate values with probabilities that add to 1. Its **expected value** 期望值 is $E(X) = \sum x\,P(X = x)$.

> **Worked example.** A game costs ¥3. The prize is ¥0, ¥2 or ¥10 with probabilities 0.5, 0.3 and 0.2. $E(X) = 0 + 0.6 + 2 = 2.6$, so the expected gain is $-0.4$ yuan per game. A game is **fair** if the expected gain is 0.

### The binomial distribution

$X \sim B(n, p)$ counts the successes in $n$ independent trials, each with the same probability of success $p$. $E(X) = np$ and $\operatorname{Var}(X) = np(1-p)$. Use the GDC's binomial pdf for $P(X = r)$ and cdf for $P(X \leq r)$.

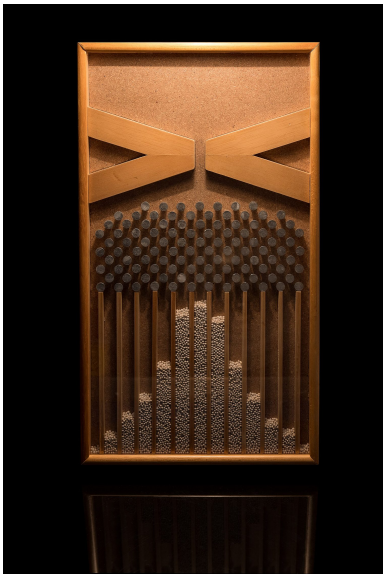
> **Worked example.** 20 % of seeds fail to grow. Five are planted, $X \sim B(5, 0.2)$. $P(X = 1) = 0.4096$, $P(X \geq 1) = 1 - 0.8^5 \approx 0.672$, and $E(X) = 1$.

## The normal distribution

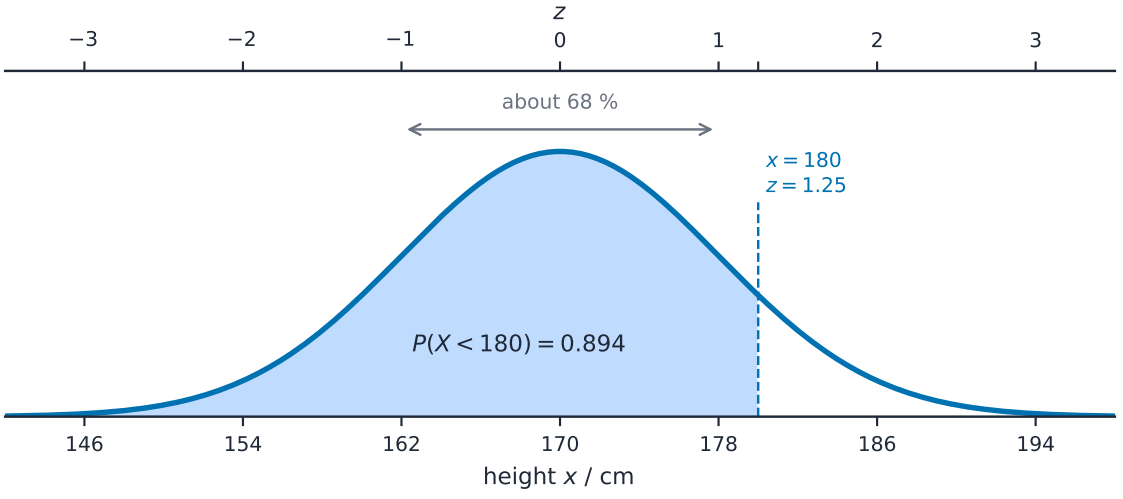
Many measurements follow a **normal distribution** 正态分布 $X \sim N(\mu, \sigma^2)$: a symmetric bell-shaped curve centred on the mean $\mu$. About 68 % of values lie within $1\sigma$ of the mean, 95 % within $2\sigma$ and 99.7 % within $3\sigma$.



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*A Galton board: balls falling through pins pile up into the bell-shaped normal curve*




*Probability is area under the curve; z counts standard deviations from the mean*

**Worked example.** Heights $X \sim N(170, 8^2)$ cm.

- $P(X < 180) = 0.894$ (by GDC; $z = \frac{180-170}{8} = 1.25$).
- $P(160 < X < 180) = 0.789$.
- The height exceeded by only 10 % of people: inverse normal, $P(X < h) = 0.9$, gives $h \approx 180.3$ cm.



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## Hypothesis testing

### The language of a test

1. The **null hypothesis** 原假设 $H_0$ says there is no effect or no association. The **alternative hypothesis** 备择假设 $H_1$ says there is.
2. Choose a **significance level** 显著性水平, usually 5 %.
3. Calculate the **p-value**: the probability, if $H_0$ were true, of a result at least as extreme as the one observed.
4. If the p-value is less than the significance level, reject $H_0$. Otherwise there is not enough evidence to reject it. A test never proves $H_0$.

### The chi-squared test for independence

The **chi-squared test** 卡方检验 for independence uses a **contingency table** 列联表 of observed frequencies. The expected frequency of each cell, if the variables were independent, is

$$E = \frac{\text{row total} \times \text{column total}}{\text{grand total}}, \qquad \chi^2_{\text{calc}} = \sum \frac{(O-E)^2}{E}$$

with **degrees of freedom** 自由度 $(\text{rows}-1)(\text{columns}-1)$. All expected frequencies must be at least 5.

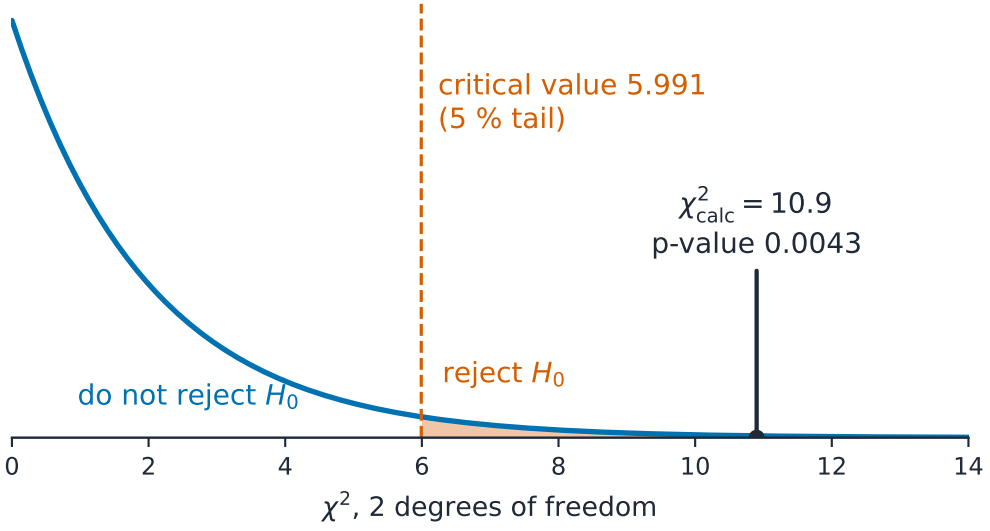
**Worked example.** 100 people at a school were asked their favourite drink.

| Observed | Tea | Coffee | Juice | Total |
|---|---|---|---|---|
| Students | 20 | 18 | 22 | 60 |
| Teachers | 20 | 17 | 3 | 40 |
| Total | 40 | 35 | 25 | 100 |

- $H_0$: favourite drink is independent of being a student or a teacher.
- Expected: students 24, 21, 15; teachers 16, 14, 10. All are at least 5.
- GDC: $\chi^2_{\text{calc}} \approx 10.9$, with $(2-1)(3-1) = 2$ degrees of freedom, p-value $\approx 0.0043$.
- $0.0043 < 0.05$, so reject $H_0$: there is evidence that favourite drink depends on the group. The largest contribution comes from teachers and juice: 3 observed, 10 expected.

You may instead compare $\chi^2_{\text{calc}}$ with the **critical value** 临界值, given in the question: here 5.991. Since $10.9 > 5.991$, reject $H_0$.


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*Reject $H_0$ when the statistic falls in the 5 % tail*

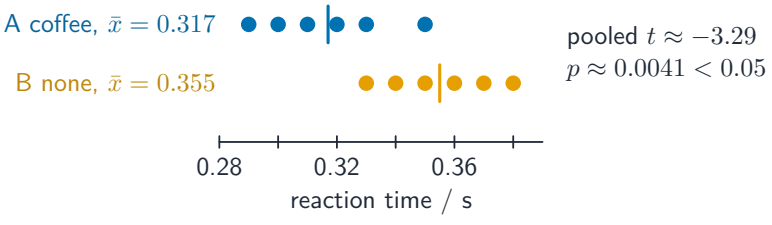
## The chi-squared goodness of fit test

A **goodness of fit test** 拟合优度检验 asks whether data follow a given distribution. The degrees of freedom are the number of classes minus 1.

> **Worked example.** A die is rolled 60 times, giving 8, 9, 12, 7, 10 and 14 for the faces 1 to 6. $H_0$: the die is fair, so each expected frequency is 10. $\chi^2_{\text{calc}} = \frac{4+1+4+9+0+16}{10} = 3.4$, with 5 degrees of freedom and p-value 0.639. Since $0.639 > 0.05$, there is not enough evidence that the die is biased.

## The t-test

The **t-test** compares the means of two independent samples from normal populations with equal variances (a **pooled** two-sample t-test). It can be **one-tailed** 单尾 ($H_1 : \mu_1 < \mu_2$) or **two-tailed** 双尾 ($H_1 : \mu_1 \neq \mu_2$). Decide which before looking at the data.




*The two samples in the t-test*

> **Worked example.** Reaction times in seconds for six students who drank coffee, $A$, and six who did not, $B$:

| | | | | | | |
|---|---|---|---|---|---|---|
| $A$ | **0.32** | **0.29** | **0.35** | **0.30** | **0.33** | **0.31** |
| $B$ | 0.36 | 0.34 | 0.38 | 0.33 | 0.37 | 0.35 |

- $H_0 : \mu_A = \mu_B$; $H_1 : \mu_A < \mu_B$, at the 5 % level.
- GDC (pooled): $\bar{x}_A \approx 0.317$, $\bar{x}_B = 0.355$, $t \approx -3.29$, p-value $\approx 0.0041$.
- $0.0041 < 0.05$, so reject $H_0$: there is evidence that the coffee group reacts faster. With only six in each group, a larger study would be needed before drawing firm conclusions.

**Common mistakes.**

- Reading a strong correlation as proof of cause.
- Predicting $x$ from the regression line of $y$ on $x$, or extrapolating far beyond the data.
- Writing *accept* $H_0$ or $H_0$ *is true*. Say: there is not enough evidence to reject $H_0$.
- Comparing the $p$-value the wrong way: reject $H_0$ when $p$ is **less than** the significance level.
- Using the wrong degrees of freedom: a $\chi^2$ test for independence on an $r \times c$ table has $(r-1)(c-1)$.
- Writing $P(X \geq 3) = 1 - P(X \leq 3)$ for a binomial variable. It is $1 - P(X \leq 2)$.
- Reading the variance as the standard deviation: $X \sim N(50, 16)$ has $\sigma = 4$.

## Practice

1. A school has 300 students in Year 10 and 200 in Year 11. How many from each year are in a stratified sample of 50?
2. For $3, 5, 7, 7, 8, 10, 12, 20$, find the median and IQR, and show that 20 is an outlier.
3. Each value in a data set with mean 9 and standard deviation 4.90 is doubled and then 3 is added. Find the new mean and standard deviation.
4. The regression line of score on hours is $y = 39.3 + 4.87x$ with $r = 0.974$, for 0 to 9 hours. Predict the score for 5.5 hours, and comment on a prediction for 15 hours.
5. Six students are ranked $1, 2, 3, 4, 5, 6$ in mathematics and $2, 1, 4, 3, 6, 5$ in physics. Find Spearman's rank correlation coefficient and interpret it.
6. In a class of 30, 18 play football, 12 basketball and 5 both. Find $P(F \cup B)$ and $P(F \mid B)$.
7. Show that playing football and playing basketball are not independent in question 6.
8. Two counters are taken without replacement from 3 red and 2 blue. Find the probability that they are the same colour.
9. A game costs ¥3, with prizes ¥0, ¥2, ¥10 with probabilities 0.5, 0.3, 0.2. Find the expected gain.
10. $X \sim B(5, 0.2)$. Find $P(X = 1)$, $P(X \geq 1)$ and $E(X)$.



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11. Heights are $N(170, 8^2)$ cm. Find $P(160 < X < 180)$ and the height exceeded by 10 % of people.
12. Find the expected frequencies for the drinks table, and the degrees of freedom.
13. Test at the 5 % level whether the die with frequencies 8, 9, 12, 7, 10, 14 is fair.
14. For the reaction-time data, state the hypotheses for a one-tailed t-test, and interpret the p-value 0.0041.

**Answers.**

1. 30 from Year 10 and 20 from Year 11.
2. Median 7.5; IQR $11 - 6 = 5$; $20 > 11 + 7.5 = 18.5$, so 20 is an outlier.
3. Mean $2(9) + 3 = 21$; standard deviation $2 \times 4.90 = 9.80$.
4. About 66 marks, reliable (strong $r$, interpolation); 15 hours gives 112, an unreliable extrapolation beyond the data and the maximum mark.
5. $r_s \approx 0.829$: strong agreement between the two subjects' rankings.
6. $\frac{5}{6}$; $\frac{5}{12}$.
7. $P(F)P(B) = 0.6 \times 0.4 = 0.24$, but $P(F \cap B) = \frac{5}{30} \approx 0.167$.
8. $0.3 + 0.1 = 0.4$.
9. $2.6 - 3 = -0.4$ yuan.
10. 0.4096; 0.672; 1.
11. 0.789; about 180.3 cm.
12. Students 24, 21, 15; teachers 16, 14, 10; 2 degrees of freedom.
13. $\chi^2 = 3.4$, 5 degrees of freedom, p-value $0.639 > 0.05$: not enough evidence that the die is biased.
14. $H_0 : \mu_A = \mu_B$, $H_1 : \mu_A < \mu_B$. $0.0041 < 0.05$, so reject $H_0$: evidence that the coffee group reacts faster.

## Exam tips

- Name the sampling method and say why it might be biased in context.
- Compare data sets with a centre **and** a spread, in context.
- Interpret $r$ by strength and direction, and remember that correlation is not causation.
- Write $H_0$ and $H_1$ in words about the context, and decide one- or two-tailed first.
- Compare the p-value with the significance level, then write the conclusion in context: "there is (not) enough evidence that …".
- Check that every expected frequency in a chi-squared test is at least 5.


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# Calculus

International Baccalaureate · IB Diploma · Mathematics: Applications and Interpretation · SL

## Scope and prerequisites

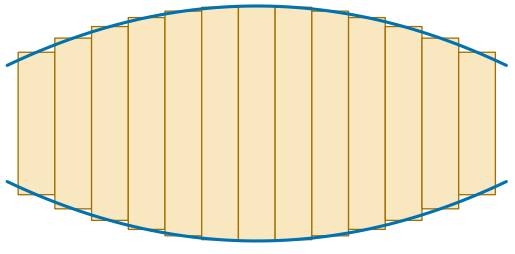
This handout teaches topic 5, Calculus, of Mathematics: Applications and Interpretation, the course first assessed in 2021: gradients and derivatives of $ax^n$, increasing and decreasing functions, tangents and normals, stationary points and optimisation, integration of $ax^n$ and areas, and the trapezoidal rule, and, for HL, further derivatives and the chain, product and quotient rules, related rates and the second derivative, further integration and volumes of revolution, kinematics, differential equations with slope fields and Euler's method, and coupled systems with phase portraits.

Prerequisites: gradients of lines and functions (topic 2), indices, and, for HL, eigenvalues and eigenvectors (topic 1).

## Gradients and derivatives

### ■ Kepler's wine casks

In 1613 the astronomer Johannes Kepler bought wine for his wedding in Linz. He watched the merchant measure each barrel by pushing a rod through the hole in its side to the far corner and reading off a length, whatever the shape of the barrel. Kepler doubted that one length could give a volume, and he wrote a whole book about it. He found volumes by cutting a solid into very thin slices and adding them up, an early form of integration. He also showed that the Austrian barrel shape gave almost the largest volume for its rod length, and that near a maximum, small changes in shape hardly change the volume, so the merchants' rule was nearly fair. That is the key idea of optimisation: at a maximum the rate of change is zero.



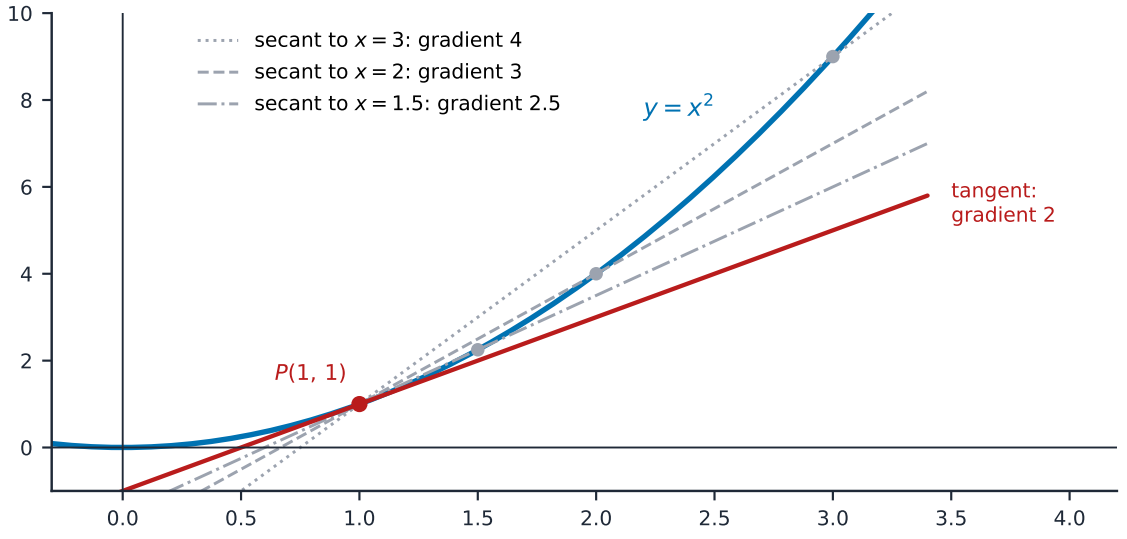

*Kepler's idea: slice and add*



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## Secants and the tangent

The gradient of a chord, or **secant** 割线, between $P(x, f(x))$ and a nearby point $Q(x+h, f(x+h))$ is $\dfrac{f(x+h)-f(x)}{h}$. As $h \to 0$ the secant approaches the **tangent** 切线 at $P$. The limit of the secant gradients is the **derivative** 导数 $f'(x)$, also written $\dfrac{dy}{dx}$.




*Secant gradients approach the tangent gradient*

The derivative has two meanings:

- the **gradient** of the curve $y = f(x)$ at a point;
- the **rate of change** of $y$ with respect to $x$: if $V$ is a volume in litres and $t$ is time in seconds, $\dfrac{dV}{dt}$ is in litres per second.

## The power rule

For any integer $n$, the derivative of $ax^n$ is $anx^{n-1}$, and the derivative of a constant is 0. Differentiate term by term.

> **Worked example.** $y = 4x^3 - 5x + \dfrac{2}{x} + 7$. Write $\dfrac{2}{x} = 2x^{-1}$. Then $\dfrac{dy}{dx} = 12x^2 - 5 - 2x^{-2} = 12x^2 - 5 - \dfrac{2}{x^2}$.

$f$ is **increasing** 递增 where $f'(x) > 0$ and **decreasing** 递减 where $f'(x) < 0$. For $f(x) = x^3 - 3x$, $f'(x) = 3x^2 - 3$, which is negative for $-1 < x < 1$: $f$ decreases there and increases elsewhere.

### Tangents and normals

At $x = a$ the tangent has gradient $m = f'(a)$ and passes through $(a, f(a))$. The **normal** 法线 is perpendicular to the tangent, with gradient $-\dfrac{1}{m}$.

> **Worked example.** For $y = x^2$ at $x = 2$: $y = 4$ and $\dfrac{dy}{dx} = 2x = 4$.

- Tangent: $y - 4 = 4(x - 2)$, so $y = 4x - 4$.
- Normal: gradient $-\frac{1}{4}$, so $y - 4 = -\frac{1}{4}(x - 2)$, or $x + 4y - 18 = 0$.

On a calculator paper, the GDC finds the gradient at a point directly (numerical derivative); write down the value and what it means.

## Stationary points and optimisation

### Local maximum and minimum points

A **stationary point** 驻点 is where $f'(x) = 0$: the tangent is horizontal. Look at the sign of $f'(x)$ on each side:



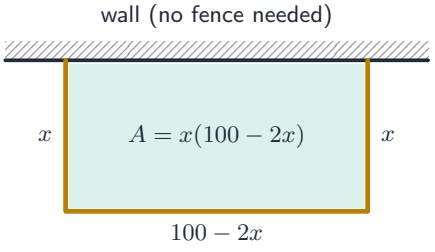
*Fenced enclosures: optimisation finds the dimensions that give the largest area for a fixed length of fence*

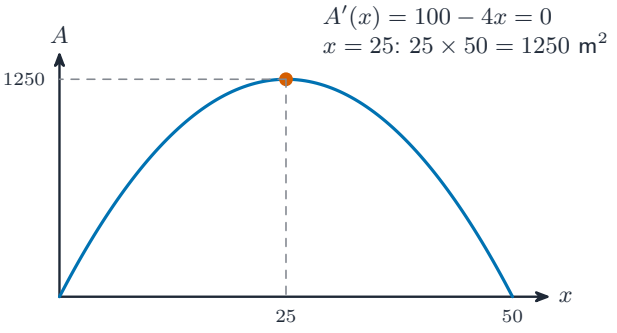
| $f'$ before | $f'$ after | Point |
|---|---|---|
| $+$ | $-$ | **local maximum** 局部最大值 |
| $-$ | $+$ | **local minimum** 局部最小值 |

**Worked example.** $f(x) = x^3 - 3x$. $f'(x) = 3x^2 - 3 = 0$ gives $x = \pm 1$. Since $f'$ changes from $+$ to $-$ at $x = -1$, $(-1, 2)$ is a local maximum; it changes from $-$ to $+$ at $x = 1$, so $(1, -2)$ is a local minimum.

## Optimisation

To make a quantity as large or as small as possible:




*Use the fixed fence length to write $A$ in one variable; $A'(x) = 0$ at $x = 25$ gives the largest pen*

1. Write it as a function of **one** variable, using any fixed condition to remove the others.
2. Differentiate and solve the derivative $= 0$, using the GDC if needed.
3. Check that it is a maximum or a minimum, and check the ends of the domain.
4. Answer the question asked, with units.

**Worked example.** A drinks can must hold 500 cm$^3$. Find the radius that uses the least metal.

- Volume: $\pi r^2 h = 500$, so $h = \dfrac{500}{\pi r^2}$.
- Surface area: $A = 2\pi r^2 + 2\pi r h = 2\pi r^2 + \dfrac{1000}{r}$, for $r > 0$.
- $\dfrac{dA}{dr} = 4\pi r - \dfrac{1000}{r^2} = 0$ gives $r^3 = \dfrac{250}{\pi}$, so $r \approx 4.30$ cm, and then $h \approx 8.60$ cm.
- $\dfrac{dA}{dr}$ changes from negative to positive, so this is a minimum: $A \approx 349$ cm$^2$.

The best can has its height equal to its diameter. Real cans are taller and thinner, because the ends are made of thicker metal and the shape must suit a hand and a fridge: the model leaves these out.

**Worked example.** A farmer has 100 m of fence to make a rectangular pen against a long wall, so only three sides need fencing. With the two sides at right angles to the wall $x$ m long, the third side is $100-2x$ and the area is $A = x(100-2x) = 100x-2x^2$, for $0 < x < 50$. $\dfrac{dA}{dx} = 100 - 4x = 0$ at $x = 25$, and $\dfrac{dA}{dx}$ changes from positive to negative there, so the largest area is $25 \times 50 = 1250$ m$^2$.

**Worked example.** A company's profit is $P(x) = -2x^2 + 120x - 1000$ yuan when it sells $x$ items a day. $P'(x) = -4x + 120 = 0$ at $x = 30$, and $P(30) = ¥800$ is the maximum daily profit.

## Integration

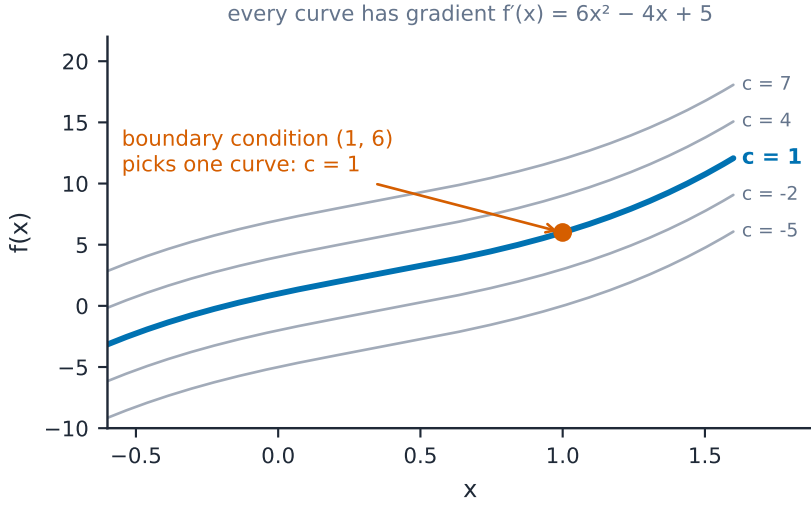
### Antiderivatives

**Integration** 积分 reverses differentiation. For $n \neq -1$:

$$\int ax^n \, dx = \frac{a}{n+1}x^{n+1} + c$$

The **constant of integration** 积分常数 $c$ appears because every constant differentiates to 0. A **boundary condition**, a known point on the curve, finds $c$.

**Worked example.** A curve has gradient $f'(x) = 6x^2 - 4x + 5$ and passes through $(1, 6)$. Then $f(x) = 2x^3 - 2x^2 + 5x + c$, and $f(1) = 5 + c = 6$, so $c = 1$ and $f(x) = 2x^3 - 2x^2 + 5x + 1$.




*Every value of c gives a curve with the same gradient; the boundary condition picks c = 1*

## Definite integrals and area

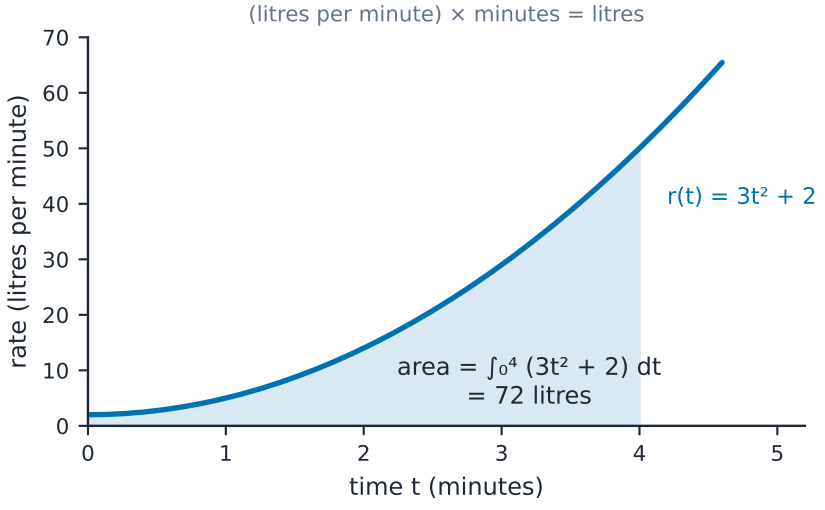
The **definite integral** 定积分 $\int_a^b f(x)\,dx$ is the area between the curve and the $x$-axis from $x = a$ to $x = b$, when $f(x) \geq 0$. On a calculator paper, find it with the GDC and write down the integral you used.

> **Worked example.** The area under $y = 4 - x^2$ from $x = 0$ to $x = 2$ is $\int_0^2 (4 - x^2)\,dx = \left[4x - \frac{x^3}{3}\right]_0^2 = 8 - \frac{8}{3} = \frac{16}{3} \approx 5.33$.

## From a rate to a total

If $r(t)$ is a rate of change, then $\int_a^b r(t)\,dt$ is the total change from $t = a$ to $t = b$: the area under a rate graph is an amount.

> **Worked example.** Water flows into a tank at $r(t) = 3t^2 + 2$ litres per minute. In the first 4 minutes the tank gains $\int_0^4 (3t^2 + 2)\,dt = \left[t^3 + 2t\right]_0^4 = 64 + 8 = 72$ litres. The units of the area are (litres per minute) $\times$ minutes $=$ litres.




*The area under a rate graph is an amount: 72 litres in 4 minutes*

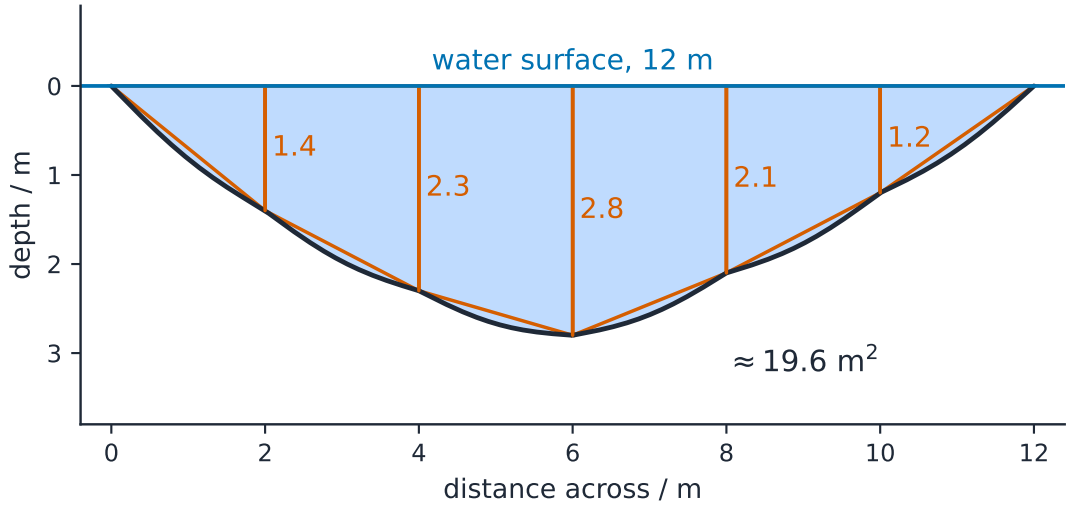
## The trapezoidal rule

When only data are known, or the function cannot be integrated, estimate the area with trapezoids. With $n$ strips of equal width $h = \dfrac{b-a}{n}$:

$$\int_a^b y\,dx \approx \frac{h}{2}\Big(y_0 + y_n + 2(y_1 + y_2 + \cdots + y_{n-1})\Big)$$

**Worked example.** A river is 12 m wide. Its depth, measured every 2 m across, is 0, 1.4, 2.3, 2.8, 2.1, 1.2 and 0 m. The cross-section area is about

$$\frac{2}{2}\Big(0 + 0 + 2(1.4 + 2.3 + 2.8 + 2.1 + 1.2)\Big) = 19.6 \text{ m}^2$$




*Trapezoids under a curve that bends downwards give an underestimate*

If the curve bends downwards (concave down), as here, the straight tops of the trapezoids lie below it, so the rule **underestimates** the area. If the curve bends upwards, it overestimates. More, narrower strips give a better estimate.

**Common mistakes.**

- Differentiating $\dfrac{1}{x^2}$ without rewriting it: write $x^{-2}$, then the derivative is $-2x^{-3}$.
- Forgetting $+C$ on an indefinite integral.
- Using the tangent gradient $m$ for a normal. The normal has gradient $-\dfrac{1}{m}$.
- Counting strips wrongly in the trapezoidal rule: $n$ strips have width $h = \dfrac{b-a}{n}$ and $n+1$ ordinates.
- Integrating straight across a root. Area below the $x$-axis gives a negative integral, so split at the roots.

- Not checking the ends of the domain in an optimisation question.

## Practice

1. Differentiate $y = 4x^3 - 5x + \dfrac{2}{x} + 7$.
2. Find where $f(x) = x^3 - 3x$ is decreasing.
3. Find the equations of the tangent and normal to $y = x^2$ at $x = 2$.
4. Find and classify the stationary points of $f(x) = x^3 - 3x$.
5. A profit is $P(x) = -2x^2 + 120x - 1000$. Find the number of items that gives the greatest profit, and the profit.
6. A can holds 500 cm$^3$. Show that $A = 2\pi r^2 + \dfrac{1000}{r}$, and find the radius that minimises $A$.
7. Find $\displaystyle\int (6x^2 - 4x + 5)\,dx$.
8. A curve has gradient $6x^2 - 4x + 5$ and passes through $(1, 6)$. Find its equation.
9. Find the area under $y = 4 - x^2$ from $x = 0$ to $x = 2$.
10. Use the trapezoidal rule with the river depths 0, 1.4, 2.3, 2.8, 2.1, 1.2, 0 m every 2 m to estimate the cross-section area.
11. Is the estimate in question 10 too large or too small? Explain.
12. The volume of water in a tank is $V = 2t^3 - 3t^2 + 50$ litres after $t$ hours. Find $\dfrac{dV}{dt}$ at $t = 4$ and say what it means.
13. Find the gradient of $y = x^2 - \dfrac{8}{x}$ at $x = 2$.
14. Explain why the derivative is zero at the maximum volume of Kepler's barrel.

**Answers.**

1. $12x^2 - 5 - \dfrac{2}{x^2}$.
2. $-1 < x < 1$.
3. Tangent $y = 4x - 4$; normal $x + 4y - 18 = 0$.
4. Maximum $(-1, 2)$; minimum $(1, -2)$.
5. 30 items; ¥800.
6. $h = \frac{500}{\pi r^2}$, so $A = 2\pi r^2 + 2\pi r \cdot \frac{500}{\pi r^2}$; $r = \sqrt[3]{250/\pi} \approx 4.30$ cm.
7. $2x^3 - 2x^2 + 5x + c$.
8. $y = 2x^3 - 2x^2 + 5x + 1$.
9. $\frac{16}{3} \approx 5.33$.
10. 19.6 m$^2$.
11. Too small: the river bed curves downwards between the measurements, so the trapezoids lie inside the true cross-section.
12. $\frac{dV}{dt} = 6t^2 - 6t = 72$: at $t = 4$ the volume is increasing at 72 litres per hour.


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13. $2x + \frac{8}{x^2} = 4 + 2 = 6$.
14. At a maximum the tangent is horizontal, so the rate of change of volume with shape is zero: small changes in shape hardly change the volume.

## Exam tips

- Rewrite $\frac{a}{x^n}$ as $ax^{-n}$ before differentiating or integrating.
- Give the meaning of a derivative with units: "litres per second".
- In optimisation, write the quantity in one variable, justify the maximum or minimum, and answer in context.
- Do not forget $+c$ in an indefinite integral; use a boundary condition to find it.
- For the trapezoidal rule, count strips, not points: 7 measurements make 6 strips.


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# School-supervised mathematical exploration

International Baccalaureate · IB Diploma · Mathematics: Applications and Interpretation · SL

## Scope and prerequisites

This handout teaches the internal assessment of Mathematics: Applications and Interpretation, the course first assessed in 2021: what the mathematical exploration is, how to choose a topic and a research question, the five assessment criteria, how to communicate mathematics in writing, how to collect data, build a model, test it and reflect on it, the working process and academic integrity, and, for HL, what criterion E expects at HL.

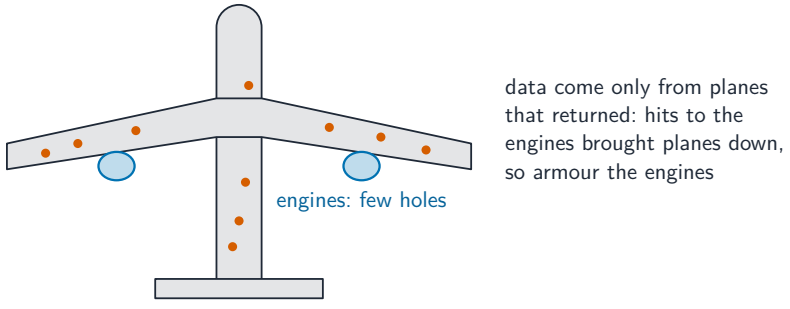
Prerequisites: none beyond the course itself. The exploration uses whatever mathematics fits your question, from any topic.

The worked example in this handout uses original, fictional data. It is not a real student's exploration and must not be copied as one.

## What the exploration is

### The holes that were not there

During the Second World War the mathematician Abraham Wald worked with a group of statisticians advising the United States military. One question, in the version of the story usually told, was where to add armour to bomber aircraft. The planes that came back had many bullet holes in the wings and the body, and few in the engines, so the obvious answer was to armour the wings and body. Wald pointed out that the data came only from planes that had **survived**. Planes hit in the engines were missing from the data because they had not come back. The armour should go where the returning planes had no holes. Wald used ordinary mathematics; what mattered was asking how the data had been collected and what that did to the conclusion. That kind of thinking is what the exploration rewards.




*Survivorship bias*

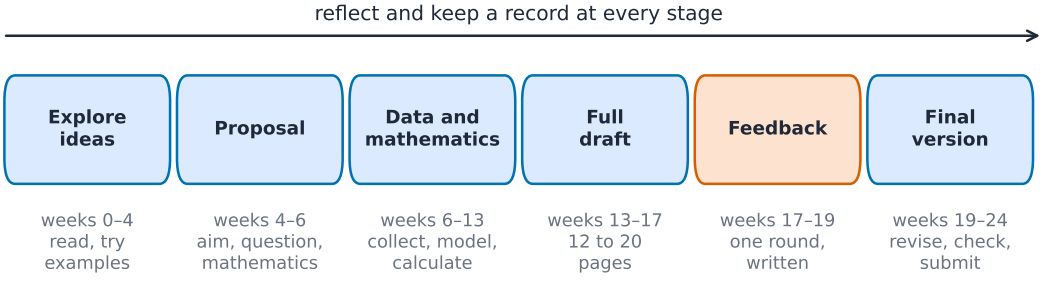


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## The task

The **mathematical exploration** 数学探究 is a piece of written work, about 12 to 20 pages long, in which you investigate an area of mathematics that interests you. It is the course's **internal assessment** 内部评估: your teacher marks it, and the IB checks a sample of marking from each school (**moderation** 评分审核). It is worth 20 % of the final grade at both SL and HL.

In Applications and Interpretation the exploration often starts from a real situation: data you collect, a model you build, or a decision you analyse. Pure topics are allowed too. Either way it is your own investigation, written for a reader who is one of your classmates.



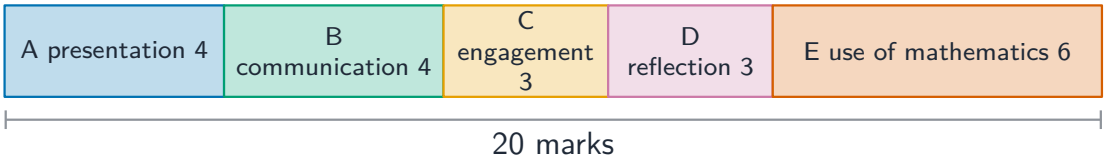

*Start early: collecting good data takes longer than you expect*



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### The five criteria

| Criterion | Marks | What it rewards |
|---|---|---|
| A **Presentation** 表达 | 4 | Organised, coherent and concise: an introduction with an aim, a clear structure and a conclusion |
| B **Mathematical communication** 数学交流 | 4 | Correct notation, symbols and terms; defined variables; labelled graphs and tables |
| C **Personal engagement** 个人投入 | 3 | Your own thinking and approach, shown throughout the work |
| D **Reflection** 反思 | 3 | Critical comments on your results, methods, assumptions and limitations |
| E **Use of mathematics** 数学运用 | 6 | Relevant, correct mathematics at the level of the course, with clear understanding |




*How the 20 marks are shared*

The total is 20 marks. Criterion E carries the most marks, but the other four together carry more.

## Choosing a topic and a research question

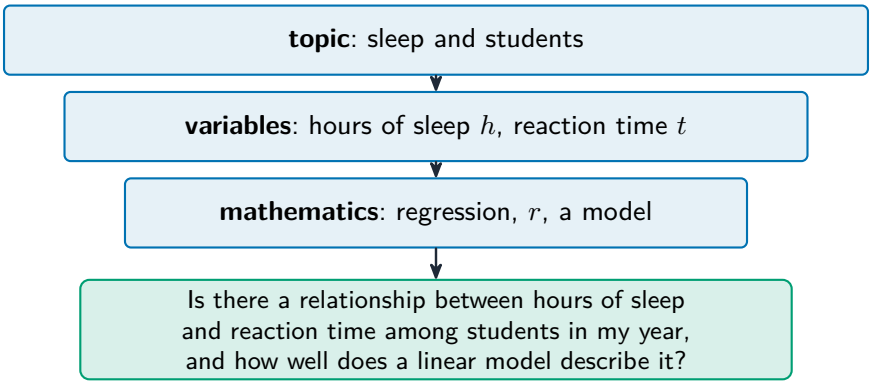
### What makes a good topic

- **Interest.** You will spend months on it. Choose something you genuinely want to know.
- **Mathematics.** The question must need mathematics from the course, used and explained by you. Collecting data and drawing one bar chart is not enough.
- **Scope.** "Statistics of sport" is too broad; one calculation is too narrow.
- **Feasibility.** You must be able to get enough good data, safely and with permission, in the time available.


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## From topic to research question

A **research question** 研究问题 names the variables and the mathematics.




*Narrowing a topic to a research question*

| Weak | Better |
|---|---|
| Sleep and students | Is there a relationship between hours of sleep and reaction time among students in my year, and how well does a linear model describe it? |
| Bus delays | Do the delays of the number 12 bus follow a normal distribution, and what time should I leave home to be on time 95 % of the time? |
| Shops in my city | How well does a Voronoi diagram of the supermarkets in my district predict which supermarket people actually use? |
| Phone prices | Which model, linear, exponential or power, best describes how the price of a phone falls with its age, judged by $R^2$ and by context? |

Good questions are specific and leave you decisions to make: which data, which model, which test.

## Collecting data

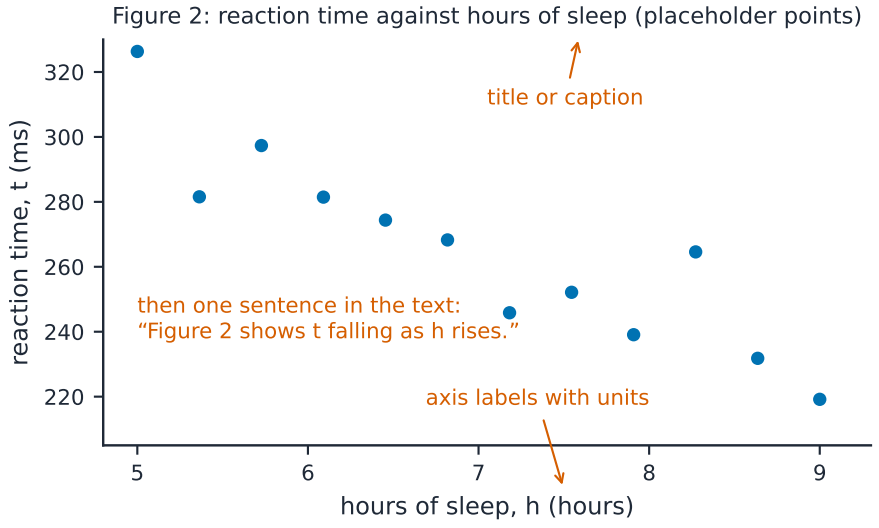
The quality of an exploration is limited by the quality of its data.

- **Primary data** 一手数据 you collect yourself show personal engagement, but plan the method: sample size, sampling technique (topic 4), and how you measure.
- **Secondary data** 二手数据 from a published source must be cited, and you must explain why it is reliable.
- Think like Wald: who is **missing** from your sample, and how could that bias the result?
- People's data need their consent, and should be anonymous.

## Communicating mathematics

### Writing for a classmate

- State the aim at the start and answer it at the end.
- Explain why each method is chosen before you use it.
- Put long tables of raw data in an appendix and keep a summary in the text.
- Every graph and table needs a title or caption, labelled axes and units, and a sentence in the text that says what it shows.
- Cite every source, including data sets and software, in a bibliography.




*A graph the reader can use: caption, labelled axes with units, and a sentence about it*

### Correct notation

Calculator and spreadsheet notation is not acceptable in the written work.

| Do not write | Write |
|---|---|
| `y = 2.3x^2 + 1.2E-3` | $y = 2.3x^2 + 1.2 \times 10^{-3}$ |
| `sqrt(x)`, `x*y` | $\sqrt{x}$, $xy$ |
| `r = -.989`, so it fits | $r = -0.989$, and explain what it means |
| `p = 3.9E-4` | p-value $= 0.00039$, less than the 5 % significance level |

Define every variable when it first appears, with its units: “Let $h$ be the hours of sleep the night before, and $t$ the reaction time in milliseconds.”



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## Modelling and testing: a worked case

### The question and the data

A student, whom we will call Mei, wonders whether sleep affects reaction time. Her research question: *Is there a relationship between hours of sleep and reaction time among students in my year, and can a model predict one from the other?*

Mei asks 12 volunteers, with their consent, to report their hours of sleep and to take an online reaction test the next morning; each time is the mean of five attempts. These are fictional data for this handout:

| $h$ **(hours)** | **5.5** | **6.0** | **6.5** | **6.5** | **7.0** | **7.0** | **7.5** | **8.0** | **8.0** | **8.5** | **9.0** | **9.0** |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $t$ (ms) | 312 | 305 | 298 | 301 | 290 | 294 | 285 | 279 | 283 | 276 | 270 | 273 |

### A linear model

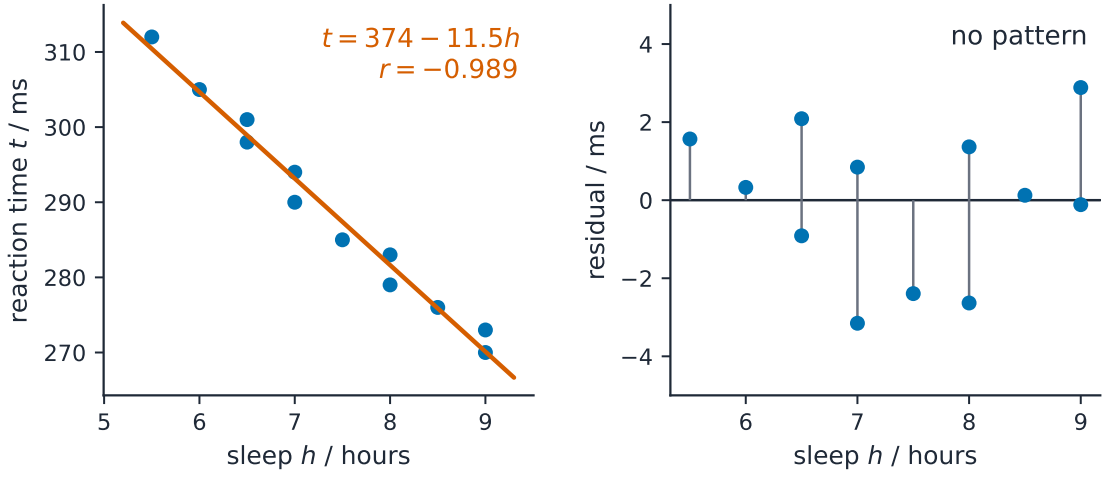
The scatter diagram looks linear, so Mei fits a **regression line** 回归直线 (topic 4):

$$t = 374 - 11.5h, \qquad r = -0.989$$

The gradient means each extra hour of sleep is associated with a reaction time about 11.5 ms shorter. The intercept, 374 ms for no sleep at all, is far outside the data and should not be interpreted.

### Testing the model

Mei checks the **residuals** 残差, observed minus predicted, using the unrounded fitted line $t = 373.78826896 - 11.51931330h$. They lie between $-3.2$ and 2.9 ms, with no pattern, so a straight line is a reasonable model within the data.




*A strong correlation, and residuals with no pattern*

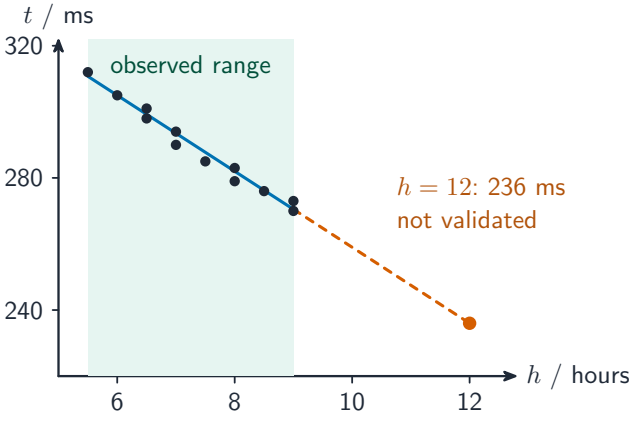


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Mei also splits the students into two groups, less than 8 hours (7 students, mean 297.9 ms) and at least 8 hours (5 students, mean 276.2 ms), and runs a one-tailed t-test. The p-value is about 0.0004, so there is strong evidence that the mean reaction time is lower in the group that slept more.

## Reflection

**Reflection** is critical thinking about the results and the method, written wherever it is relevant.




*Extrapolation beyond the data is not validated*



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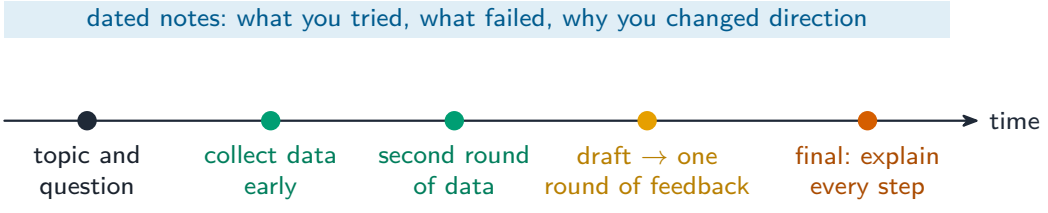
| Descriptive (weak) | Critical (strong) |
|---|---|
| The model is accurate. | Within $5.5 \leq h \leq 9$ the observed residuals are all within 3.2 ms; this describes fit to the sample, not guaranteed prediction accuracy. At $h = 12$ it predicts 236 ms, outside the observed sleep range: this extrapolation has not been validated. |
| Sleep makes you faster. | The data show correlation, not causation. Students who sleep more may also drink less coffee or be more relaxed; an experiment that controlled sleep would be needed. |
| My sample was small. | The 12 volunteers were from one class, so the sample is not random and may not represent my year: people who volunteer may care more about sleep. |
| The sleep data were good. | Hours of sleep were self-reported, so their validity is uncertain. A sleep-tracking watch for a few students would test how accurate the reports were. |
| I did two tests. | Splitting $h$ into two groups threw away information; the regression uses every value, so it is the stronger method here. |

Good reflection names an **assumption** 假设, says what effect it has, and suggests what would test it.

## Integrity and the working process

### Working steadily

- Keep dated notes of what you tried, what failed and why you changed direction.
- Collect the data early: a second round of collection is often needed.
- Your teacher gives written feedback on **one** draft. Use it fully.
- Check the length: far too long is not concise, far too short rarely has enough mathematics.




*Steady work: early data, one feedback draft, notes all the way*


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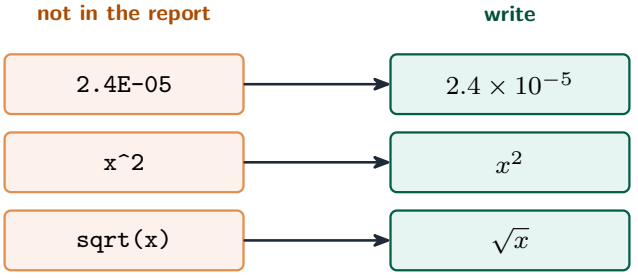
### Academic integrity

- Never invent data, change results to make them look better, or copy a sample exploration.
- Cite sources of ideas, data, images and software, in the text and in the bibliography.
- If you use an AI tool, cite it as you would any other source, following your school's policy. Presenting its output as your own work is academic misconduct.
- Collect data about people only with their consent, and keep it anonymous.

Expect to be asked to explain any part of your work. If you cannot explain a step, it should not be in the exploration.

**Common mistakes.**

- Choosing a topic so wide that no mathematics is done. Narrow it to one question you can answer with your data.
- Collecting too little data, or data you cannot cite.
- Using calculator notation in the report: write $2.4 \times 10^{-5}$, not 2.4E-05.
- Graphs without labelled axes, units or a source for the data.
- Fitting a model without testing it, for example with residuals or a new data point.
- A reflection that only repeats the results. Say what they mean, how reliable they are and what limits them.




*Write mathematics in the report, not calculator notation*

## Practice

1. Rewrite the topic "Statistics in basketball" as a focused research question that names the mathematics.
2. Rewrite in correct notation: `y = 4.1x^3 - 2E-4`, `sqrt(2)*pi`, `r = .87`.
3. Using $t = 374 - 11.5h$, predict the reaction time for $h = 7.5$, and find the residual for the student with $t = 285$.
4. Explain why Mei should not interpret the intercept 374 ms.
5. Explain how Wald's insight about the bombers applies to a survey of students who reply to an online questionnaire.

6. Which criterion, A to E, does each feature mainly affect? (a) Axes with no units; (b) no conclusion; (c) a wrong p-value; (d) a comment on self-reported data; (e) designing your own data collection.
7. Explain why "sleep makes you faster" is not supported by Mei's data, and what would test it.
8. Turn this into critical reflection: "I could have used more data."
9. A student downloads a data set and uses an AI tool to explain a test. What must they do?
10. Explain why splitting a continuous variable into two groups can make a test weaker.

**Answers.**

1. For example: "Is there a relationship between a player's height and their rebounds per game in my school league, and is it linear?" Any answer must name variables and a method.
2. $y = 4.1x^3 - 2 \times 10^{-4}$; $\sqrt{2}\,\pi$; $r = 0.87$.
3. $374 - 11.5(7.5) \approx 288$ ms, so the residual is $285 - 288 \approx -3$ ms (about $-2.4$ ms with the unrounded line).
4. $h = 0$ is far outside the data ($5.5 \leq h \leq 9$); predicting there is extrapolation, and a person with no sleep may not behave like the model at all.
5. Only students who chose to reply are in the data; those who did not reply may differ, for example in interest in the topic, so the results may be biased. It is the missing data that matter.
6. (a) B; (b) A; (c) E; (d) D; (e) C.
7. The data show correlation only: another variable, such as caffeine or stress, may affect both. An experiment in which sleep is controlled, with other variables kept the same, would test cause and effect.
8. For example: "With 12 students from one class, one unusual student changes $r$ noticeably; a random sample of about 40 from the whole year would make the model more reliable and representative."
9. Cite both, in the text and in the bibliography, and explain why the data are reliable. The AI tool's output must not be presented as their own work.
10. Grouping replaces each value by its group, so differences within a group are lost; a method that uses every value, such as regression, uses all the information.

## Exam tips

- Choose a question you care about, and check that it needs mathematics from the course.
- Plan data collection: sample size, sampling method, consent and who might be missing.



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- Write for a classmate: define every variable, label every graph, and explain every method.
- Never use calculator notation in the written work.
- Look at residuals, not only $r$, before trusting a model; never interpret far outside the data.
- Reflect critically throughout: name assumptions, their effects and how to test them.
- Cite every source, including software and AI tools; never invent or alter data.



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