

Exercise walk-through

5.2

Root finding and numerical integration

International Baccalaureate · IB Diploma · Mathematics: Applications and Interpretation

5.2 Root finding and numerical integration

Name: _____ Class: _____ Date: _____ **Total: 13 marks**

KEY WORDS iteration

Objective

Root finding and numerical integration. Original classroom practice for IB SL; not an official IB paper.

1 Key knowledge and worked examples

■ How reliable is an approximation?

- A nonlinear equation has no convenient exact solution. An approximation still needs evidence that it is reliable.
- This lesson studies **iteration** : A repeated update in which each new approxi-

mation is calculated from the previous one.

■ The method

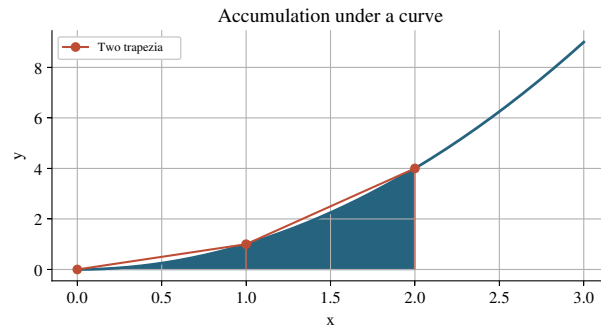
- A continuous function with opposite signs at two endpoints has a root between them.
 - Newton's method uses $x_{\text{next}} = x - f(x)/f'(x)$, with a nonzero derivative.
 - The trapezium rule approximates a definite integral using endpoint heights.
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■ A worked case

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Example:

For $f(x)=x^2-2$ and $x_0=1.5$, Newton gives $x_1=1.5-(2.25-2)/3=1.4166667$. With $y=x^2$ on $[0,2]$ and two equal strips, $h=1$ and trapezium area $=(1/2)(0+2\times 1+4)=3$; the exact area is $8/3$.



■ Where it goes wrong

- A sign change across a discontinuity does not prove a root.
- Iteration can diverge or cycle.
- The trapezium rule's overestimate or underestimate depends on curvature, not just whether the function increases.

Not true: Every sign change proves a root, including one across a discontinuity. A sign change shows a root only if the function is continuous there: $1/x$ changes sign across $x=0$ but has no root.

■ Applying it

- Give a stopping criterion and report sensible accuracy.
- Confirm the approximated root by a sign bracket around the stated rounded answer; explain any failure of the chosen iteration.

■ In your course

- Applications and Interpretation SL, the course first assessed in 2021.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

Key:

- **Iteration:** A repeated update in which each new approximation is calculated from the previous one.
 - A continuous function with opposite signs at two endpoints has a root between them.
 - Watch for: A sign change across a discontinuity does not prove a root.
-

■ Write from a Newton step to 1.5

- For $x^2 - 2$, take one Newton step from 2.
- A sign change across a discontinuity does not prove a root.

Exercises 2.1-2.3

2.1 After one Newton step for x^2-2 from $x_0=1.5$, find x_1 . Give it to 3 significant figures. Show the relation, substitution and units. [2]

2.2 Use two strips for the trapezium estimate of integral x^2 from 0 to 2. Show the relation, substitution and units. [2]

2.3 Find the exact integral of x^2 from 0 to 2. Give it to 3 significant figures. Show the relation, substitution and units. [2]

Exercises 2.1-2.3

2.1 After one Newton step for x^2-2 from $x_0=1.5$, find x_1 . Give it to 3 significant figures. Show the relation, substitution and units. [2]

2.1 $x_1=1.5-(1.5^2-2)/(2 \times 1.5)=17/12 \approx 1.416667$.

2.2 Use two strips for the trapezium estimate of integral x^2 from 0 to 2. Show the relation, substitution and units. [2]

2.2 $h=1$ with heights 0,1,4 gives $(1/2)(0+2 \times 1+4)=3$.

2.3 Find the exact integral of x^2 from 0 to 2. Give it to 3 significant figures. Show the relation, substitution and units. [2]

2.3 The exact integral is $[x^3/3]$ from 0 to 2 $=8/3$.

Exercises 2.4·2.5

2.4 Decide whether this claim is true and justify: Every sign change proves a root, including one across a discontinuity. [2]

2.5 $f(x)=1/x$ changes sign between -1 and 1. Why does this NOT prove a root in $[-1,1]$? Choose one answer.

- **A** The function is discontinuous at 0, so the sign-change rule does not apply
- **B** The interval does not contain 0, so the sign change happens elsewhere
- **C** A sign change never indicates a root unless the function is a polynomial

[1]

Exercises 2.4-2.5

2.4 Decide whether this claim is true and justify: Every sign change proves a root, including one across a discontinuity. [2]

2.4 False. A sign change shows a root only if the function is continuous there: $1/x$ changes sign across $x=0$ but has no root.

2.5 $f(x)=1/x$ changes sign between -1 and 1. Why does this NOT prove a root in $[-1,1]$? Choose one answer.

- **A** The function is discontinuous at 0, so the sign-change rule does not apply
- **B** The interval does not contain 0, so the sign change happens elsewhere
- **C** A sign change never indicates a root unless the function is a polynomial [1]

2.5 A The intermediate value reasoning requires continuity on the whole interval. At the asymptote $x=0$ the function jumps from negative to positive with no root existing.

Exercise 3.1

3.1 Apply one Newton step to $f(x)=x^3-2x+2$ from $x_0=0$ and then one more. What happens, and what does it teach about convergence? [4]

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3.1 $x_1=0-2/(-2)=1$; $x_2=1-1/1=0$. It cycles 0, 1 rather than converging. Newton requires suitable starting values/derivatives; use a bracketed method or inspect the function rather than assuming every iteration converges.