Calculus

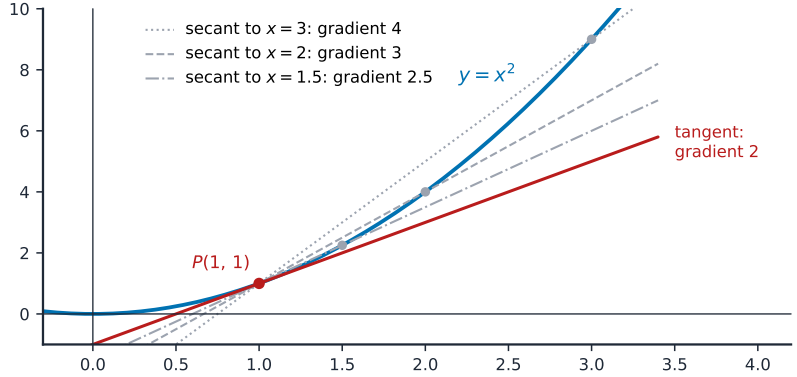
Topic 5 · SL and HL

Kepler's wine casks

> At a maximum the rate of change is zero.

- Linz, 1613; one rod length for every barrel.
- Thin slices give volumes; near a maximum, volume barely changes.
- This unit: derivatives $\to$ optimisation $\to$ integrals $\to$ equations.

The derivative



- Gradient of the **tangent**; rate of change with units; $\frac{d}{dx}(ax^n) = anx^{n-1}$.

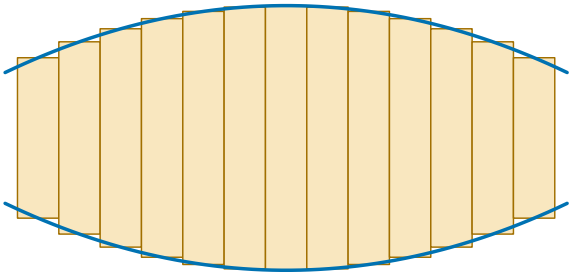
Tangent and normal

1. **Point.** $y = x^2$ at $x = 2$ gives $(2, 4)$
2. **Tangent.** $m = 4$, so $y = 4x - 4$
3. **Normal.** $m = -\frac{1}{4}$, so $x + 4y - 18 = 0$

Decision

Gradient from the **derivative**, point from the curve

Volume by slicing



volume $\approx \sum \pi r^2 \, \Delta x$
thinner slices: closer;
the limit is an integral

In the limit, an integral

The best can

1. **Area.** $A = 2\pi r^2 + \frac{1000}{r}$
2. **Derivative.** $4\pi r - \frac{1000}{r^2} = 0$
3. **Answer.** $r \approx 4.30$ cm, $A \approx 349$ cm^2

Decision

Show it is a minimum

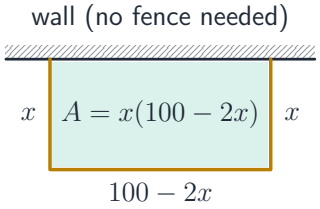
Check: maximum or minimum?

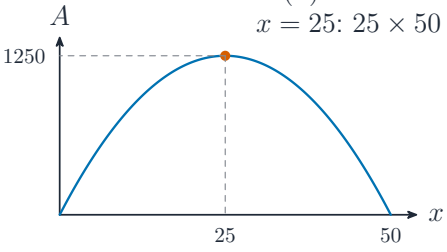
Your turn

f' changes from $-$ to $+$ at $x = 1$. What is $(1, -2)$?

A. A local minimum
B. A local maximum
C. A point of inflection
D. Not a stationary point

The largest pen against a wall





$A'(x) = 100 - 4x = 0$

$x = 25$: $25 \times 50 = 1250$ m^2

Maximum area 1250 m^2

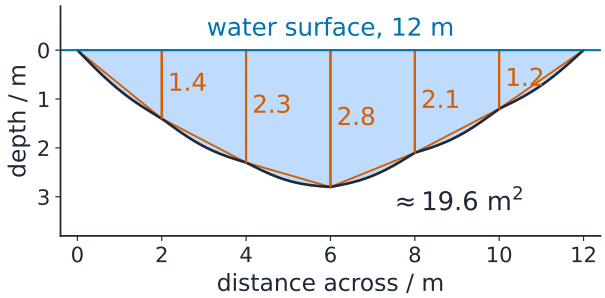
Integration

1. **Integrate.** $f(x) = 2x^3 - 2x^2 + 5x + c$
2. **Point** $(1, 6)$**.** $5 + c = 6$, so $c = 1$
3. **Area.** $\int_0^2 (4 - x^2)\,dx = \frac{16}{3}$

Decision

Write the integral you put into the GDC

The trapezoidal rule

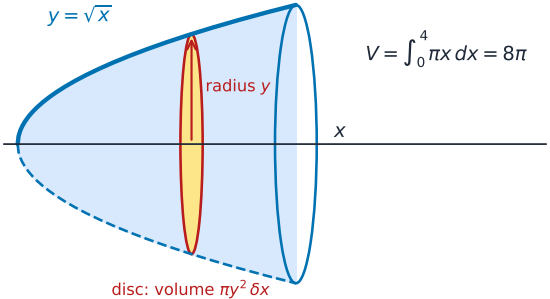


- $\frac{h}{2}\big(y_0 + y_n + 2(y_1 + \cdots + y_{n-1})\big)$.
- Concave down $\to$ underestimate.

Rules of differentiation

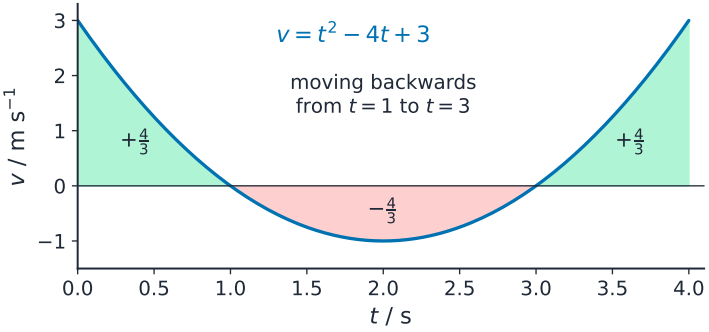
Function	Derivative
e^{3x^2}	$6xe^{3x^2}$
$x^2 \sin x$	$2x \sin x + x^2 \cos x$
$\frac{\ln x}{x}$	$\frac{1-\ln x}{x^2}$

Volumes of revolution



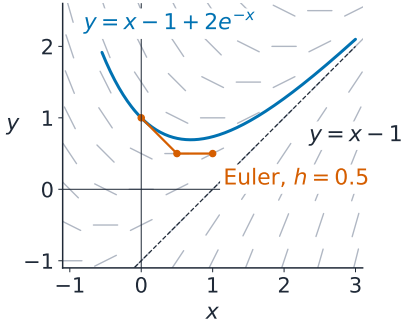
- $V = \pi \int y^2 \, dx$ about the x-axis; each thin disc has volume $\pi y^2 \, \delta x$.

Kinematics



- $v = \frac{ds}{dt}$, $a = \frac{dv}{dt}$.
- Distance $\int |v|\, dt$; **displacement** $\int v\, dt$.

Slope fields and Euler



- $y_{n+1} = y_n + h\frac{dy}{dx}$.
- Smaller h, better estimate.

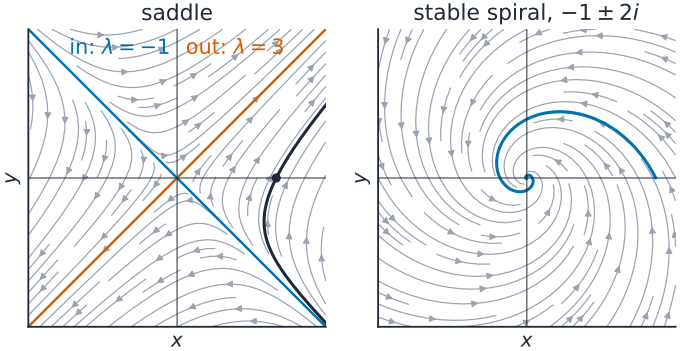
Separating variables

1. **Separate.** $\int \frac{1}{y}\,dy = \int x\,dx$
2. **Integrate.** $\ln y = \frac{x^2}{2} + c$
3. **Solve.** $y = 2e^{x^2/2}$

Decision

Check by differentiating

Phase portraits



- Eigenvalues decide the shape.
- Opposite signs $\rightarrow$ saddle; $\alpha \pm \beta i$, $\alpha \neq 0 \rightarrow$ spiral.

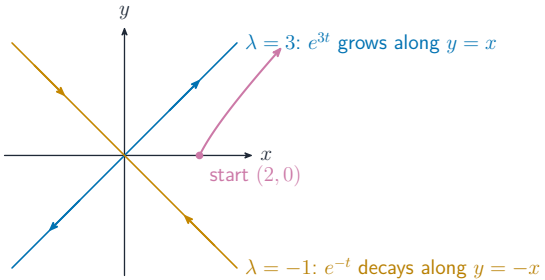
Solving a coupled system

1. **Eigen.** $\lambda = 3, -1$ with $(1,1)$, $(1,-1)$
2. **Start.** $(2,0) = (1,1) + (1,-1)$
3. **Solution.** $x = e^{3t} + e^{-t}$, $y = e^{3t} - e^{-t}$

Decision

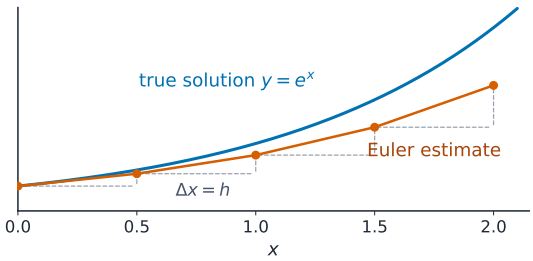
The largest eigenvalue wins in the long run

A phase portrait



A saddle: trajectories curve out towards $y = x$

Euler's method



$$\frac{dy}{dx} = y,\ y(0) = 1: \quad y_{n+1} = y_n + h\,y_n$$

Smaller steps reduce the error

Exam tips

> Write the function in one variable before you differentiate.

- Rewrite $\frac{a}{x^n}$ as ax^{-n}.
- Units for every rate of change.
- $+c$ in every indefinite integral.

Check yourself

Try these

Find and classify the stationary points of $x^3 - 3x$.

Find the curve with gradient $6x^2 - 4x + 5$ through $(1, 6)$.

Estimate the river area with the trapezoidal rule.