Geometry and trigonometry

Topic 3 · SL and HL

A map that stopped an epidemic

> Geometry measures what cannot be measured directly.

- London, 1854: cholera deaths clustered round one pump.
- Snow drew the area nearest that pump.
- This unit: 3D $\to$ triangles $\to$ sectors $\to$ Voronoi diagrams.

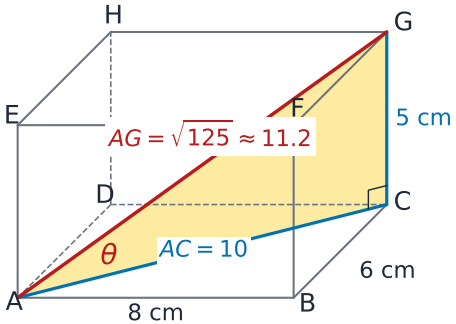
Solids

1. **Pyramid.** $V = \frac{1}{3}(36)(4) = 48$ cm^3
2. **Face height.** $\sqrt{4^2 + 3^2} = 5$ cm
3. **Surface.** $36 + 4 \times \frac{1}{2}(6)(5) = 96$ cm^2

Decision

Check curved surface or total

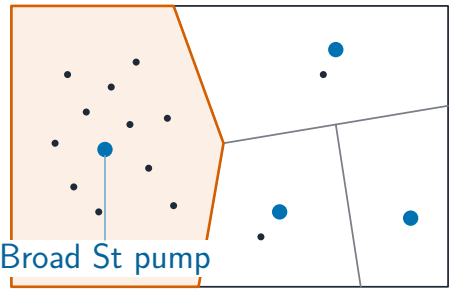
Line and plane



$\tan\theta = \frac{5}{10}$
$\theta \approx 26.6°$

- The angle is between the line and its shadow.
- $\tan\theta = \frac{5}{10}$, so $\theta \approx 26.6°$.

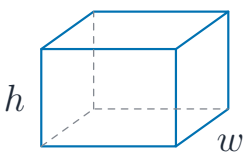
Nearest-pump cells



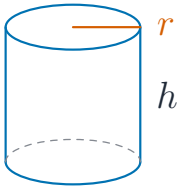
each cell: points nearest one pump; deaths (dots) cluster in one cell (schematic, not Snow's data)

A **Voronoi diagram** points to the source

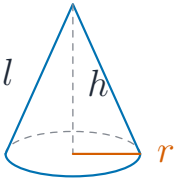
Volumes and surface areas of solids



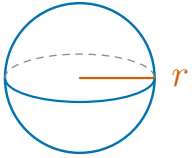
cuboid



cylinder



cone



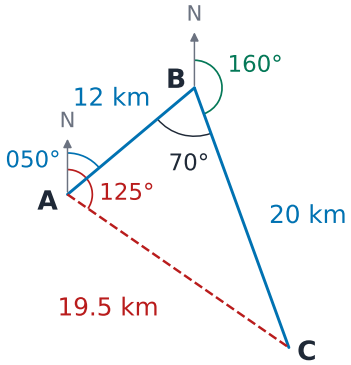
sphere

Use the formula booklet

Sine and cosine rules

Rule	Formula
Sine rule	$\frac{a}{\sin A} = \frac{b}{\sin B}$
Cosine rule	$c^2 = a^2 + b^2 - 2ab\cos C$
Area	$\frac{1}{2}ab\sin C$

Bearings



- Clockwise from north, three figures; $AC \approx 19.5$ km on a **bearing** of 125°.

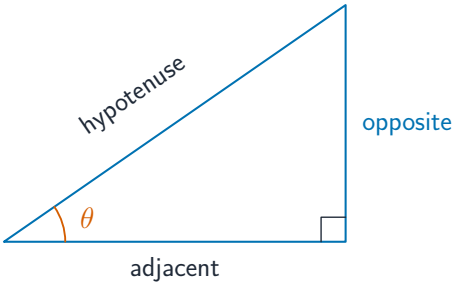
Check: which rule?

Your turn

Which rule finds the largest angle?

A. The cosine rule
B. The sine rule
C. $\frac{1}{2}ab\sin C$
D. Pythagoras' theorem

Naming the sides



SOH CAH TOA

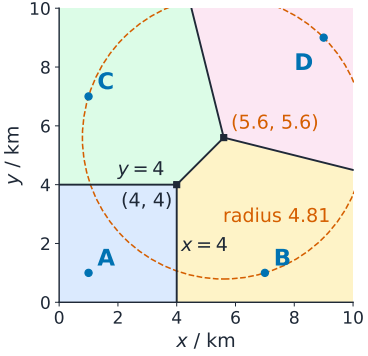
Sectors and bisectors

1. **Arc.** $\frac{40}{360} \times 30\pi \approx 10.5$ cm
2. **Area.** $\frac{40}{360} \times 225\pi \approx 78.5$ cm^2
3. **Bisector of** $(2, 1)$**,** $(6, 5)$**.** $y = -x + 7$

Decision

Every point on the bisector is equidistant

Voronoi diagrams



- Edges are perpendicular bisectors.
- A **vertex** is equidistant from three sites.

Using a Voronoi diagram

1. **Vertices.** $(4, 4)$ is 4.24 km; $(5.6, 5.6)$ is 4.81 km
2. **Boundary.** best about 4.61 km
3. **Choose.** $(5.6, 5.6)$, furthest from all wells

Decision

Compare vertices, boundary and corners

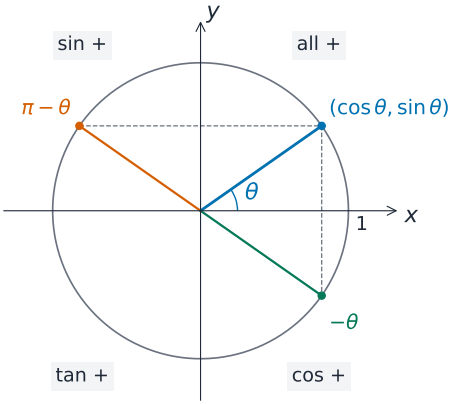
Radians and the unit circle

1. **Obtuse.** $\sin\theta = 0.6$ gives $\cos\theta = -0.8$
2. **Equation.** $\sin x = 0.5$ gives $x = \frac{\pi}{6}, \frac{5\pi}{6}$
3. **Graph.** $2\cos x = x$ at $x \approx 1.03$

Decision

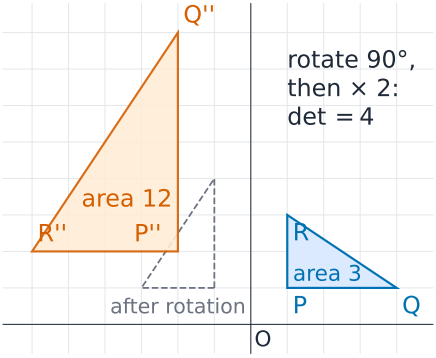
Radian mode on the GDC

Symmetry on the unit circle



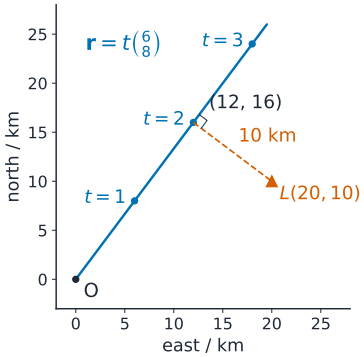
All, sin, tan, cos: which are positive

Matrix transformations



- A then B: use BA.
- Area scales by $|\det|$.

Closest approach



- $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$; speed $|\mathbf{v}|$.
- Minimise the distance squared.

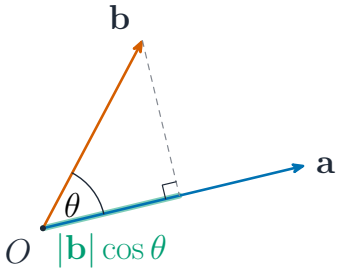
Scalar and vector products

1. **Dot.** $(2, 1, 0) \cdot (1, 3, 2) = 5$
2. **Angle.** $\cos\theta = \frac{5}{\sqrt{70}}$, $\theta \approx 53.3°$
3. **Cross.** $(2, -4, 5)$, triangle area ≈ 3.35

Decision

Zero dot product means perpendicular

The angle between vectors

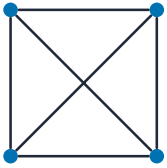


The **scalar product** measures alignment

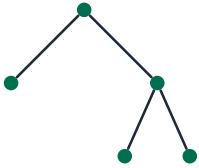
Graphs

Route	Exists when
Eulerian circuit	**connected**, all degrees even
Eulerian trail	exactly two odd vertices
Hamiltonian cycle	no simple test

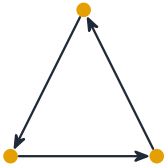
Graph vocabulary



complete K_4: 6 edges



tree: 5 vertices, 4 edges

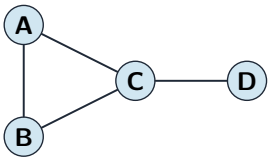


directed graph

sum of degrees $= 2\times$ number of edges

A **directed graph** has one-way edges

Counting walks with M^3

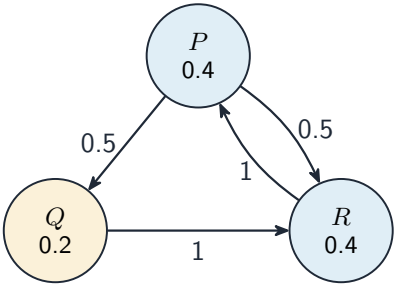


$$M = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad M^3 = \begin{pmatrix} 2 & 3 & \mathbf{4} & 1 \\ 3 & 2 & 4 & 1 \\ 4 & 4 & 2 & 3 \\ 1 & 1 & 3 & 0 \end{pmatrix}$$

Row A, column C of M^3 is 4: the four walks of length 3 are $ABAC$, $ACAC$, $ACBC$ and $ACDC$

The walks ABAC, ACAC, ACBC and ACDC

A steady state for page rank



$q = 0.5p, \quad r = 0.5p + q, \quad p = r$

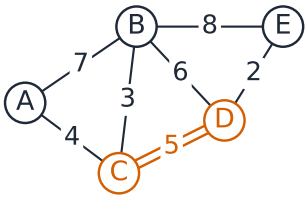
$p + q + r = 1$

steady state $(0.4,\ 0.2,\ 0.4)$

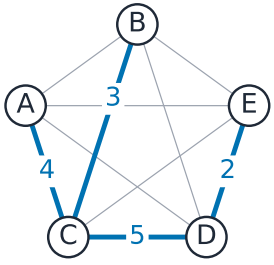
The long-run share of time on each page

Network algorithms

odd C, D: repeat CD
35 + 5 = 40



minimum spanning tree
2 + 3 + 4 + 5 = 14



- Kruskal or Prim gives the **minimum spanning tree**.
- Postman repeats the shortest route between odd vertices.

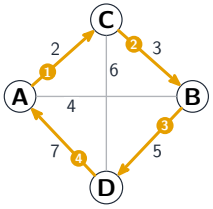
Bounds for the salesman

1. **Upper.** $ACBDEA = 26$
2. **Lower.** delete A, **tree** 10, add $4 + 7$, so 21
3. **Bounds.** $21 \le T \le 26$

Decision

State the algorithm and start **vertex**

Bounds for the travelling salesman



nearest neighbour from A: A–C–B–D–A $= 2+3+5+7 = 17$
upper bound
delete A: MST of B, C, D $= 3+5 = 8$
plus AC + AB $= 2+4$: lower bound 14
$14 \leq$ optimal tour ≤ 17

The best tour lies between 14 and 17

Exam tips

> Draw the diagram first, then calculate.

- Draw the right-angled triangle on its own.
- North line at every **bearing** point.
- Voronoi **edge** = **perpendicular bisector**.

Check yourself

Try these

Find the angle between a space diagonal of an 8 by 6 by 5 cuboid and its base.

Find the perpendicular bisector of $(2, 1)$ and $(6, 5)$.

A pizza slice has radius 15 cm and angle 40°. Find its area.