Number and algebra

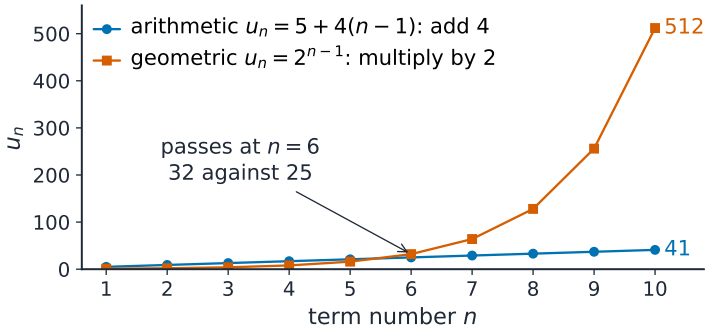
Topic 1 · SL and HL

Franklin's thousand pounds

> Interest on interest grows geometrically.

- £1000 lent at 5 % for 100 years.
- $1000 \times 1.05^{100} \approx 131\,501$.
- This unit: sequences $\to$ finance $\to$ error $\to$ logs $\to$ matrices.

Add or multiply?



- $u_n = u_1 + (n-1)d$; $u_n = u_1 r^{n-1}$.
- A p % rise means $r = 1 + \frac{p}{100}$.

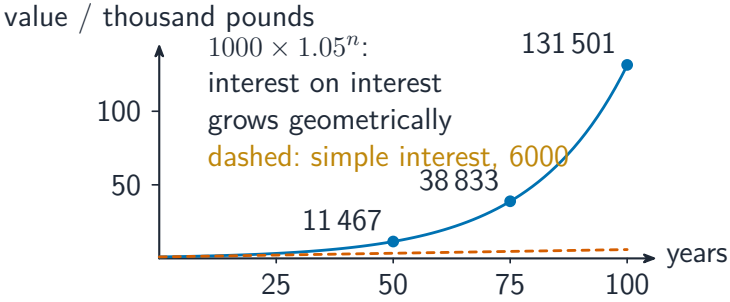
When does it pass 4000?

1. **Model.** $u = 2400 \times 1.03^n$
2. **Solve.** $2400 \times 1.03^n > 4000$ gives $n > 17.3$
3. **Answer.** 18 years

Decision

Give a whole number and check both sides

Compound growth over a century

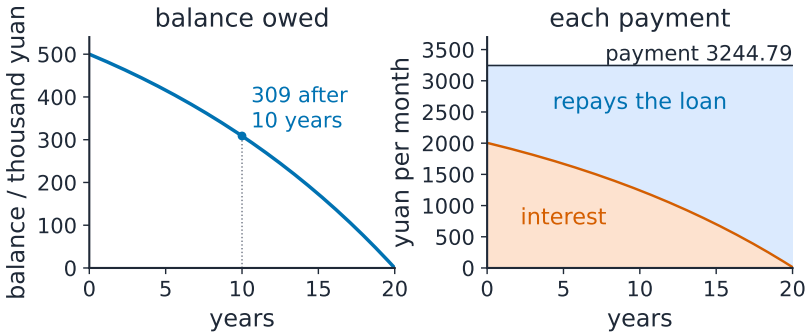


Compounding pulls away over time

The TVM solver

Value	**¥500 000 loan, 20 years**
N, $I\%$	240 payments, 4.8 %
PV, FV	500 000, 0
P/Y, C/Y	12, 12
PMT	-3244.79 a month

Amortisation



- First payment ¥2000 interest, ¥1244.79 repayment.
- Total interest about ¥278 749.

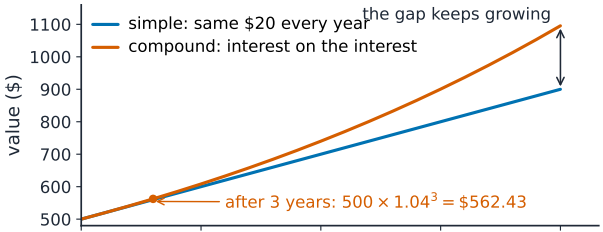
Check: which TVM sign?

Your turn

For a loan you receive and then repay, which signs are correct?

A. PV positive, PMT negative
B. PV negative, PMT negative
C. PV positive, PMT positive
D. Both zero

Simple and compound interest

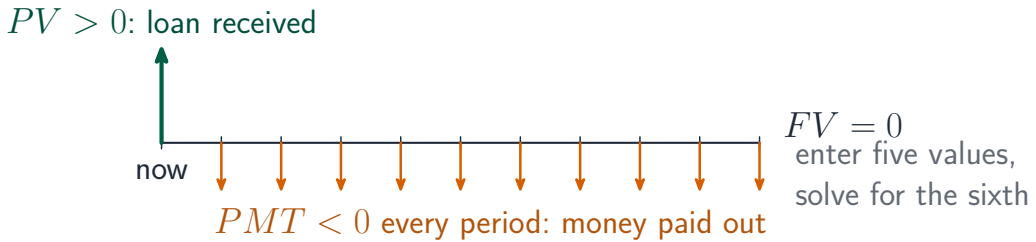


simple: $A = P(1 + rt)$

compound: $A = P(1 + r)^n$

The gap keeps growing

Signs in a finance solver



Money in is positive, money out negative

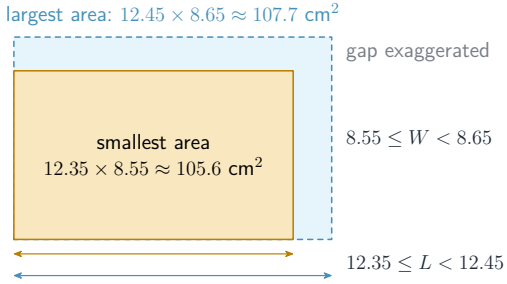
Bounds and error

1. **Lower area.** $12.35 \times 8.55 \approx 105.6\ \text{cm}^2$
2. **Upper area.** $12.45 \times 8.65 \approx 107.7\ \text{cm}^2$
3. **Error.** $\varepsilon = \left|\frac{v_A - v_E}{v_E}\right| \times 100\%$

Decision

Answers to 3 s.f., money to 2 d.p.

Bounds on an area



The true area lies between them

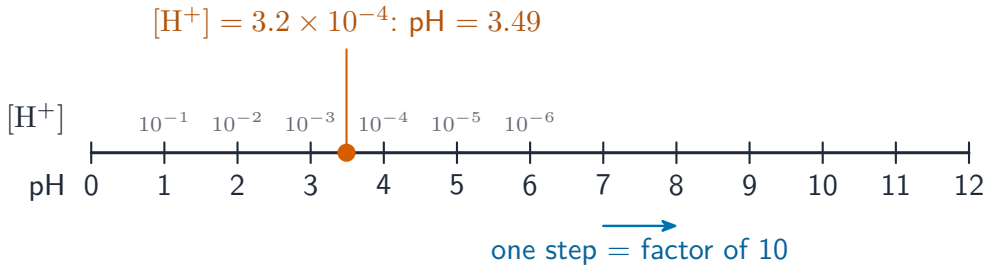
Logarithms

1. **Base 10.** $10^x = 250,\ x = \log 250 \approx 2.40$
2. **Base e.** $e^{0.2t} = 5,\ t = 5 \ln 5 \approx 8.05$
3. **pH.** $-\log(3.2 \times 10^{-4}) \approx 3.49$

Decision

Logarithms undo exponentials

The pH scale is logarithmic



One pH step is a factor of 10

Solving with technology

1. **Enter.** $2t + 3c + j = 73,\ t + 2c + 2j = 66,\ 3t + c + j = 53$
2. **Solve.** $t = 8,\ c = 14,\ j = 15$
3. **Check.** $16 + 42 + 15 = 73$

Decision

A check catches entry errors

A system from three orders

order	equation	check with $t = 8$, $c = 14$, $j = 15$
2 tea, 3 coffee, 1 juice	$2t + 3c + j = 73$	$16 + 42 + 15 = 73$ ✓
1 tea, 2 coffee, 2 juice	$t + 2c + 2j = 66$	$8 + 28 + 30 = 66$ ✓
3 tea, 1 coffee, 1 juice	$3t + c + j = 53$	$24 + 14 + 15 = 53$ ✓

Check the solution in every equation

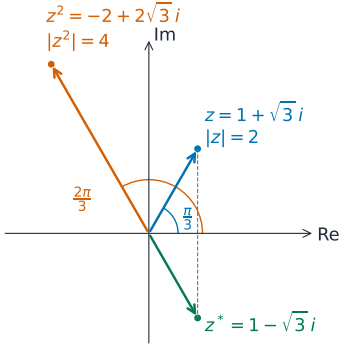
Laws of logarithms

1. **Take ln.** $(x+2)\ln 3 = x\ln 5$
2. **Collect.** $x(\ln 5 - \ln 3) = 2\ln 3$
3. **Solve.** $x \approx 4.30$

Decision

Bring the exponent down, then collect

Complex numbers



- $z = re^{i\theta}$; multiply moduli, add arguments.
- $3\sin t + 4\cos t = 5\sin(t + 0.927)$.

Inverse matrices

1. **Matrix form.** $\begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} X = \begin{pmatrix} 7 \\ 18 \end{pmatrix}$
2. **Inverse.** $A^{-1} = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}$
3. **Solve.** $X = A^{-1}B$, so $x = 3$, $y = 1$

Decision

Multiply by A^{-1} on the left

Multiplying matrices

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix}$$

$1 \times 6 + 2 \times 8 = 22$

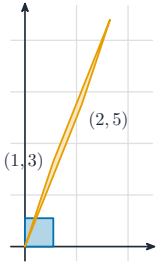
row of A $\times$ column of B

A: $2 \times \mathbf{2}$ B: $\mathbf{2} \times 2$ AB: 2×2

inner sizes must match

The inner sizes must match

The determinant as an area factor

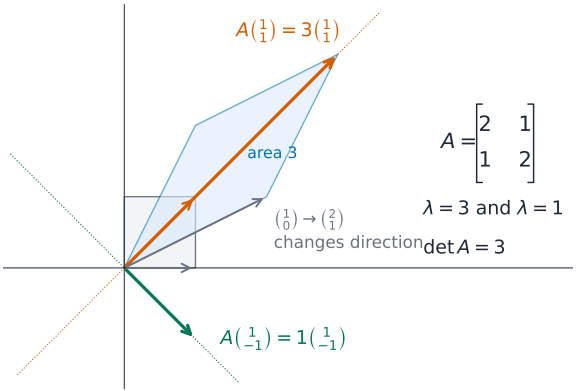


$$A = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}, \quad \det A = 2 \cdot 3 - 1 \cdot 5 = 1$$

the unit square (area 1) maps to a parallelogram of area $|\det A| = 1$; $\det A = 0$ squashes it flat: no inverse

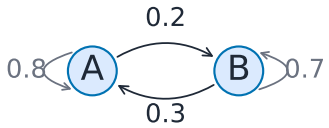
> The **determinant** scales every area

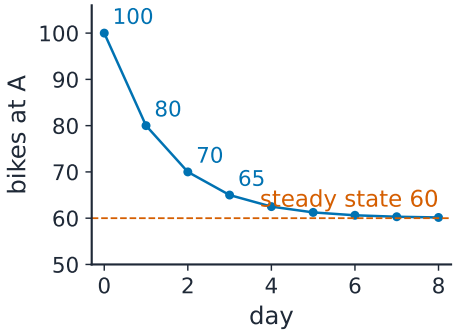
Eigenvectors



- $A\mathbf{v} = \lambda\mathbf{v}$; $\det(A - \lambda I) = 0$.

The steady state





- $A^n = PD^nP^{-1}$.
- **Eigenvalue** 1 gives the long run.

Exam tips

> Decide first whether the change adds or multiplies.

- Write down every TVM value and sign.
- Whole numbers for “how many years”.
- Never write GDC notation like 3.07E-4.

Check yourself

Try these

A car loan of ¥120 000 is repaid monthly over 5 years at 6 %. Find the payment.

Find the bounds of the area of a square of side 6.2 cm to 1 d.p.

Find the sum of the first 10 terms of $6 + 9 + 13.5 + \ldots$