Course Companion

International Baccalaureate · IB Diploma · Mathematics: Applications and Interpretation · HL

Every topic handout in one volume

全部专题讲义合集

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Number and algebra

International Baccalaureate · IB Diploma · Mathematics: Applications and Interpretation · HL

Scope and prerequisites

This handout teaches topic 1, Number and algebra, of Mathematics: Applications and Interpretation, the course first assessed in 2021: arithmetic and geometric sequences and series and their applications, compound interest, depreciation, loans and annuities with technology, standard form, approximation, bounds and percentage error, exponents and logarithms, and solving equations with technology, and, for HL, the laws of logarithms, rational exponents, complex numbers, matrices, and eigenvalues and eigenvectors. It is written for both SL and HL students. HL students also apply the further results in the sections marked AHL.

Prerequisites: percentages, negative numbers, substitution into formulae, and solving linear equations. You need a graphic display calculator (GDC) with a finance (TVM) solver and an equation solver.

Sequences and growth

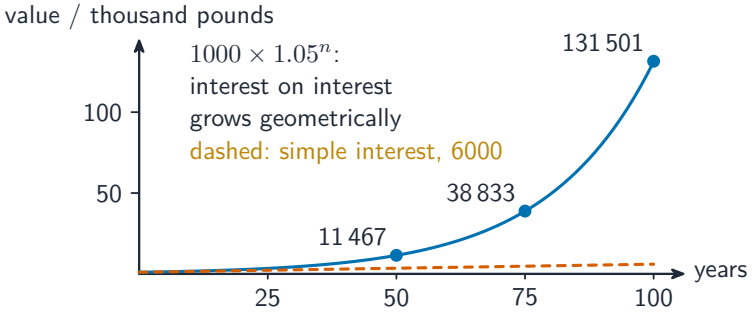
Franklin's thousand pounds

When Benjamin Franklin died in 1790, he left £1000 each to the cities of Boston and Philadelphia. The money was to be lent to young workers at 5 % interest, and the cities were not to touch it for 100 years. Franklin did the calculation himself: at 5 % a year, £1000 would grow to about £131 000 in 100 years. He was right about the arithmetic, since $1000 \times 1.05^{100} \approx 131\,501$. In reality the funds grew more slowly, because not every loan was repaid, but 200 years later they were still worth millions of dollars. Money that earns interest on its interest grows geometrically. This topic is about sequences like this one and the financial tools built from them.

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A population growing by the same percentage each year is a geometric sequence



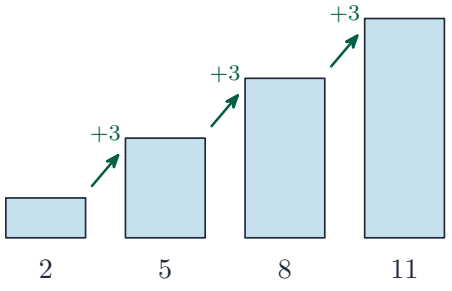
Franklin's £1000 at 5% compound interest

■ Arithmetic sequences

A **sequence** 数列 is an ordered list of numbers, its terms $u_1, u_2, u_3, \ldots$ An **arithmetic sequence** 等差数列 adds the same **common difference** 公差 d each time. The formula booklet gives

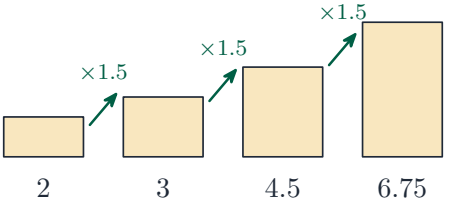
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arithmetic: add d each step



$u_n = a + (n-1)d$: equal steps

geometric: multiply by r each step



$u_n = ar^{n-1}$: growing steps

An arithmetic sequence climbs in equal steps; a geometric one multiplies by a ratio

$$u_n = u_1 + (n-1)d, \qquad S_n = \frac{n}{2}\big(2u_1 + (n-1)d\big) = \frac{n}{2}(u_1 + u_n)$$

where S_n is the **series** 级数, the sum of the first n terms.

> **Worked example.** A theatre has 20 seats in the first row and 2 more in each row behind, for 25 rows. The last row has $u_{25} = 20 + 24(2) = 68$ seats, and the theatre has $S_{25} = \frac{25}{2}(20 + 68) = 1100$ seats.

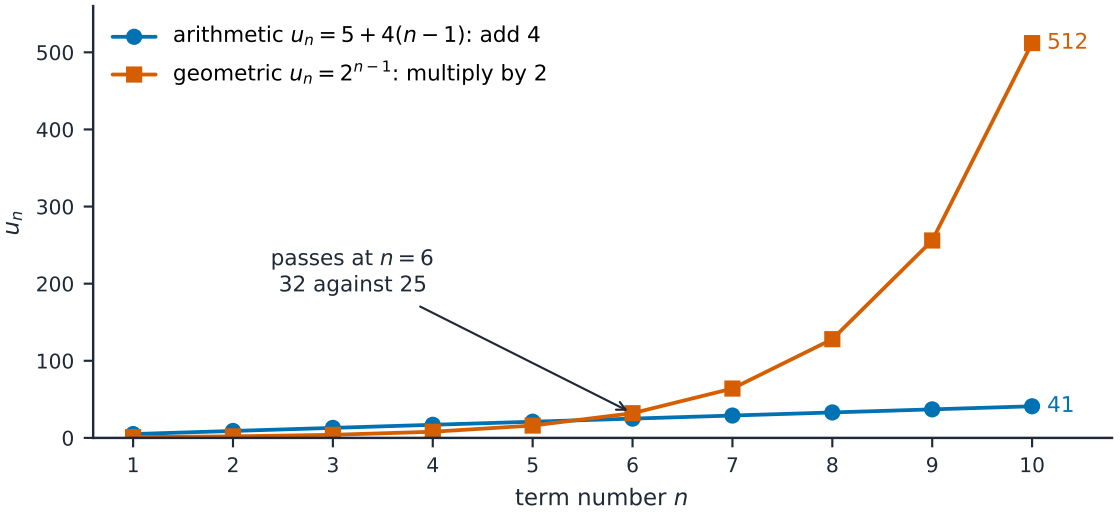
Arithmetic sequences model anything that changes by a fixed amount: simple interest, a salary with a fixed yearly rise, the cost of a taxi ride per kilometre. Their graph is a set of points on a straight line.

Geometric sequences

A **geometric sequence** 等比数列 multiplies by the same **common ratio** 公比 r each time:

$$u_n = u_1 r^{n-1}, \qquad S_n = \frac{u_1(r^n - 1)}{r - 1} = \frac{u_1(1 - r^n)}{1 - r}, \quad r \neq 1$$

A growth of p % a year means multiplying by $r = 1 + \frac{p}{100}$; a decrease of p % means $r = 1 - \frac{p}{100}$.



Adding gives a straight line; multiplying eventually wins

Worked example. A town of 2400 people grows by 3 % a year.

- After 10 years: $2400 \times 1.03^{10} \approx 3225$ people.
- It first exceeds 4000 when $2400 \times 1.03^{n} > 4000$. Use the GDC: graph $y = 2400 \times 1.03^{x}$ and $y = 4000$, or use a table, to find $n > 17.3$. So it takes 18 years.

Population models like this rarely stay geometric for long, because space and resources run out. A model is only as good as its **assumptions** 假设: here, a constant growth rate.

Financial mathematics

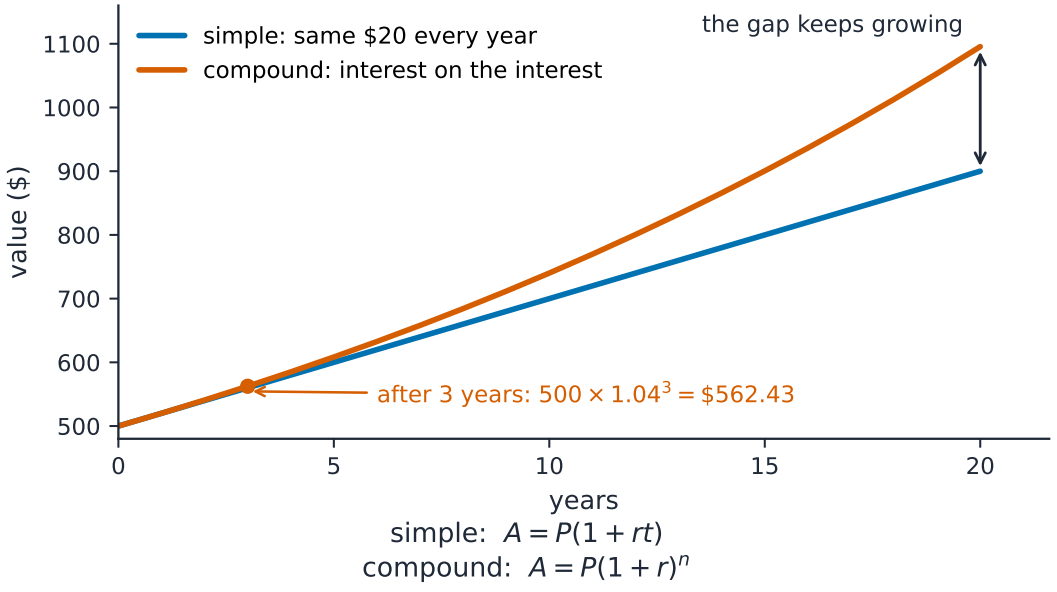
■ Compound interest and depreciation

With **compound interest** 复利 the interest is added to the balance and then earns interest itself. A present value PV invested at r % a year, compounded k times a year, for n years, becomes

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Shop discounts and interest rates are both percentage changes: multiply by a factor, again and again



Simple interest grows in a straight line; compound interest grows faster every year

$$FV = PV\left(1 + \frac{r}{100k}\right)^{kn}$$

Depreciation 折旧 is the opposite: a car or machine loses r % of its value each year, so $V = V_0\left(1 - \frac{r}{100}\right)^n$.

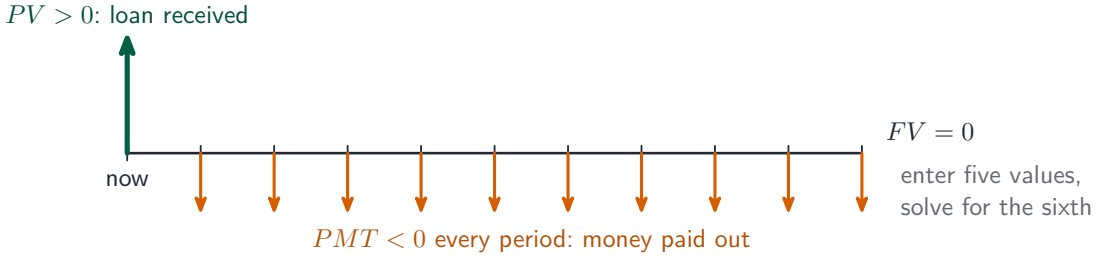
Worked example. ¥10 000 is invested at 4 % a year for 5 years. Compounded annually it becomes $10\,000 \times 1.04^5 \approx$ ¥12 166.53; compounded monthly, $10\,000\left(1+\frac{4}{1200}\right)^{60} \approx$ ¥12 209.97.

Worked example. A car bought for ¥200 000 loses 15 % of its value a year. After 4 years it is worth $200\,000 \times 0.85^4 \approx$ ¥104 401.

Inflation 通货膨胀 means prices rise, so money buys less. If savings earn 3 % while prices rise 2 %, the **real** value of the savings rises by only about 1 % a year.

The finance solver

Most financial questions in this course are answered with the GDC's finance solver, usually called the TVM (time value of money) solver. It links six values:



The finance solver's sign convention

Variable	Meaning
N	total number of payments or compounding periods
$I\%$	interest rate per year, as a percentage
PV	present value
PMT	payment each period
FV	future value
P/Y, C/Y	payments and compounding periods per year

Enter any five and solve for the sixth. **Sign convention:** money you pay out is negative, and money you receive is positive. Always write down the values you entered: they earn method marks, and a wrong sign is the most common error.

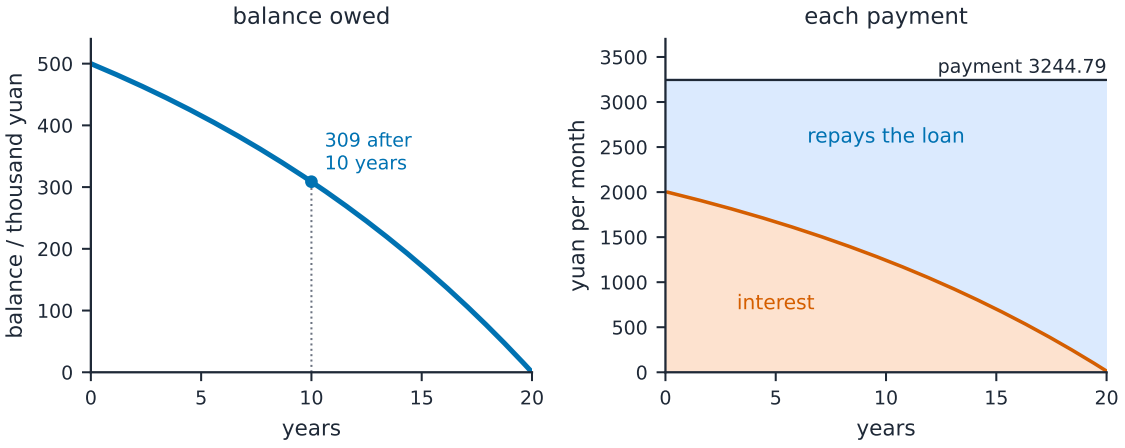
Loans and annuities

A loan repaid by equal regular payments is **amortised** 分期偿还: each payment covers the interest due and repays part of the loan.

Worked example. A family borrows ¥500 000 to buy a flat at 4.8 % a year, compounded monthly, repaid monthly over 20 years.

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- TVM: $N = 240$, $I\% = 4.8$, $PV = 500\,000$, $FV = 0$, $P/Y = C/Y = 12$. Solve for $PMT \approx -3244.79$: a payment of ¥3244.79 a month.
- Total paid: $240 \times 3244.79 \approx$ ¥778 749, so the interest is about ¥278 749.
- The first month's interest is $500\,000 \times 0.004 =$ ¥2000, so only ¥1244.79 of the first payment repays the loan.



Early payments are mostly interest

An **annuity** 年金 is a series of equal regular payments. Saving ¥2000 a month at 3 % a year, compounded monthly, for 10 years gives (TVM with $PV = 0$, $PMT = -2000$, $N = 120$) a future value of about ¥279 483, of which ¥240 000 was paid in.

Approximation and error

Standard form and rounding

A number in **standard form** 科学记数法 is written $a \times 10^k$, where $1 \le a < 10$ and k is an integer: $3\,600\,000 = 3.6 \times 10^6$ and $0.000\,307 = 3.07 \times 10^{-4}$. Your GDC may show 3.07E−4, but you must write 3.07×10^{-4}.



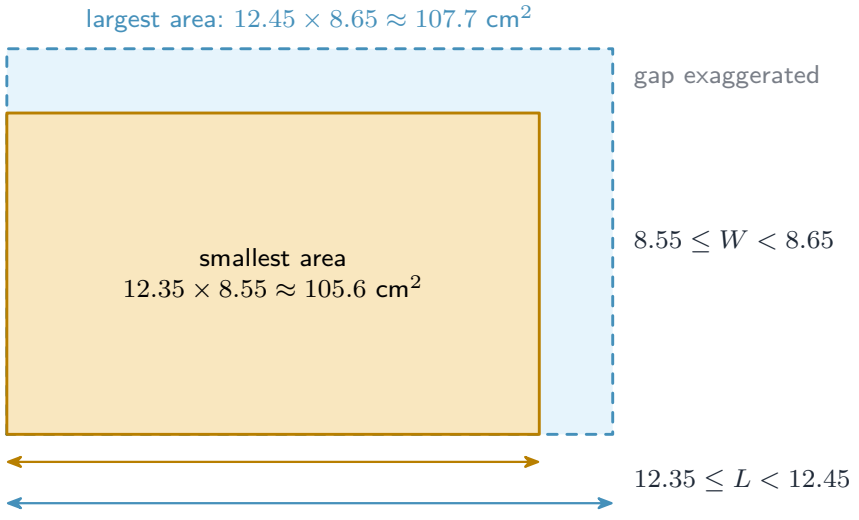
Vernier calipers read to 0.02 mm: every measurement carries an upper and lower bound

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Round to **decimal places** (digits after the point) or to **significant figures** 有效数字 (digits from the first non-zero digit). In IB exams, give final answers exactly or to 3 significant figures unless the question says otherwise. Keep full calculator values in the working: rounding too early is a common way to lose accuracy marks.

Bounds

A measurement rounded to a given accuracy lies between a **lower bound** 下界 and an **upper bound** 上界. A length of 12.4 cm, to 1 decimal place, lies in $12.35 \leq L < 12.45$.



Lower bounds give the smallest area, upper bounds the largest.

> **Worked example.** A rectangle is 12.4 cm by 8.6 cm, both to 1 decimal place. The smallest possible area is $12.35 \times 8.55 \approx 105.6$ cm^2, and the largest is $12.45 \times 8.65 \approx 107.7$ cm^2.

For a difference or a quotient, think about which bounds make it largest: the largest value of $a - b$ uses the upper bound of a and the lower bound of b.

Percentage error

When an approximate value v_A is used for an exact value v_E,

$$\varepsilon = \left| \frac{v_A - v_E}{v_E} \right| \times 100\%$$

> **Worked example.** Using $\frac{22}{7}$ for π gives $\varepsilon = \left| \dfrac{3.142857 - 3.141593}{3.141593} \right| \times 100\% \approx 0.0402\%$.

Divide by the **exact** value, not the approximate one.

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Exponents and logarithms

■ Laws of exponents

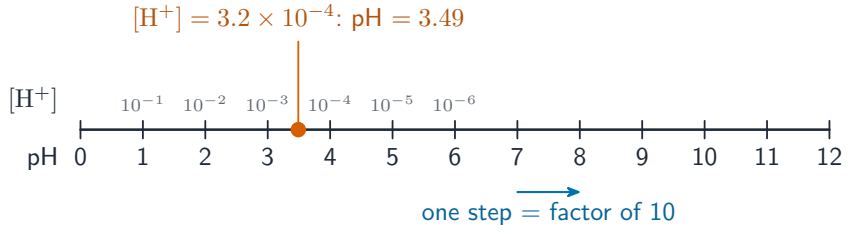
For integer exponents:

$$a^m \times a^n = a^{m+n}, \qquad \frac{a^m}{a^n} = a^{m-n}, \qquad (a^m)^n = a^{mn}, \qquad a^0 = 1, \qquad a^{-n} = \frac{1}{a^n}$$

For example, $\dfrac{(3x^2y)^3}{9x^4y} = \dfrac{27x^6y^3}{9x^4y} = 3x^2y^2$.

■ Logarithms

A **logarithm** 对数 answers the question "what power?":



A logarithmic scale: pH

$$a^x = b \iff x = \log_a b$$

Your GDC has two bases: $\log x$ means $\log_{10} x$, and the **natural logarithm** 自然对数 $\ln x$ means $\log_e x$, where $e \approx 2.718$. So $\log 1000 = 3$ and $\ln e^2 = 2$.

> **Worked example.** Solve $10^x = 250$: $x = \log 250 \approx 2.40$. Solve $e^{0.2t} = 5$: $0.2t = \ln 5$, so $t = 5 \ln 5 \approx 8.05$.

Logarithmic scales compress huge ranges of numbers. The acidity of a solution is $\text{pH} = -\log[\text{H}^+]$: when $[\text{H}^+] = 3.2 \times 10^{-4}$, the pH is $-\log(3.2 \times 10^{-4}) \approx 3.49$. A change of 1 in pH means a factor of 10 in $[\text{H}^+]$.

Equations with technology

■ Systems of linear equations

The GDC's equation solver solves systems of up to three linear equations in three unknowns. Write the equations in the standard form $ax + by + cz = d$ first, and write down the system you entered.

order	equation	check with $t=8$, $c=14$, $j=15$
2 tea, 3 coffee, 1 juice	$2t+3c+j=73$	$16+42+15=73$ ✓
1 tea, 2 coffee, 2 juice	$t+2c+2j=66$	$8+28+30=66$ ✓
3 tea, 1 coffee, 1 juice	$3t+c+j=53$	$24+14+15=53$ ✓

Write the system, solve, then check every equation

Worked example. A café sells tea, coffee and juice. Order 1 (2 teas, 3 coffees, 1 juice) costs ¥73; order 2 (1 tea, 2 coffees, 2 juices) costs ¥66; order 3 (3 teas, 1 coffee, 1 juice) costs ¥53. With t, c and j the prices in yuan:

$$2t+3c+j=73, \qquad t+2c+2j=66, \qquad 3t+c+j=53$$

The solver gives $t=8$, $c=14$, $j=15$. Check in every equation: $16+42+15=73$, $8+28+30=66$ and $24+14+15=53$. Substituting back catches entry errors, which are the most common mistake with the solver.

Polynomial equations

The polynomial solver finds the real (and, at HL, complex) roots of polynomial equations.

Worked example. $x^3-2x^2-5x+6=0$ has roots $x=-2$, 1 and 3. Check: $(x+2)(x-1)(x-3)$ expands to the original polynomial.

For other equations, such as $2^x=x+3$, graph both sides and use the intersection tool.

AHL: Laws of logarithms and exponents

Laws of logarithms

$$\log_a xy=\log_a x+\log_a y, \qquad \log_a\frac{x}{y}=\log_a x-\log_a y, \qquad \log_a x^m=m\log_a x$$

with change of base $\log_a x=\dfrac{\ln x}{\ln a}$.

Worked example. $\log_2 48-\log_2 3=\log_2 16=4$, and $3\ln x-\ln x^2=\ln\dfrac{x^3}{x^2}=\ln x$.

Worked example. Solve $3^{x+2}=5^x$: $(x+2)\ln 3=x\ln 5$, so $x=\dfrac{2\ln 3}{\ln 5-\ln 3}\approx 4.30$.

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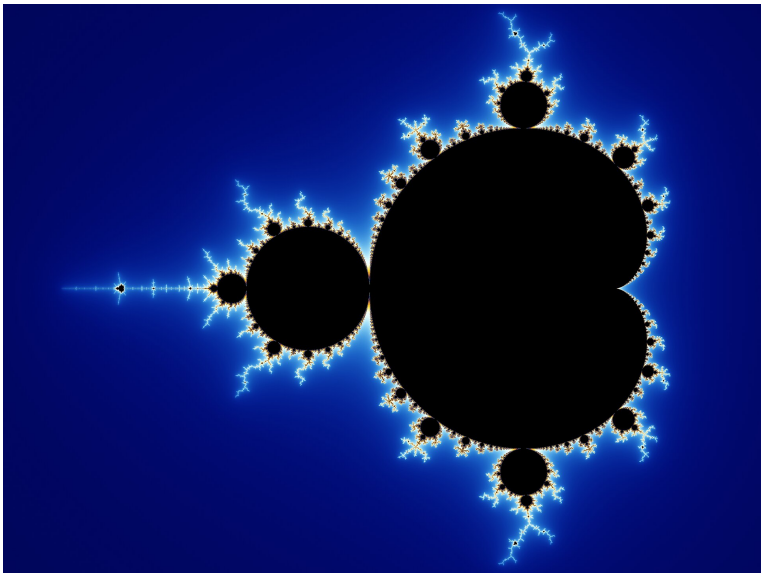
Rational exponents

$a^{\frac{1}{n}} = \sqrt[n]{a}$ and $a^{\frac{m}{n}} = (\sqrt[n]{a})^m$. For example, $8^{\frac{2}{3}} = 4$ and $(27x^6)^{\frac{2}{3}} = 9x^4$.

AHL: Complex numbers

Cartesian, polar and exponential form

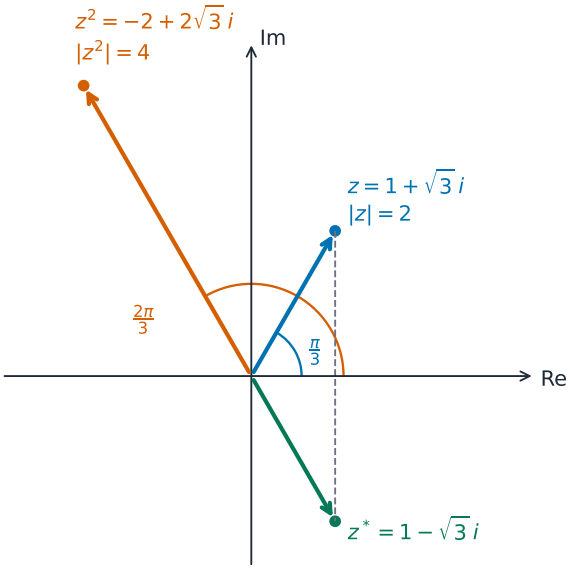
A **complex number** 复数 is $z = a + bi$, where $i^2 = -1$; a is the real part and b the imaginary part. On an **Argand diagram** 阿尔冈图 it is the point (a, b). Its **modulus** 模 is $r = |z| = \sqrt{a^2 + b^2}$, and its **argument** 辐角 θ is the angle from the positive real axis:



The Mandelbrot set: the complex numbers c for which repeating $z \to z^2 + c$, starting from $z = 0$, stays bounded

$$z = a + bi = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

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Multiplying multiplies moduli and adds arguments

- Add and subtract in Cartesian form.
- Multiply and divide in polar form: multiply (divide) the moduli and add (subtract) the arguments. So z^2 has twice the argument of z.
- The **complex conjugate** 共轭复数 of $a + bi$ is $a - bi$.

> **Worked example.** $z = 1 + \sqrt{3}\,i = 2e^{\frac{\pi}{3}i}$ and $w = 2i = 2e^{\frac{\pi}{2}i}$. Then $zw = 4e^{\frac{5\pi}{6}i}$ and $\dfrac{z}{w} = e^{-\frac{\pi}{6}i}$. Sketch each point first to choose the quadrant for the argument.

Adding waves

Complex numbers make it easy to add sinusoidal functions of the same frequency, as in alternating current. Since $\sin t$ is the imaginary part of e^{it},

$$3\sin t + 4\cos t = \operatorname{Im}\big((3+4i)e^{it}\big)$$

and $3+4i = 5e^{0.927i}$, so $3\sin t + 4\cos t = 5\sin(t+0.927)$: a single wave with amplitude 5 and phase shift 0.927.

AHL: Matrices

Matrix arithmetic

A **matrix** 矩阵 is a rectangular array of numbers. An $m \times n$ matrix has m rows and n columns.

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$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} \quad = \quad \begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix}$$

$1 \times 6 + 2 \times 8 = 22$

row of A $\times$ column of B

A: $2 \times \mathbf{2}$ $\quad$ B: $\mathbf{2} \times 2$ $\quad$ AB: 2×2

inner sizes must match

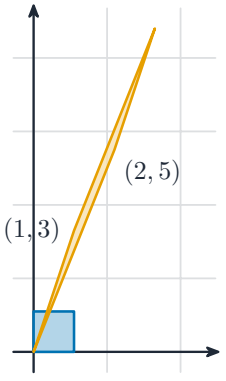
Each entry of AB is a row of A times a column of B; the inner sizes must match.

- Add and subtract matrices of the same size entry by entry.
- To multiply AB, each entry is a row of A times a column of B; the number of columns of A must equal the number of rows of B. In general $AB \neq BA$.
- The **identity matrix** 单位矩阵 I has 1s on the diagonal and 0s elsewhere, and $AI = IA = A$.

Worked example. $\begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 3-4 & 0+2 \\ 1-8 & 0+4 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -7 & 4 \end{pmatrix}$.

Determinant and inverse

For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the **determinant** 行列式 is $\det A = ad - bc$, and if $\det A \neq 0$ the **inverse matrix** 逆矩阵 is



$A = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$, $\det A = 2 \cdot 3 - 1 \cdot 5 = 1$

the unit square (area 1) maps to a parallelogram of area $|\det A| = 1$; $\det A = 0$ squashes it flat: no inverse

The determinant is the area scale factor

$$A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \qquad AA^{-1} = A^{-1}A = I$$

Use the GDC for larger matrices. A matrix with determinant 0 has no inverse.

Worked example. Solve $2x + y = 7$, $5x + 3y = 18$. In matrix form $AX = B$ with $A = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$, $\det A = 1$, so

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$$X = A^{-1}B = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}\begin{pmatrix} 7 \\ 18 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Multiply by A^{-1} on the left: $X = A^{-1}B$, not BA^{-1}.

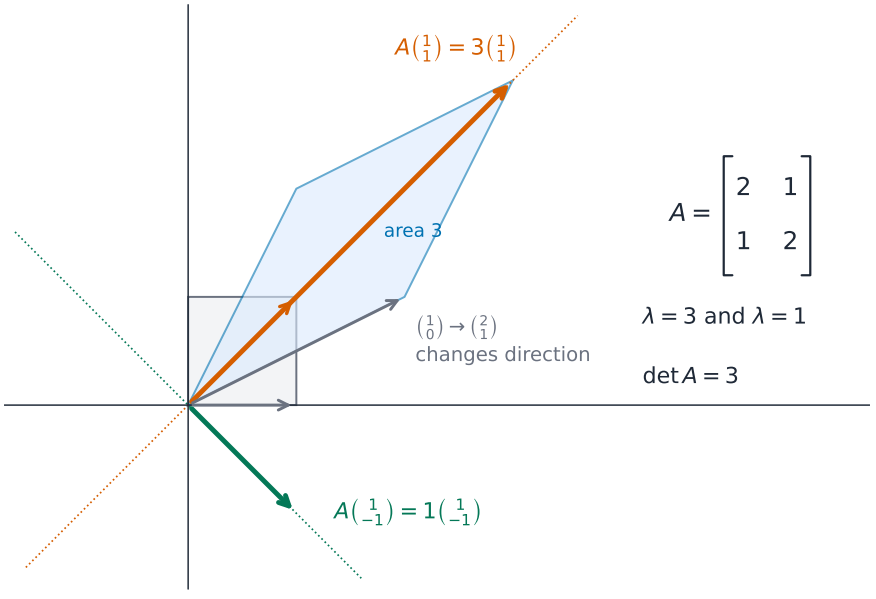
AHL: Eigenvalues and eigenvectors

What an eigenvector is

Most vectors change direction when multiplied by a matrix. An **eigenvector** 特征向量 $\mathbf{v}$ only stretches: $A\mathbf{v} = \lambda\mathbf{v}$, where the number λ is its **eigenvalue** 特征值. The eigenvalues solve the **characteristic polynomial** 特征多项式

$$\det(A - \lambda I) = 0$$

Worked example. For $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$: $(2-\lambda)^2 - 1 = 0$, so $\lambda^2 - 4\lambda + 3 = 0$ and $\lambda = 3$ or $\lambda = 1$. For $\lambda = 3$, $(A - 3I)\mathbf{v} = \mathbf{0}$ gives $-x + y = 0$, so $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. For $\lambda = 1$, $x + y = 0$, so $\mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.



Eigenvectors stay on their own line

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■ Diagonalisation and powers

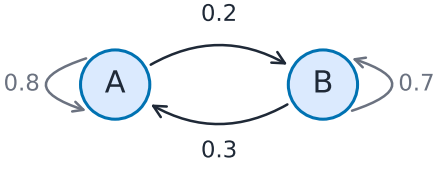
If P has the eigenvectors as its columns and D the eigenvalues on its diagonal, then $A = PDP^{-1}$ and

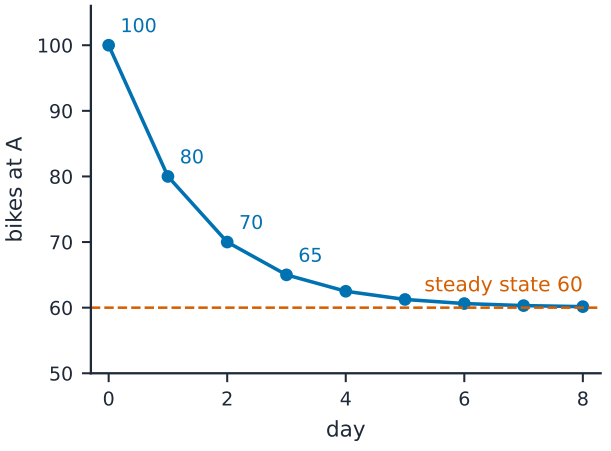
$$A^n = PD^nP^{-1}$$

Powers of D are easy: raise each diagonal entry to the power n. This explains what happens in the long run.

> **Worked example.** A bike-share scheme has two stations. Each day, 80 % of the bikes at A stay and 20 % move to B; 70 % of those at B stay and 30 % move to A. With $\mathbf{s}_n$ the numbers at A and B after n days, $\mathbf{s}_{n+1} = T\mathbf{s}_n$ where $T = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix}$.

- Eigenvalues: $\lambda^2 - 1.5\lambda + 0.5 = 0$, so $\lambda = 1$ or 0.5, with eigenvectors $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.
- Start with 100 bikes at A: $\begin{pmatrix} 100 \\ 0 \end{pmatrix} = 20\begin{pmatrix} 3 \\ 2 \end{pmatrix} + 40\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.
- So $\mathbf{s}_n = 20\begin{pmatrix} 3 \\ 2 \end{pmatrix} + 40(0.5)^n\begin{pmatrix} 1 \\ -1 \end{pmatrix}$: the number at A is $60 + 40(0.5)^n$, giving 100, 80, 70, 65, …
- As n grows, $(0.5)^n \to 0$: in the long run 60 bikes are at A and 40 at B, the **steady state** 稳态.





The eigenvalue 1 gives the steady state; the eigenvalue 0.5 sets how fast it is reached

Common mistakes.

- Taking the measured value as an upper bound: 3.5 cm to the nearest 0.1 cm lies between 3.45 cm and 3.55 cm.

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- Dividing by the approximate value in a percentage error. Use $\left|\dfrac{v_A - v_E}{v_E}\right| \times 100\%$.
- Writing 34×10^3 as standard form. The number in front must satisfy $1 \le a < 10$: 3.4×10^4.
- Using $u_n = u_1 + nd$ for an arithmetic sequence. It is $u_n = u_1 + (n-1)d$.
- Giving PV and FV the same sign in the finance solver. Money paid out and money received have opposite signs.
- Setting N to the number of years when interest is paid monthly. N counts periods, so 5 years is $N = 60$.
- Rounding in the middle of a calculation. Keep full calculator values and round only the final answer.
- **HL:** writing $\log(a+b) = \log a + \log b$. The product law needs a product.
- **HL:** assuming $\mathbf{AB} = \mathbf{BA}$, or writing $(\mathbf{AB})^{-1} = \mathbf{A}^{-1}\mathbf{B}^{-1}$. The inverse is $\mathbf{B}^{-1}\mathbf{A}^{-1}$.
- **HL:** looking for an inverse when $\det \mathbf{M} = 0$. Such a matrix has no inverse.
- **HL:** writing $\mathbf{M}^n = \mathbf{P}^{-1}\mathbf{D}^n\mathbf{P}$. With $\mathbf{M} = \mathbf{PDP}^{-1}$, the power is $\mathbf{PD}^n\mathbf{P}^{-1}$.

Practice

1. Write 0.000 307 in standard form, and find $(2.4 \times 10^6) \times (5 \times 10^{-4})$ in standard form.
2. The side of a square is 6.2 cm, correct to 1 decimal place. Find the lower and upper bounds of its area.
3. A student measures g as 9.6 m/s^2. The accepted value is 9.81 m/s^2. Find the percentage error.
4. Kai saves ¥50 in week 1, ¥55 in week 2, ¥60 in week 3, and so on. Find the amount saved in week 20 and the total after 20 weeks.
5. A town of 2400 people grows by 3 % a year. Find the population after 10 years, and the number of complete years until it first exceeds 4000.
6. Find the sum of the first 10 terms of $6 + 9 + 13.5 + \ldots$
7. ¥8000 is invested at 2.5 % a year, compounded monthly. Find its value after 6 years.
8. A machine costs ¥60 000 and depreciates by 12 % a year. Find its value after 5 years.
9. A car loan of ¥120 000 is repaid monthly over 5 years at 6 % a year, compounded monthly. Find the monthly payment and the total interest paid.
10. Wei saves ¥1500 a month for 15 years at 3.6 % a year, compounded monthly. Find the final value, and how much of it is interest.
11. Use technology to solve $2x + 3y - z = 5$, $x - y + 2z = 5$, $3x + y + z = 9$.
12. Use technology to solve $x^3 - 2x^2 - 5x + 6 = 0$.
13. Solve $10^x = 250$ and $e^{0.2t} = 5$.

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14. Find the pH of a solution with $[\mathrm{H}^+] = 3.2 \times 10^{-4}$, using $\mathrm{pH} = -\log[\mathrm{H}^+]$.
15. **AHL.** Write $\log_2 48 - \log_2 3$ as a number, and $3\ln x - \ln x^2$ as a single logarithm.
16. **AHL.** Simplify $(27x^6)^{\frac{2}{3}}$.
17. **AHL.** $z = 1 + \sqrt{3}\,i$ and $w = 2i$. Write both in exponential form, and find zw and $\dfrac{z}{w}$ in exponential form.
18. **AHL.** Write $5\sin t + 12\cos t$ in the form $R\sin(t + \alpha)$.
19. **AHL.** $A = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$. Find AB, $\det A$ and A^{-1}.
20. **AHL.** Use A^{-1} from question 19 to solve $3x + 2y = 12$, $x + 4y = 14$.
21. **AHL.** Find the eigenvalues and eigenvectors of $\begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$.
22. **AHL.** Each year 90 % of a region's town dwellers stay in the town and 10 % move to the country; 20 % of country dwellers move to the town and 80 % stay. Find the long-term proportion living in the town.

Answers.

1. 3.07×10^{-4}; 1.2×10^{3}.
2. $6.15^2 = 37.8225$ cm^2 and $6.25^2 = 39.0625$ cm^2, so $37.8225 \le A < 39.0625$ cm^2.
3. $\varepsilon = \dfrac{|9.6 - 9.81|}{9.81} \times 100\% \approx 2.14\%$.
4. $u_{20} = 50 + 19(5) = ¥145$; $S_{20} = 10(50 + 145) = ¥1950$.
5. $2400 \times 1.03^{10} \approx 3225$; $1.03^n > \frac{5}{3}$ gives $n > 17.3$, so 18 years.
6. $r = 1.5$: $S_{10} = \dfrac{6(1.5^{10} - 1)}{0.5} \approx 680$.
7. $8000\left(1 + \frac{2.5}{1200}\right)^{72} \approx ¥9293.22$.
8. $60\,000 \times 0.88^5 \approx ¥31\,663.92$.
9. TVM: $N = 60$, $I\% = 6$, $PV = 120\,000$, $FV = 0$, $P/Y = C/Y = 12$: $PMT \approx ¥2319.94$. Interest $= 60 \times 2319.94 - 120\,000 \approx ¥19\,196$.
10. TVM: $N = 180$, $I\% = 3.6$, $PV = 0$, $PMT = -1500$, $P/Y = C/Y = 12$: $FV \approx ¥357\,310$. Paid in $180 \times 1500 = ¥270\,000$, so the interest is about ¥87 310.
11. $x = 2$, $y = 1$, $z = 2$.
12. $x = -2,\ 1,\ 3$.
13. $x = \log 250 \approx 2.40$; $t = 5\ln 5 \approx 8.05$.
14. $\mathrm{pH} = -\log(3.2 \times 10^{-4}) \approx 3.49$.
15. **AHL.** $\log_2 16 = 4$; $\ln \dfrac{x^3}{x^2} = \ln x$.
16. **AHL.** $27^{\frac{2}{3}} x^{6\times\frac{2}{3}} = 9x^4$.
17. **AHL.** $z = 2e^{\frac{\pi}{3}i}$, $w = 2e^{\frac{\pi}{2}i}$; $zw = 4e^{\frac{5\pi}{6}i}$; $\dfrac{z}{w} = e^{-\frac{\pi}{6}i}$.
18. **AHL.** $5 + 12i = 13e^{1.176i}$, so $5\sin t + 12\cos t = 13\sin(t + 1.18)$.

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19. **AHL.** $AB = \begin{pmatrix} -1 & 2 \\ -7 & 4 \end{pmatrix}$; $\det A = 12 - 2 = 10$; $A^{-1} = \frac{1}{10}\begin{pmatrix} 4 & -2 \\ -1 & 3 \end{pmatrix}$.

20. **AHL.** $\frac{1}{10}\begin{pmatrix} 4 & -2 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 12 \\ 14 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 20 \\ 30 \end{pmatrix}$, so $x = 2$, $y = 3$.

21. **AHL.** $\lambda^2 - 7\lambda + 10 = 0$, so $\lambda = 5$ with eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and $\lambda = 2$ with eigenvector $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

22. **AHL.** $T = \begin{pmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{pmatrix}$. The steady state satisfies $0.1x = 0.2y$, so $x : y = 2 : 1$: two thirds live in the town.

Exam tips

- Write down every value you enter in the TVM solver, and check the sign of each payment.
- Give answers to 3 significant figures unless told otherwise, but money to 2 decimal places.
- For "how many years" questions, give a whole number and check the values either side.
- In percentage error, divide by the exact value.
- Write 3.07×10^{-4}, never the GDC's 3.07E−4.
- Substitute solutions back into every equation of a system to catch entry errors.
- **HL:** sketch a complex number before finding its argument.
- **HL:** for a long-run question with a transition matrix, find the eigenvector for eigenvalue 1.

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Functions

International Baccalaureate · IB Diploma · Mathematics: Applications and Interpretation · HL

Scope and prerequisites

This handout teaches topic 2, Functions, of Mathematics: Applications and Interpretation, the course first assessed in 2021: equations of straight lines, parallel and perpendicular lines, functions, domain, range and inverse, key features of graphs found with technology, and modelling with linear, quadratic, exponential, direct and inverse variation, cubic and sinusoidal functions, and, for HL, composite and inverse functions, transformations of graphs, models with limits such as Newton's law of cooling and logistic growth, and straightening data with logarithmic scales. It is written for both SL and HL students. HL students also apply the further results in the sections marked AHL.

Prerequisites: substitution into formulae, solving linear equations, plotting points, and topic 1 (sequences, exponents and logarithms).

Straight lines and linear models

A law that held for fifty years

In 1965 the engineer Gordon Moore noticed that the number of components on a computer chip had been doubling every year, and he predicted that it would carry on. In 1975 he revised the rate to doubling about every two years. "Moore's law" is not a law of nature, only a **model** 模型: a function chosen to describe data and make predictions. It held remarkably well for decades and guided the planning of the whole chip industry, until the components became so small that physical limits began to slow it. This topic is about choosing such functions, using them, and judging how far to trust them.

Gradient and the equation of a line

The **gradient** 斜率 of the line through (x_1, y_1) and (x_2, y_2) is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{rise}}{\text{run}}$$

A line can be written in three forms:

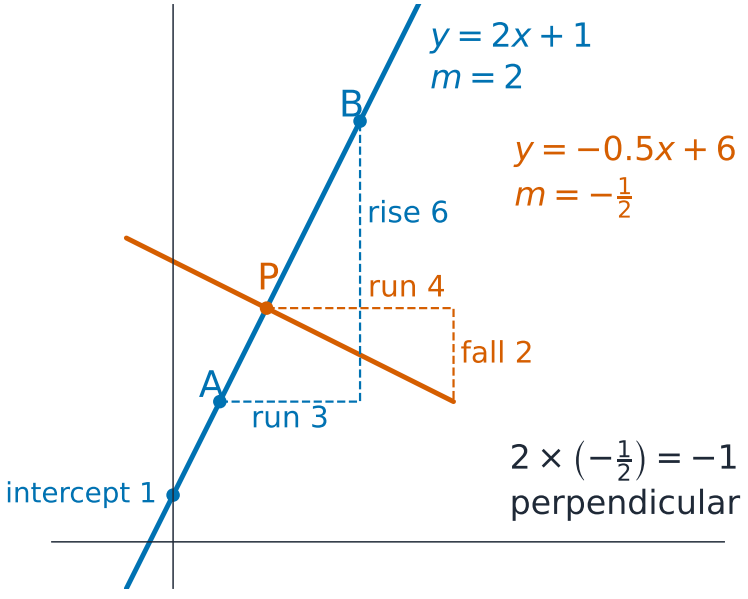
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Form	Equation	Use it when
Gradient-intercept	$y = mx + c$	you know m and the y-intercept c
Point-gradient	$y - y_1 = m(x - x_1)$	you know m and one point
General	$ax + by + d = 0$	the question asks for integer coefficients

Parallel 平行 lines have equal gradients. **Perpendicular** 垂直 lines have gradients that multiply to -1: if one gradient is m, the other is $-\frac{1}{m}$.

> **Worked example.** Find the line through $A(1, 3)$ and $B(4, 9)$, and the line through $P(2, 5)$ perpendicular to it.

- $m = \frac{9-3}{4-1} = 2$, so $y - 3 = 2(x - 1)$, that is $y = 2x + 1$, or $2x - y + 1 = 0$.
- The perpendicular gradient is $-\frac{1}{2}$, so $y - 5 = -\frac{1}{2}(x - 2)$, that is $y = -0.5x + 6$.

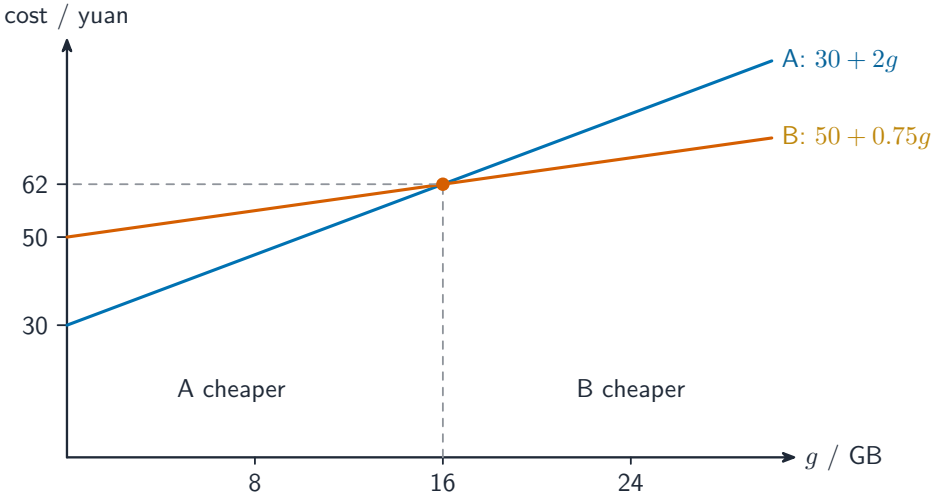


Perpendicular gradients multiply to -1

Linear models

A **linear model** 线性模型 $y = mx + c$ describes a quantity that changes at a constant rate. The gradient is the rate of change, and the intercept is the value when $x = 0$.

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Two linear models cross at 16 GB

Worked example. A plumber charges $C = 120 + 85h$ yuan for a job lasting h hours. The gradient 85 is the charge per hour, and the intercept 120 is a fixed call-out charge. A 3.5-hour job costs $120 + 85(3.5) = ¥417.50$.

Worked example. Phone plan A costs $30 + 2g$ yuan a month for g GB of data, and plan B costs $50 + 0.75g$. They cost the same when $30 + 2g = 50 + 0.75g$, so $g = 16$ GB. Plan A is cheaper below 16 GB and plan B above it. This is the same problem as finding where two lines intersect, which a GDC does directly.

Functions and their graphs

Domain, range and notation

A **function** 函数 gives exactly one output for each input. In $f(x) = 2x - 5$, x is the input and $f(x)$ the output.

- The **domain** 定义域 is the set of allowed inputs. In a model it is usually limited by the context: a time cannot be negative.
- The **range** 值域 is the set of outputs that the function actually produces.

Worked example. $f(x) = 2x - 5$ with domain $0 \le x \le 6$ has range $-5 \le f(x) \le 7$.

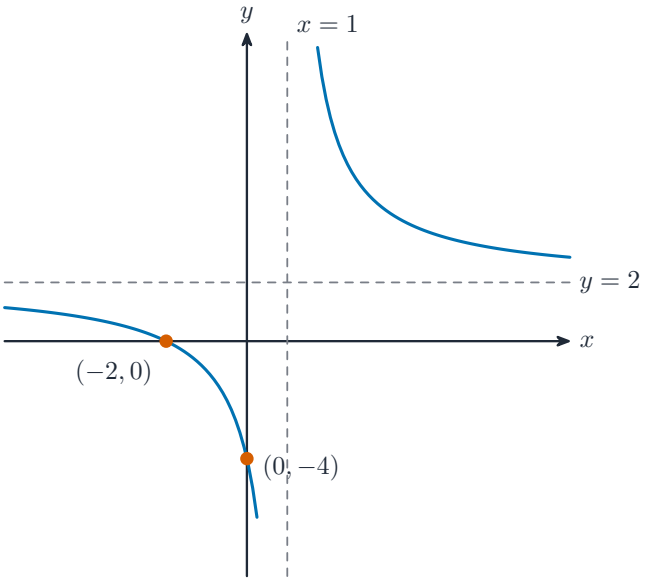
Inverse functions

An **inverse function** 反函数 f^{-1} undoes f: if $f(a) = b$, then $f^{-1}(b) = a$. The graph of $y = f^{-1}(x)$ is the reflection of $y = f(x)$ in the line $y = x$, and the domain and range swap.

For $f(x) = 2x - 5$: $f(4) = 3$, so $f^{-1}(3) = 4$. To undo "multiply by 2, then subtract 5", add 5 and then divide by 2: $f^{-1}(x) = \frac{x+5}{2}$.

Key features with technology

Use the GDC to sketch a graph and find its key features. Copy the window you used, and label the features on your sketch.



Key features of f(x) = 6/(x − 1) + 2

- **Intercepts:** where the graph meets the axes. The x-intercepts are the **zeros** 零点 of the function.
- **Maximum and minimum points**, also called turning points.
- **Asymptotes** 渐近线: lines that the graph approaches as an input tends to a finite value or infinity. A graph may cross a horizontal asymptote.
- **Intersections** of two graphs, which solve equations such as $f(x) = g(x)$.

Worked example. $f(x) = \dfrac{6}{x-1} + 2$ has a vertical asymptote $x = 1$ (where the denominator is 0) and a horizontal asymptote $y = 2$. Its y-intercept is $f(0) = -4$, and its x-intercept is where $\frac{6}{x-1} = -2$, that is $x = -2$.

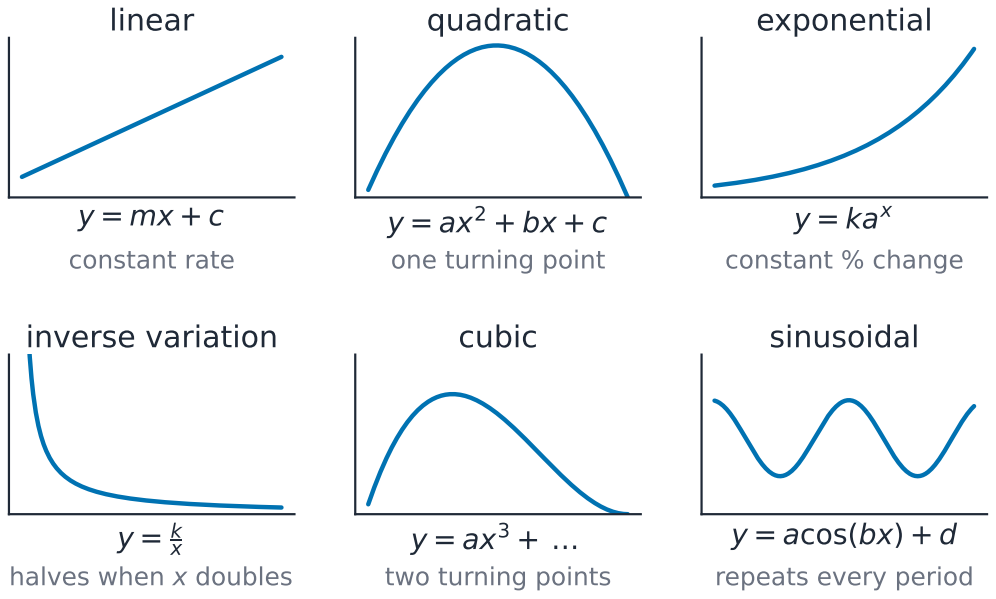
Worked example. $y = x^2 - 4$ and $y = 2x$ intersect where $x^2 - 2x - 4 = 0$. The GDC gives $(3.24, 6.47)$ and $(-1.24, -2.47)$.

A family of models

Choosing the shape

Each type of function has a typical shape and a typical story. Choose a model by thinking about how the quantity changes, not only by how the points look.

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Match the shape to how the quantity changes

Model	Equation	Typical context
Linear	$f(x) = mx + c$	fixed charge plus a rate
Quadratic	$f(x) = ax^2 + bx + c$	thrown objects, area, profit
Exponential	$f(x) = ka^x + c$ or $ke^{rx} + c$	growth or decay by a fixed percentage
Direct or inverse variation	$f(x) = ax^n$, n positive or negative	$y \propto x^2$, $y \propto \frac{1}{x}$
Cubic	$f(x) = ax^3 + bx^2 + cx + d$	volumes
Sinusoidal	$f(x) = a\sin(bx) + d$ or $a\cos(bx) + d$	tides, daylight, rotation

Quadratic models

A **quadratic model** 二次模型 has one turning point, on its **axis of symmetry** 对称轴 $x = -\frac{b}{2a}$.

> **Worked example.** A ball is thrown so that its height is $h = -4.9t^2 + 14t + 1.5$ metres after t seconds.

- Maximum at $t = -\frac{14}{2(-4.9)} \approx 1.43$ s, height $h \approx 11.5$ m.
- It lands when $h = 0$: the GDC gives $t \approx 2.96$ s. The other root is negative, outside the domain.

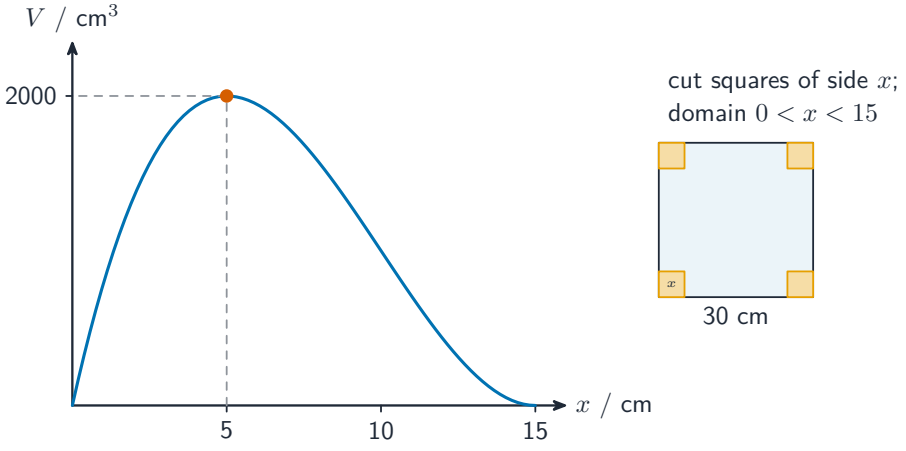
Exponential models

A quantity that changes by a fixed **percentage** each step follows an **exponential model** 指数模型. With $a > 1$ it grows; with $0 < a < 1$ it decays.

> **Worked example.** 200 mg of a medicine is in the blood, and 15 % is removed each hour, so $A = 200 \times 0.85^t$. After 3 hours $A \approx 122.8$ mg. It falls below 50 mg when $0.85^t < 0.25$: the GDC gives $t > 8.53$ hours.

Variation and cubic models

If y is **directly proportional** 正比例 to x^n, then $y = kx^n$; if y is **inversely proportional** 反比例 to x^n, then $y = \dfrac{k}{x^n}$. Find k from one pair of values.



The open-box model and its maximum

> **Worked example.** The intensity of light I is inversely proportional to the square of the distance d. If $I = 12$ when $d = 2$, then $12 = \frac{k}{4}$, so $k = 48$ and $I = \frac{48}{d^2}$. At $d = 4$, $I = 3$: doubling the distance divides the intensity by 4.

> **Worked example.** Squares of side x cm are cut from the corners of a 30 cm square sheet, which is folded into an open box. $V = x(30 - 2x)^2$ with domain $0 < x < 15$. The GDC's maximum is at $x = 5$, giving $V = 2000$ cm^3.

Sinusoidal models

A **sinusoidal model** 正弦模型 $f(x) = a\sin(bx) + d$ or $a\cos(bx) + d$ repeats with **period** 周期 $\dfrac{360°}{b}$ (or $\frac{2\pi}{b}$ in radians). Its **amplitude** 振幅 is $|a|$, and its **principal axis** 主轴 is $y = d$. The maximum is $d + |a|$ and the minimum is $d - |a|$.

Worked example. The depth of water in a harbour is $h = 1.8\cos(30t°) + 4.2$ metres, t hours after midnight. The maximum depth is 6 m and the minimum 2.4 m; the period is $\frac{360}{30} = 12$ hours, so high tide comes at midnight and noon.

Testing and improving models

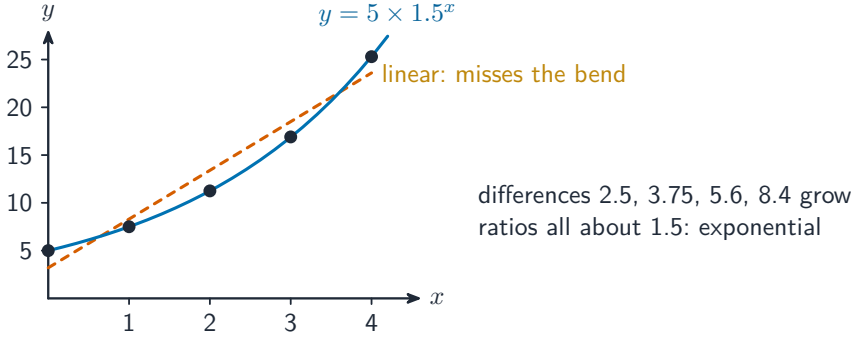
Developing a model

Modelling is a cycle:

1. Identify the variables and make **assumptions** 假设.
2. Choose a function and find its parameters, from given values or by regression with technology.
3. Test it: compare its predictions with data, and check that its values make sense.
4. Reflect: change the assumptions or the function, and test again.

Is the model reasonable?

- **Domain.** A population model $P = 3000 \times 1.04^t$ fitted to 10 years of data predicts more than 150 000 people after 100 years. Nothing in the data supports that: real populations are limited by space and resources.
- **Values.** A quadratic model for the height of a ball gives negative heights after it lands; restrict the domain.
- **Fit.** A good model's predictions are close to the data, with no pattern in the errors. Topic 4 measures this with residuals and R^2.



Growing differences, constant ratios: exponential

Worked example. Data: $x = 0, 1, 2, 3, 4$ and $y = 5, 7.5, 11.25, 16.9, 25.3$. The differences grow (2.5, 3.75, 5.6, 8.4), so a linear model is wrong. The ratios are all about 1.5, so try $y = 5 \times 1.5^x$: it gives 5, 7.5, 11.25, 16.875 and 25.3, a very close fit.

Predicting within the data (**interpolation** 内插) is safer than predicting beyond it (**extrapolation** 外推).

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AHL: Composite and inverse functions

Composite functions

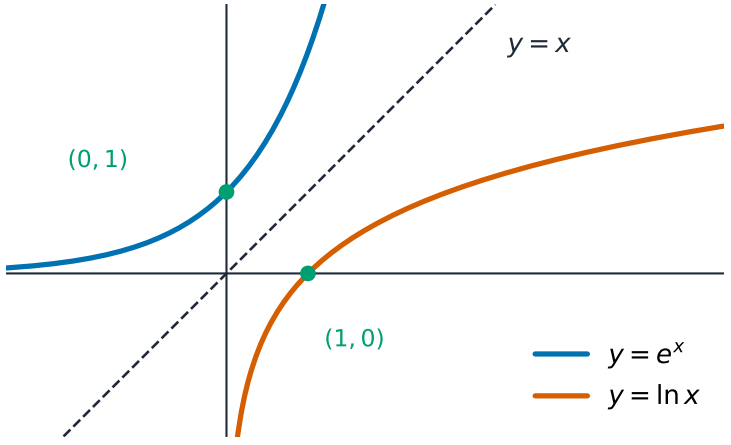
The **composite function** 复合函数 $(f \circ g)(x) = f(g(x))$ applies g first, then f. Order matters.

> **Worked example.** $f(x) = 2x+1$ and $g(x) = x^2-3$. Then $(f \circ g)(x) = 2(x^2-3)+1 = 2x^2 - 5$, but $(g \circ f)(x) = (2x+1)^2 - 3$. For example, $(g \circ f)(2) = 25 - 3 = 22$.

In context: if a currency conversion is g and a commission is f, the order in which they are applied changes the result.

Finding an inverse

To find f^{-1}, write $y = f(x)$, swap x and y, and rearrange for y.



ln and exp inverse · reflect in $y = x$

e^x and $\ln x$ undo each other, so each is the other reflected in $y = x$.

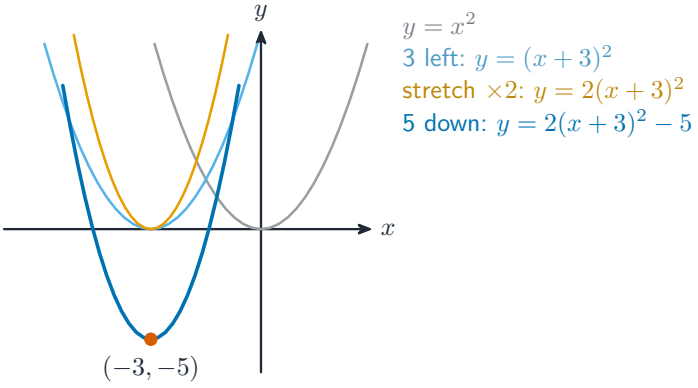
> **Worked example.** $f(x) = \dfrac{x+4}{3}$. Swap: $x = \dfrac{y+4}{3}$, so $y = 3x - 4$ and $f^{-1}(x) = 3x - 4$. Check: $f(f^{-1}(x)) = \dfrac{(3x-4)+4}{3} = x$.

A function has an inverse only if it is **one-to-one**: each output comes from one input. $g(x) = x^2$ is not, but with domain $x \geq 0$ it is, and then $g^{-1}(x) = \sqrt{x}$.

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Transformations of graphs

Transformation	New graph
translation up by d	$y = f(x) + d$
translation right by c	$y = f(x - c)$
vertical stretch, factor p	$y = pf(x)$
horizontal stretch, factor $\frac{1}{q}$	$y = f(qx)$
reflection in the x-axis	$y = -f(x)$
reflection in the y-axis	$y = f(-x)$



Building $y = 2(x + 3)^2 - 5$ step by step

Worked example. $y = 2(x + 3)^2 - 5$ comes from $y = x^2$ by a translation 3 to the left, a vertical stretch with factor 2, and a translation 5 down. The vertex moves from $(0, 0)$ to $(-3, -5)$. Apply the stretch before the vertical translation: stretching after moving down would move the vertex too.

AHL: Models with limits

Exponential models with an asymptote

Many quantities approach a limit instead of growing forever. **Newton's law of cooling** says a hot object cools at a rate proportional to the difference between its temperature and the room's, giving

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A cooling cup of coffee approaches room temperature but never passes it: a model with a horizontal asymptote

$$T = T_{\text{room}} + (T_0 - T_{\text{room}})e^{-kt}$$

Worked example. A cup of coffee at 90 °C cools in a room at 20 °C: $T = 20 + 70e^{-0.05t}$, with t in minutes.

- After 10 minutes: $T = 20 + 70e^{-0.5} \approx 62.5$ °C.
- It reaches 40 °C when $e^{-0.05t} = \frac{2}{7}$, so $t = 20 \ln 3.5 \approx 25.1$ minutes.
- As $t \to \infty$, $T \to 20$: the line $T = 20$ is a horizontal asymptote.

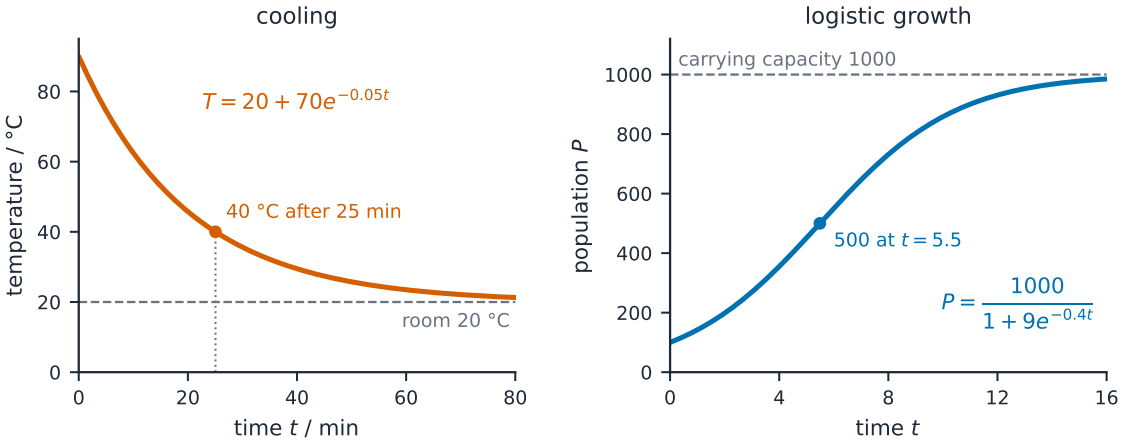
■ Logistic models

A **logistic model** 逻辑斯谛模型 grows almost exponentially at first and then levels off at a **carrying capacity** 环境容量 L:

$$f(x) = \frac{L}{1 + Ce^{-kx}}$$

Worked example. $P = \dfrac{1000}{1 + 9e^{-0.4t}}$. At $t = 0$, $P = \frac{1000}{10} = 100$. As $t \to \infty$, $P \to 1000$. It reaches half the capacity, 500, when $9e^{-0.4t} = 1$, that is $t = \frac{\ln 9}{0.4} \approx 5.49$. This is where it grows fastest.

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Both curves approach a horizontal asymptote

Other models

- **Natural logarithm** models $f(x) = a + b\ln x$ grow ever more slowly, such as the perceived loudness of a sound.
- **Sinusoidal** models with a horizontal shift, $f(x) = a\sin\big(b(x-c)\big)+d$. For example, daylight hours in a northern city might be $D = 2.5\sin\big(0.0172(t-80)\big)+12.2$, with t the day of the year: the longest day has $12.2 + 2.5 = 14.7$ hours.

AHL: Straightening data with logarithms

Semi-log graphs

If $y = ka^x$, taking logarithms gives



A seismograph: earthquake magnitude is logarithmic, so taking logs turns huge ranges into straight lines

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$$\log y = \log k + x \log a$$

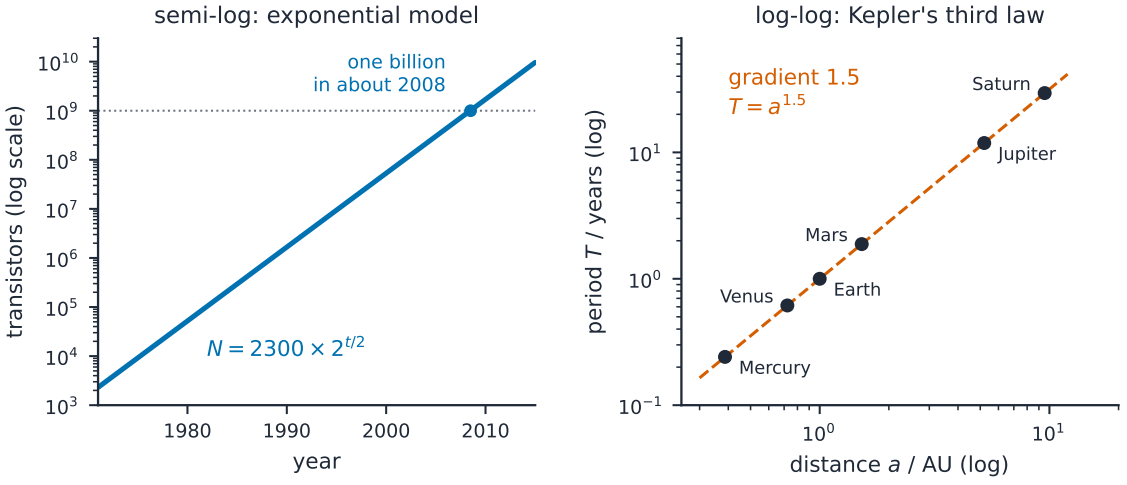
so a graph of $\log y$ against x, a **semi-log graph** 半对数图, is a straight line with gradient $\log a$ and intercept $\log k$. Moore's law as a model, $N = 2300 \times 2^{t/2}$ with t years after 1971, gives a straight line of gradient $\frac{\log 2}{2} \approx 0.151$. The model passes one billion when $2^{t/2} = \frac{10^9}{2300}$, that is $t \approx 37.5$: about 2008.

■ Log-log graphs

If $y = ax^n$, then

$$\log y = \log a + n \log x$$

so a graph of $\log y$ against $\log x$, a **log-log graph** 双对数图, is a straight line with gradient n and intercept $\log a$. In 1619 Johannes Kepler found a law of this kind for the planets: the square of the orbital period is proportional to the cube of the mean distance from the Sun. In years and astronomical units, $T = a^{1.5}$.



A straight line on logarithmic axes reveals the model

> **Worked example.** Neptune's mean distance is 30.07 AU. Kepler's law predicts $T = 30.07^{1.5} \approx 165$ years, close to the observed 164.8 years.

> **Worked example.** Data give the line $\ln y = 0.693x + 1.1$. Then $y = e^{1.1}e^{0.693x} \approx 3.00 \times 2^x$, since $e^{0.693} \approx 2$.

Common mistakes.

- Reading $f^{-1}(x)$ as $\dfrac{1}{f(x)}$. It is the inverse function.
- Mixing up domain (the inputs) and range (the outputs).

- Using a model far outside the data it was built from, such as a linear growth model for 50 years.
- Reading the base of $y = k\,a^x$ as the rate: $a = 1.05$ means 5% growth per unit, and $a = 0.92$ means an 8% fall.
- Taking b in $y = a\sin(bx) + d$ as the period. The period is $\dfrac{360^\circ}{b}$, the amplitude is $|a|$ and the principal axis is $y = d$.
- Reporting a model without checking it against the data and the context.
- **HL:** composing in the wrong order: $(f \circ g)(x) = f(g(x))$ applies g first.
- **HL:** reading the gradient of a semi-log graph as a. For $y = k\,a^x$, $\ln y = \ln k + x \ln a$, so the gradient is $\ln a$.
- **HL:** for $y = a\,x^b$ on a log-log graph, the gradient is b and the intercept is $\ln a$, not a.
- **HL:** forgetting that a logistic model levels off at its carrying capacity L.

Practice

1. Find the equation of the line through $(-2, 7)$ and $(4, -2)$ in the forms $y = mx + c$ and $ax + by + d = 0$.
2. Find the line through $(8, 3)$ perpendicular to $y = 4x - 1$.
3. A plumber charges $C = 120 + 85h$ yuan. Interpret the 120 and the 85, and find the cost of a 3.5-hour job.
4. $f(x) = 2x - 5$ for $0 \le x \le 6$. State the range, find $f^{-1}(x)$ and $f^{-1}(3)$.
5. For $f(x) = \dfrac{6}{x - 1} + 2$, state the asymptotes and find the intercepts.
6. Use technology to find where $y = x^2 - 4$ meets $y = 2x$.
7. A ball's height is $h = -4.9t^2 + 14t + 1.5$ metres. Find the maximum height and when the ball lands.
8. A medicine follows $A = 200 \times 0.85^t$ mg. Find A after 3 hours, and when it first falls below 50 mg.
9. y is inversely proportional to x^2, and $y = 12$ when $x = 2$. Find y when $x = 4$.
10. Squares of side x are cut from a 30 cm square sheet to make an open box. Write the volume, state its domain and find the largest volume.
11. A tide follows $h = 1.8\cos(30t^\circ) + 4.2$. Find the maximum and minimum depths and the period.
12. Which model fits $x = 0, 1, 2, 3, 4$ and $y = 5, 7.5, 11.25, 16.9, 25.3$? Give its equation.
13. A town's population is modelled by $P = 3000 \times 1.04^t$, fitted to data for $0 \le t \le 10$. Explain why the model should not be used for $t = 100$.
14. Phone plan A costs $30 + 2g$ and plan B costs $50 + 0.75g$ yuan for g GB. When is plan B cheaper?
15. **AHL.** $f(x) = 2x + 1$ and $g(x) = x^2 - 3$. Find $(f \circ g)(x)$ and $(g \circ f)(2)$.

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16. **AHL.** Find $f^{-1}(x)$ for $f(x) = \dfrac{x+4}{3}$, and verify that $f(f^{-1}(x)) = x$.
17. **AHL.** Describe the transformations that map $y = x^2$ onto $y = 2(x+3)^2 - 5$, and state the new vertex.
18. **AHL.** Coffee cools by $T = 20 + 70e^{-0.05t}$. Find the starting temperature, the temperature after 10 minutes, when it reaches 40 °C, and its long-term temperature.
19. **AHL.** $P = \dfrac{1000}{1 + 9e^{-0.4t}}$. Find $P(0)$, the carrying capacity, and when $P = 500$.
20. **AHL.** A log-log graph of data gives the line $\log y = 0.477 + 1.5 \log x$. Find the model $y = ax^n$.
21. **AHL.** A semi-log graph gives $\ln y = 0.693x + 1.1$. Write y in the form $k \times a^x$.
22. **AHL.** Use $T = a^{1.5}$ to predict the orbital period of Neptune, at 30.07 AU.

Answers.

1. $m = \frac{-2-7}{4-(-2)} = -1.5$, so $y = -1.5x + 4$, or $3x + 2y - 8 = 0$.
2. Gradient $-\frac{1}{4}$: $y - 3 = -\frac{1}{4}(x - 8)$, so $y = -0.25x + 5$.
3. ¥120 is a fixed call-out charge and ¥85 is the charge per hour. Cost ¥417.50.
4. Range $-5 \le f(x) \le 7$; $f^{-1}(x) = \frac{x+5}{2}$; $f^{-1}(3) = 4$.
5. Asymptotes $x = 1$ and $y = 2$; y-intercept $(0, -4)$; x-intercept $(-2, 0)$.
6. $(3.24, 6.47)$ and $(-1.24, -2.47)$.
7. Maximum 11.5 m at $t \approx 1.43$ s; it lands at $t \approx 2.96$ s.
8. $A \approx 122.8$ mg; it falls below 50 mg after about 8.53 hours.
9. $k = 48$, so $y = \frac{48}{16} = 3$.
10. $V = x(30 - 2x)^2$, $0 < x < 15$; maximum $V = 2000$ cm^3 at $x = 5$.
11. Maximum 6 m, minimum 2.4 m, period 12 hours.
12. Ratios are about 1.5, so exponential: $y = 5 \times 1.5^x$.
13. It is fitted only to $0 \le t \le 10$; at $t = 100$ it predicts over 150 000 people, an extrapolation that ignores limits on space and resources.
14. $30 + 2g > 50 + 0.75g$ when $g > 16$: plan B is cheaper for more than 16 GB.
15. **AHL.** $(f \circ g)(x) = 2x^2 - 5$; $(g \circ f)(2) = g(5) = 22$.
16. **AHL.** $f^{-1}(x) = 3x - 4$; $f(3x - 4) = \frac{3x-4+4}{3} = x$.
17. **AHL.** Translate 3 left, stretch vertically by factor 2, translate 5 down; vertex $(-3, -5)$.
18. **AHL.** 90 °C; about 62.5 °C; $t = 20 \ln 3.5 \approx 25.1$ minutes; 20 °C.
19. **AHL.** $P(0) = 100$; carrying capacity 1000; $t = \frac{\ln 9}{0.4} \approx 5.49$.
20. **AHL.** $n = 1.5$ and $a = 10^{0.477} \approx 3.00$, so $y = 3x^{1.5}$.
21. **AHL.** $y = e^{1.1}(e^{0.693})^x \approx 3.00 \times 2^x$.
22. **AHL.** $30.07^{1.5} \approx 165$ years.

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Exam tips

- Give a line in the form the question asks for; general form needs integer coefficients.
- For perpendicular lines, use the negative reciprocal of the gradient.
- Sketch GDC graphs with labelled axes, key points and asymptotes, and state the window if it matters.
- State the domain of a model from its context.
- Choose a model from how the quantity changes, and test it against the data.
- Interpret parameters in context, with units: "85 is the charge in yuan per hour".
- **HL:** in $f \circ g$, apply g first.
- **HL:** to find a power model, plot $\log y$ against $\log x$; for an exponential model, plot $\log y$ against x.

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Geometry and trigonometry

International Baccalaureate · IB Diploma · Mathematics: Applications and Interpretation · HL

Scope and prerequisites

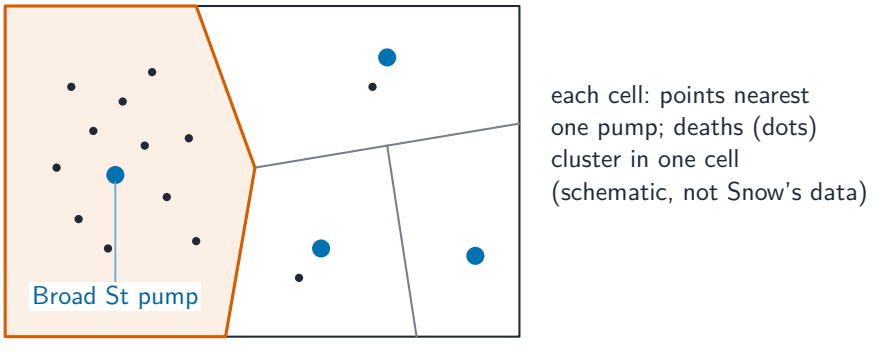
This handout teaches topic 3, Geometry and trigonometry, of Mathematics: Applications and Interpretation, the course first assessed in 2021: distance, volume, surface area and angles in three dimensions, right-angled and non-right-angled triangles, bearings, arcs and sectors, perpendicular bisectors and Voronoi diagrams, and, for HL, radians and the unit circle, matrix transformations, vectors and motion, and graph theory with its algorithms. It is written for both SL and HL students. HL students also apply the further results in the sections marked AHL.

Prerequisites: Pythagoras' theorem, the gradient and equation of a line (topic 2), and, for HL, matrices (topic 1).

Geometry in three dimensions

A map that stopped an epidemic

In August 1854 cholera killed more than 500 people in ten days in Soho, London. Most doctors believed the disease spread through bad air. The doctor John Snow marked each death on a street map and noticed that they clustered around one public water pump, on Broad Street. In a later version of his map he drew a line round the area that was closer, by walking distance, to the Broad Street pump than to any other pump, and most of the deaths lay inside it. The pump handle was removed, and the outbreak, already slowing, ended. Snow's line is an early example of a Voronoi diagram, which divides a map into the regions nearest to each of a set of points. This topic is about measuring space: distances, angles, areas and regions.



Nearest-pump regions (schematic)

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A version of John Snow's 1854 map: each dot is a death and each cross a pump; the deaths cluster around the Broad Street pump

■ Distance and midpoint

In three dimensions a point has coordinates (x, y, z). For points (x_1, y_1, z_1) and (x_2, y_2, z_2):

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}, \qquad M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right)$$

Worked example. For $P(1, 2, 3)$ and $Q(4, 6, 15)$, the differences are 3, 4 and 12, so $PQ = \sqrt{9 + 16 + 144} = 13$. The **midpoint** 中点 is $(2.5, 4, 9)$.

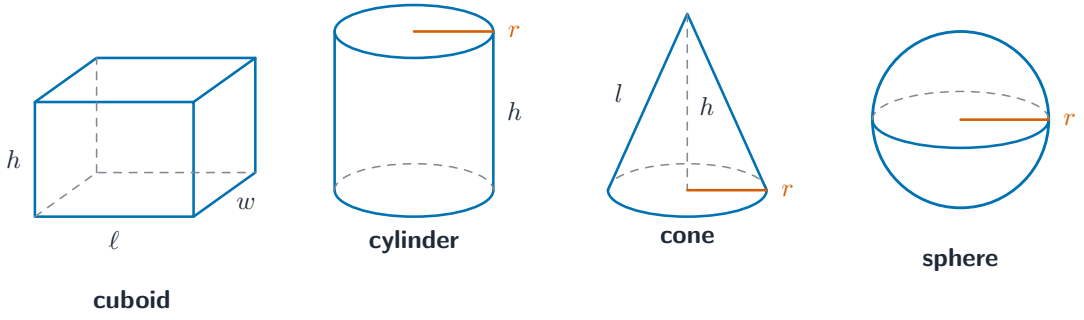
■ Volume and surface area

The formula booklet gives these. Here r is a radius, h a perpendicular height and l the **slant height** 斜高 of a cone.

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The pyramids of Giza: a square-based pyramid has volume one third of base area times height



Common solids and the dimensions (r, h, ℓ, slant l) used in their volume and surface-area formulas.

Solid	Volume	Surface area
Prism, cylinder	area of cross-section × length; cylinder $\pi r^2 h$	cylinder curved surface $2\pi r h$
Pyramid 棱锥	$\frac{1}{3}\times$ base area $\times h$	add the areas of the faces
Cone 圆锥	$\frac{1}{3}\pi r^2 h$	curved surface $\pi r l$
Sphere 球	$\frac{4}{3}\pi r^3$	$4\pi r^2$

A **hemisphere** 半球 is half a sphere: volume $\frac{2}{3}\pi r^3$, curved surface $2\pi r^2$.

Worked example. A pyramid has a square base of side 6 cm and a height of 4 cm.

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- Volume: $\frac{1}{3}(36)(4) = 48$ cm^3.
- Each triangular face has height $\sqrt{4^2 + 3^2} = 5$ cm, from the right-angled triangle between the apex, the centre of the base and the middle of a side.
- Surface area: $36 + 4 \times \frac{1}{2}(6)(5) = 96$ cm^2.

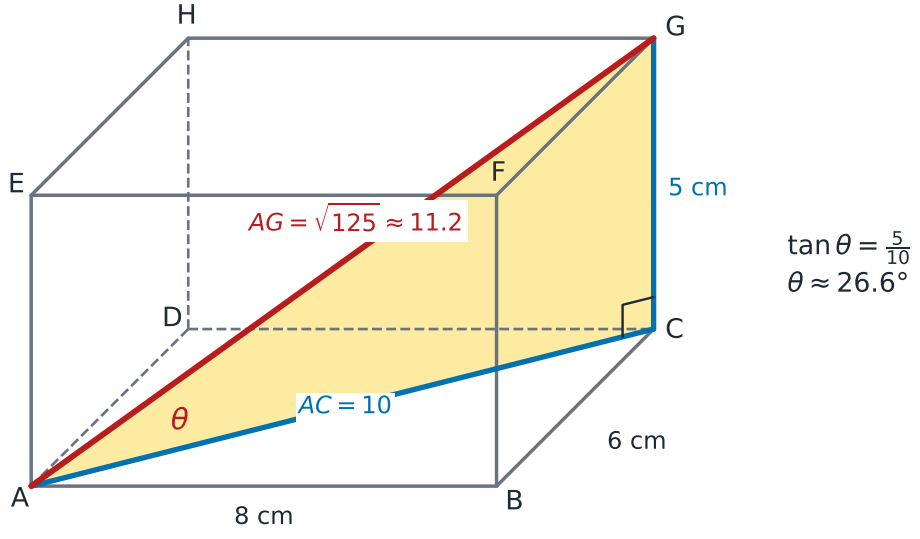
> **Worked example.** A grain silo is a cylinder of radius 3 m and height 10 m with a hemisphere on top. Its volume is $\pi(9)(10) + \frac{2}{3}\pi(27) = 108\pi \approx 339$ m^3. The curved surfaces that need painting have area $2\pi(3)(10) + 2\pi(9) = 78\pi \approx 245$ m^2. Split a **compound solid** into simple solids, and add only the faces that are on the outside.

The angle between a line and a plane

To find the angle between a line and a plane, drop a perpendicular from a point on the line to the plane. The angle is between the line and its "shadow" in the plane, and it sits in a right-angled triangle.

> **Worked example.** A cuboid is 8 cm by 6 cm by 5 cm. Find the angle between the **space diagonal** 体对角线 AG and the base.

- Base diagonal: $AC = \sqrt{8^2 + 6^2} = 10$ cm.
- $CG = 5$ cm is vertical, so triangle ACG has a right angle at C.
- $\tan\theta = \frac{5}{10}$, so $\theta \approx 26.6°$, and $AG = \sqrt{125} \approx 11.2$ cm.

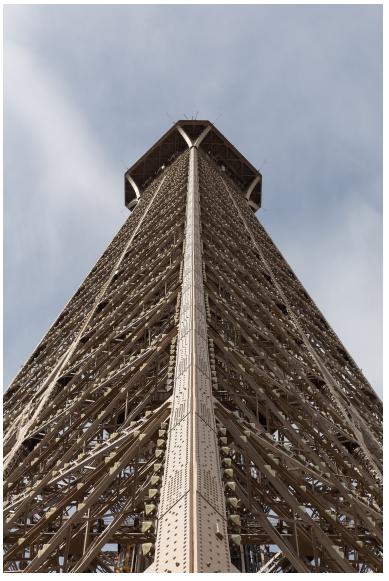


Find the right-angled triangle that contains the angle

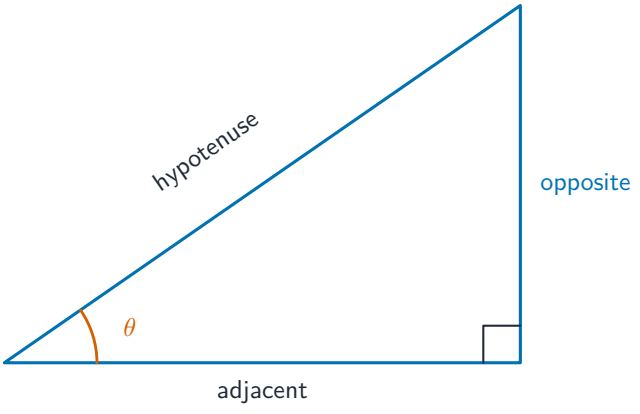
Triangle trigonometry

■ Right-angled triangles

Label the sides from the angle θ you are using: the **hypotenuse** 斜边 is opposite the right angle, the opposite side faces θ, and the adjacent side is next to it.



Looking up at a tower: an angle of elevation and a measured distance give its height



Name the sides from the angle θ: the opposite is across from it, the adjacent next to it, and the hypotenuse opposite the right angle (SOH-CAH-TOA).

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}, \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}, \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

- An **angle of elevation** 仰角 is measured up from the horizontal; an **angle of depression** 俯角 is measured down from the horizontal.

- A **bearing** 方位角 is an angle measured clockwise from north, written with three figures: 050°, 160°, 305°.

> **Worked example.** From a point 40 m from the foot of a tower, the angle of elevation of the top is 32°. The height is $40\tan 32° \approx 25.0$ m.

> **Worked example.** From the top of a cliff 60 m high, the angle of depression of a boat is 18°. The angle of elevation of the cliff top from the boat is also 18° (alternate angles), so the boat is $\frac{60}{\tan 18°} \approx 185$ m from the foot of the cliff.

■ The sine rule, the cosine rule and area

For any triangle with sides a, b, c opposite angles A, B, C:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}, \qquad c^2 = a^2 + b^2 - 2ab\cos C, \qquad \text{Area} = \frac{1}{2}ab\sin C$$

- Use the **sine rule** 正弦定理 when you know a side and the angle opposite it.
- Use the **cosine rule** 余弦定理 when you know two sides and the angle between them, or all three sides. To find an angle, rearrange: $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$.

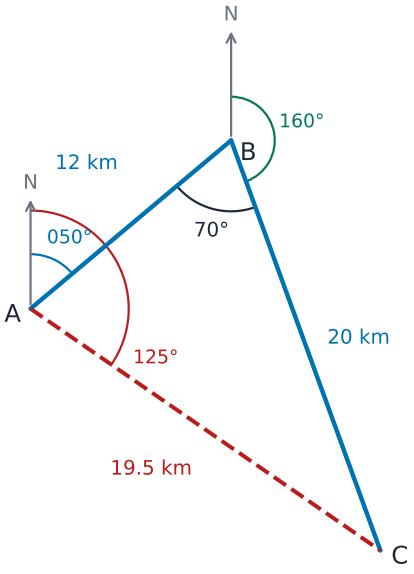
> **Worked example.** In triangle ABC, $A = 48°$, $B = 75°$ and $a = 12$ cm. Then $b = \dfrac{12\sin 75°}{\sin 48°} \approx 15.6$ cm, $C = 57°$, and the area is $\frac{1}{2}(12)(15.6)\sin 57° \approx 78.5$ cm^2.

> **Worked example.** A triangle has sides 7, 8 and 10 m. The largest angle is opposite the longest side: $\cos C = \dfrac{49 + 64 - 100}{2(7)(8)} = \dfrac{13}{112}$, so $C \approx 83.3°$.

> **Worked example (bearings).** A ship sails 12 km from A on a bearing of 050° to B, then 20 km on a bearing of 160° to C. Find the distance and bearing of C from A, and the area of triangle ABC.

- At B, the bearing back to A is $050° + 180° = 230°$, so angle $ABC = 230° - 160° = 70°$.
- Cosine rule: $AC^2 = 12^2 + 20^2 - 2(12)(20)\cos 70° \approx 379.8$, so $AC \approx 19.5$ km.
- Angle at A: $\cos A = \dfrac{12^2 + 19.49^2 - 20^2}{2(12)(19.49)} \approx 0.265$, so $A \approx 74.6°$.
- Bearing of C from A: $050° + 74.6° \approx 125°$.
- Area: $\frac{1}{2}(12)(20)\sin 70° \approx 113$ km^2.

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Draw a north line at every point where a bearing is measured

At SL you will not meet the ambiguous case of the sine rule: when you find an angle with it, the question will make clear whether the angle is acute or obtuse.

Arcs, sectors and perpendicular bisectors

Arc length and sector area in degrees

An **arc** 弧 is part of a circle, and a **sector** 扇形 is the region between an arc and two radii. A sector with angle $\theta°$ is the fraction $\frac{\theta}{360}$ of the whole circle, so

$$\text{arc length } l = \frac{\theta}{360} \times 2\pi r, \qquad \text{sector area } A = \frac{\theta}{360} \times \pi r^2$$

Worked example. A pizza of radius 15 cm is cut into slices with angle 40°. Each slice has arc $\frac{40}{360} \times 30\pi \approx 10.5$ cm, area $\frac{40}{360} \times 225\pi \approx 78.5$ cm^2, and perimeter $15 + 15 + 10.5 = 40.5$ cm.

The perpendicular bisector

The **perpendicular bisector** 垂直平分线 of AB is the line through the midpoint of AB at right angles to it. Every point on it is the same distance from A as from B.

Worked example. For $P(2, 1)$ and $Q(6, 5)$: the midpoint is $(4, 3)$, the gradient of PQ is $\frac{5-1}{6-2} = 1$, so the bisector has gradient -1 and equation $y - 3 = -(x - 4)$, that is $y = -x + 7$.

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Voronoi diagrams

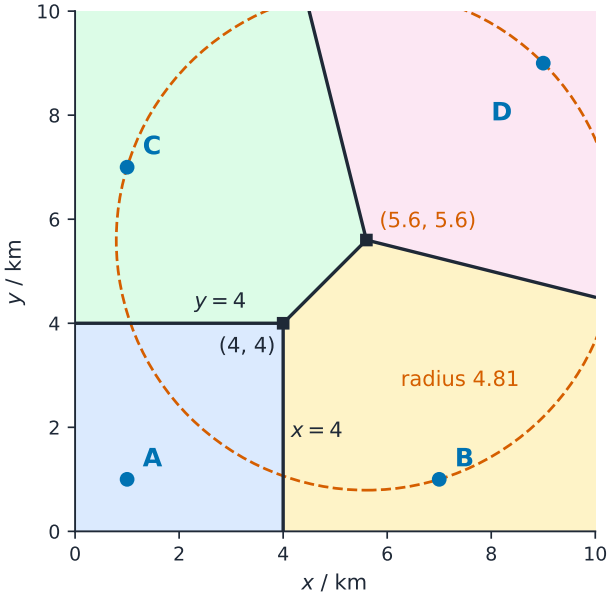
Sites, cells, edges and vertices

A **Voronoi diagram** 沃罗诺伊图 divides a region into cells, one for each **site** 站点, such as a school, a weather station or a mobile phone mast. Each **cell** 单元 contains the points closer to its site than to any other.

- An **edge** 边 is part of the perpendicular bisector of two neighbouring sites.
- A **vertex** 顶点 is where three edges meet. It is the same distance from three sites.

Worked example. Four wells are at $A(1,1)$, $B(7,1)$, $C(1,7)$ and $D(9,9)$, in kilometres, in a farming region $0 \le x \le 10$, $0 \le y \le 10$.

- The bisector of A and B is $x = 4$, and the bisector of A and C is $y = 4$. They meet at the vertex $(4,4)$, which is $\sqrt{3^2+3^2} \approx 4.24$ km from A, B and C.
- The bisector of B and C passes through their midpoint $(4,4)$ with gradient 1 (because BC has gradient -1): it is $y = x$.
- The bisector of B and D passes through $(8,5)$ with gradient $-\frac{1}{4}$: $y = -0.25x + 7$. It meets $y = x$ at the vertex $(5.6, 5.6)$, which is about 4.81 km from B, C and D.



Each cell holds the points nearest its well; the largest empty circle is centred on a vertex

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Using a Voronoi diagram

- **Nearest neighbour interpolation** 最近邻插值: to estimate a value at a point, use the value at the site of the cell that contains it. A farm at $(3, 2)$ is in cell A, so its rainfall is estimated by the rainfall at well A.
- **Adding a site:** draw the perpendicular bisectors between the new site and its neighbours, then remove the parts of the old edges that now lie inside the new cell.
- **The largest empty circle**, sometimes called the toxic waste dump problem: to place something as far as possible from every site, check the vertices, where edges meet the boundary, and the corners of the region. In the example, the vertex $(5.6, 5.6)$ is 4.81 km from its nearest wells, more than $(4, 4)$ at 4.24 km and more than any boundary point (the best is about 4.61 km), so it is the best place for a new storage tank that must be as far from the wells as possible.

AHL: Radians and the unit circle

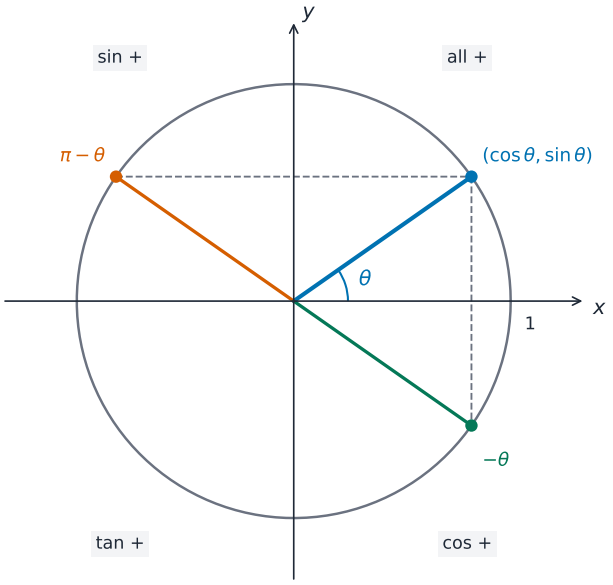
Radians

A **radian** 弧度 is the angle at the centre of a circle made by an arc equal in length to the radius. A full turn is 2π radians, so



A turning Ferris wheel: angle in radians times radius gives the arc each car travels

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Reflections in the unit circle give the symmetry rules

$$\pi \text{ radians} = 180°, \qquad l = r\theta, \qquad A = \frac{1}{2}r^2\theta \quad (\theta \text{ in radians})$$

Worked example. $150° = 150 \times \frac{\pi}{180} = \frac{5\pi}{6} \approx 2.62$ radians. A garden sprinkler sprays a sector of radius 8 m through 1.2 radians: the arc is $8 \times 1.2 = 9.6$ m and the area $\frac{1}{2}(64)(1.2) = 38.4$ m^2. Set the GDC to radian mode before you use these formulae.

The unit circle

On the **unit circle** 单位圆, radius 1 and centre the origin, the point at angle θ anticlockwise from the positive x-axis is $(\cos\theta, \sin\theta)$. This defines sine and cosine for any angle, and gives

$$\cos^2\theta + \sin^2\theta = 1, \qquad \tan\theta = \frac{\sin\theta}{\cos\theta}$$

Worked example. θ is obtuse and $\sin\theta = 0.6$. Then $\cos^2\theta = 1 - 0.36 = 0.64$, and cosine is negative for an obtuse angle, so $\cos\theta = -0.8$ and $\tan\theta = -0.75$.

Worked example. Solve $3\sin x + 1 = 2.5$ for $0 \le x \le 2\pi$. Then $\sin x = 0.5$, so $x = \frac{\pi}{6}$ or $\pi - \frac{\pi}{6} = \frac{5\pi}{6}$: by symmetry of the unit circle, two angles have the same height. An equation such as $2\cos x = x$ has no algebraic method: graph both sides and find the intersection, $x \approx 1.03$.

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AHL: Matrix transformations

■ Matrices that move shapes

Multiplying a position vector by a 2×2 matrix moves the point. The formula booklet gives these **transformation matrices** 变换矩阵, centred at the origin:

Transformation	Matrix
reflection 反射 in $y = (\tan\theta)x$	$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$
rotation 旋转 by θ anticlockwise	$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$
enlargement 放大, scale factor k	$\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$
horizontal **stretch** 伸缩, factor k	$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$
vertical stretch, factor k	$\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$

A translation adds a vector, so the general transformation is $X' = AX + \mathbf{b}$.

■ Composite transformations and area

To apply A first and then B, use the single matrix BA: the first transformation is written on the right. The **determinant** 行列式 tells you how area changes:

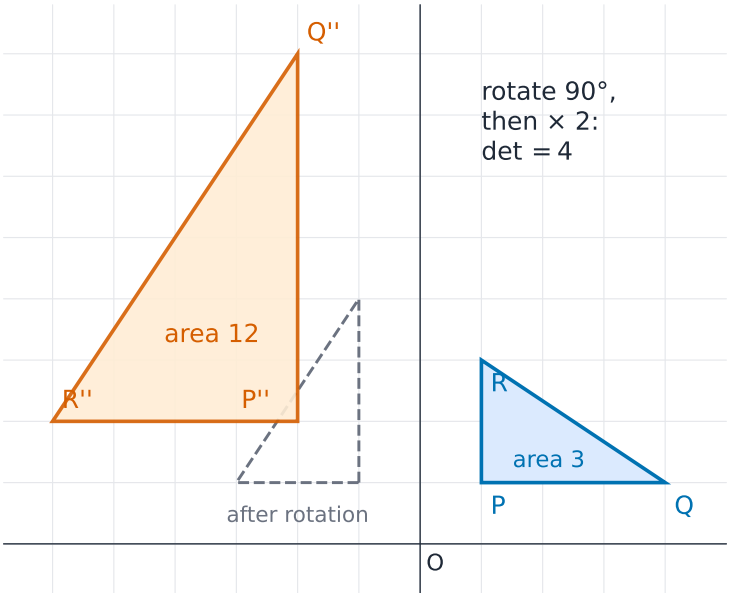
$$\text{area of image} = |\det A| \times \text{area of object}$$

A negative determinant means the shape has been flipped over, as in a reflection.

Worked example. The triangle $P(1,1)$, $Q(4,1)$, $R(1,3)$ has area 3. It is rotated $90°$ anticlockwise about O and then enlarged with scale factor 2.

- Rotation: $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Enlargement: $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.
- Single matrix: enlargement times rotation $= \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$, with determinant 4.
- Images: $P''(-2,2)$, $Q''(-2,8)$, $R''(-6,2)$, and the new area is $4 \times 3 = 12$.

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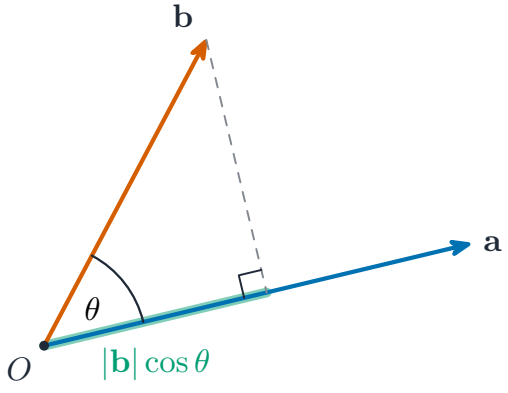


Rotation keeps the area; the enlargement multiplies it by $\det = 4$

AHL: Vectors and motion

■ Vectors, magnitude and the scalar product

A **vector** 向量 has size and direction. In components, $\mathbf{v} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$ has **magnitude** 模 $|\mathbf{v}| = \sqrt{3^2 + 4^2} = 5$, and the **unit vector** 单位向量 in its direction is $\frac{1}{5}\mathbf{v} = \begin{pmatrix} 0.6 \\ -0.8 \end{pmatrix}$. The vector from A to B is $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$, where $\mathbf{a}$ and $\mathbf{b}$ are **position vectors** 位置向量.



The scalar product picks out $|\mathbf{b}|\cos\theta$*, how far* $\mathbf{b}$ *reaches along* $\mathbf{a}$.

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The **scalar product** 数量积 is

$$\mathbf{v}\cdot\mathbf{w} = v_1w_1 + v_2w_2 + v_3w_3 = |\mathbf{v}||\mathbf{w}|\cos\theta$$

so it finds the angle between two vectors, and $\mathbf{v}\cdot\mathbf{w} = 0$ means they are perpendicular.

Worked example. For $\mathbf{v} = \begin{pmatrix}2\\1\\0\end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix}1\\3\\2\end{pmatrix}$: $\mathbf{v}\cdot\mathbf{w} = 2+3+0 = 5$, $|\mathbf{v}| = \sqrt{5}$, $|\mathbf{w}| = \sqrt{14}$, so $\cos\theta = \frac{5}{\sqrt{70}}$ and $\theta \approx 53.3°$.

The vector product

The **vector product** 向量积 $\mathbf{v}\times\mathbf{w}$ is a vector perpendicular to both, with magnitude $|\mathbf{v}||\mathbf{w}|\sin\theta$, the area of the parallelogram they make.

$$\mathbf{v}\times\mathbf{w} = \begin{pmatrix}v_2w_3 - v_3w_2\\v_3w_1 - v_1w_3\\v_1w_2 - v_2w_1\end{pmatrix}$$

Worked example. For the vectors above, $\mathbf{v}\times\mathbf{w} = \begin{pmatrix}2\\-4\\5\end{pmatrix}$, with magnitude $\sqrt{45} \approx 6.71$. The triangle with these two sides has area $\frac{1}{2}\sqrt{45} \approx 3.35$.

Lines and motion

The **vector equation of a line** 直线的向量方程 is $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$: start at the point with position vector $\mathbf{a}$ and move any multiple λ of the direction $\mathbf{b}$. In kinematics the parameter is time:

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

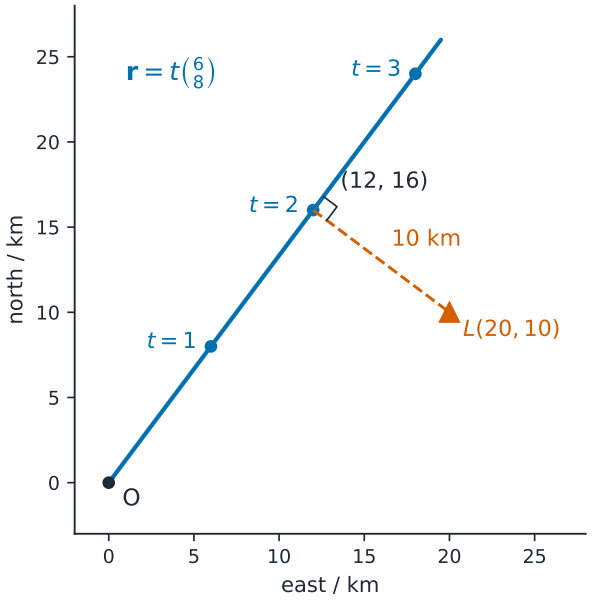
where $\mathbf{r}_0$ is the starting position and $\mathbf{v}$ the **velocity vector** 速度向量. The speed is $|\mathbf{v}|$.

Worked example. A boat leaves harbour O with velocity $\begin{pmatrix}6\\8\end{pmatrix}$ km/h, so its speed is 10 km/h and its position is $\mathbf{r} = t\begin{pmatrix}6\\8\end{pmatrix}$. A lighthouse is at $L(20, 10)$. When is the boat closest to it?

- Vector from L to the boat: $\begin{pmatrix}6t-20\\8t-10\end{pmatrix}$.
- Distance squared: $(6t-20)^2 + (8t-10)^2 = 100t^2 - 400t + 500$, a quadratic with its minimum at $t = 2$.

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- At $t = 2$ the boat is at $(12, 16)$, and the distance is $\sqrt{400 - 800 + 500} = 10$ km. Check: $\begin{pmatrix} -8 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ 8 \end{pmatrix} = 0$, so the closest line is perpendicular to the path.



At the closest approach the line to L is perpendicular to the path

Worked example. A second ship moves by $\mathbf{r} = \begin{pmatrix} 30 \\ 5 \end{pmatrix} + t\begin{pmatrix} -6 \\ 8 \end{pmatrix}$. The vector from the boat to the ship is $\begin{pmatrix} 30 - 12t \\ 5 \end{pmatrix}$, which is shortest when $t = 2.5$: the two pass 5 km apart, 2.5 hours after leaving.

■ Motion with variable velocity

When the velocity changes, the position is not a straight line. A ball thrown with velocity $\begin{pmatrix} 15 \\ 20 \end{pmatrix}$ m/s moves by $\mathbf{r} = \begin{pmatrix} 15t \\ 20t - 4.9t^2 \end{pmatrix}$. It lands when $20t - 4.9t^2 = 0$, at $t \approx 4.08$ s, $15 \times 4.08 \approx 61.2$ m away, and its greatest height is about 20.4 m, at $t \approx 2.04$ s.

AHL: Graphs and adjacency matrices

■ The bridges of Königsberg

In 1736 Leonhard Euler was asked whether a walk through the city of Königsberg could cross each of its seven bridges exactly once. He replaced the four areas of land with points and the bridges with lines, and showed that it could not be done. In doing so he started **graph theory**: the study of what is connected to what, ignoring exact

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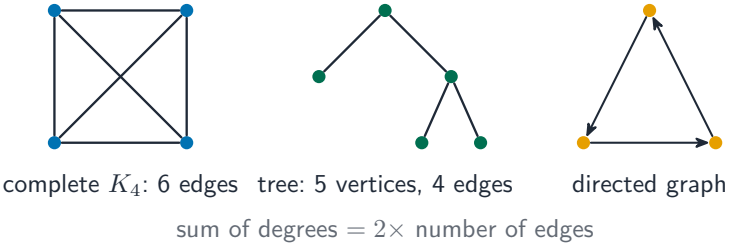
shapes and distances.



A metro map is a graph: stations are vertices and the tracks between them are edges

Words for graphs

A **graph** 图 is a set of **vertices** joined by **edges**. The **degree** 度 of a vertex is the number of edges that meet there. Adding up all the degrees counts every edge twice, so the sum of the degrees is twice the number of edges.



Complete graph, tree and directed graph

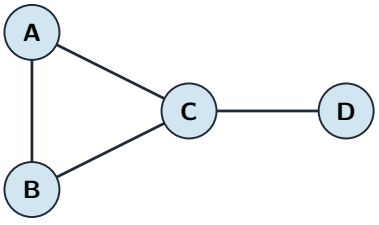
- A **simple graph** has no loops and no repeated edges. In a **complete graph** 完全图 every pair of vertices is joined: K_n has $\frac{n(n-1)}{2}$ edges.
- In a **weighted graph** 加权图 each edge has a number, such as a distance, time or cost. In a **directed graph** 有向图 the edges have arrows.
- A graph is **connected** 连通 if every vertex can be reached from every other. A **tree** 树 is a connected graph with no cycles; with n vertices it has $n-1$ edges.
- A **walk** is any route along edges. A **trail** repeats no edge, a **path** repeats no vertex, a **circuit** is a trail that ends where it started, and a **cycle** is a path that ends where it started.

Kind of route	Uses	Exists when
Eulerian circuit 欧拉回路	every edge once, back to the start	connected, all degrees even
Eulerian trail 欧拉迹	every edge once, ends elsewhere	connected, exactly two odd vertices
Hamiltonian cycle 哈密顿圈	every vertex once, back to the start	no simple test: search

In Königsberg the four land areas had degrees 5, 3, 3 and 3. With four odd vertices there is no Eulerian trail, which is exactly what Euler proved.

Adjacency matrices and walks

The **adjacency matrix** 邻接矩阵 M has, in row i and column j, the number of edges from vertex i to vertex j. The number of walks of length k from i to j is the entry in M^k.



$$M = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad M^3 = \begin{pmatrix} 2 & 3 & \mathbf{4} & 1 \\ 3 & 2 & 4 & 1 \\ 4 & 4 & 2 & 3 \\ 1 & 1 & 3 & 0 \end{pmatrix}$$

Row A, column C of M^3 is 4: the four walks of length 3 are $ABAC$, $ACAC$, $ACBC$ and $ACDC$

The entry in row A, column C of M^3 counts the walks of length 3 from A to C

Worked example. A graph has edges AB, AC, BC and CD. With the vertices in the order A, B, C, D:

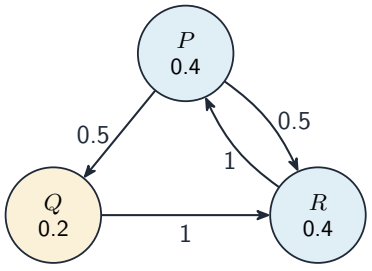
$$M = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad M^3 = \begin{pmatrix} 2 & 3 & 4 & 1 \\ 3 & 2 & 4 & 1 \\ 4 & 4 & 2 & 3 \\ 1 & 1 & 3 & 0 \end{pmatrix}$$

There are 4 walks of length 3 from A to C: $ABAC$, $ACAC$, $ACBC$ and $ACDC$.

Transition matrices for directed graphs

A random walk on a directed graph, such as a reader clicking links between web pages, is a Markov chain (topic 4). The **transition matrix** 转移矩阵 has, in each column, the probabilities of leaving that vertex along each edge.

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$q = 0.5p,\ \ r = 0.5p + q,\ \ p = r$
$p + q + r = 1$
steady state $(0.4,\ 0.2,\ 0.4)$

A random reader's long-run shares

Worked example. Page P links to Q and R, page Q links to R, and page R links to P. A reader picks a link at random. In the long run the reader spends a share p of the time on P, q on Q and r on R, where $q = 0.5p$, $r = 0.5p + q$ and $p = r$. So $p = r$ and $q = 0.5p$, and since $p + q + r = 1$, the steady state is $p = 0.4$, $q = 0.2$, $r = 0.4$. Ranking pages by this share is the idea behind early search engines.

AHL: Network algorithms

Minimum spanning trees

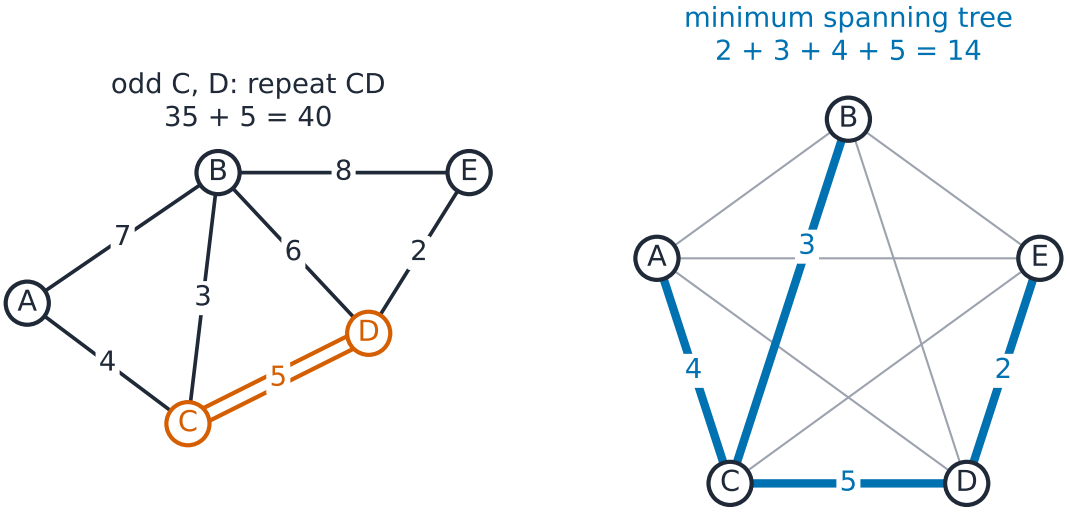
A **spanning tree** 生成树 joins all the vertices of a graph with no cycles. A **minimum spanning tree** 最小生成树 is the one with the least total weight: for example, the cheapest set of cables that connects every building.

The table gives the distances, in km, between five villages.

	A	B	C	D	E
A	–	7	4	9	11
B	7	–	3	6	8
C	4	3	–	5	10
D	9	6	5	–	2
E	11	8	10	2	–

- **Kruskal's algorithm** 克鲁斯卡尔算法: take the edges in order of weight, and keep each one unless it makes a cycle. $DE = 2$, $BC = 3$, $AC = 4$, $CD = 5$; $BD = 6$ is rejected because B, C and D are already joined. Total 14 km.
- **Prim's algorithm** 普里姆算法: start at any vertex and repeatedly add the shortest edge that reaches a new vertex. It suits a table: from A, add $AC = 4$, then $CB = 3$, then $CD = 5$, then $DE = 2$. Same tree, total 14 km.

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Left: the postman repeats CD. Right: the minimum spanning tree weighs 14

The Chinese postman problem

The **Chinese postman problem** 中国邮递员问题 asks for the shortest route that uses every edge at least once and returns to the start, like a postman covering every street. It was first studied by the Chinese mathematician Guan Meigu in 1960.

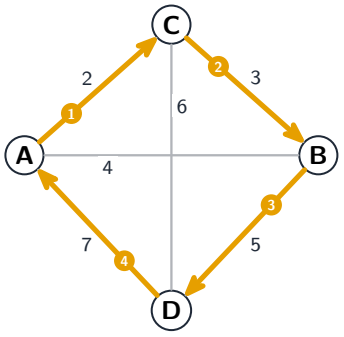
1. Find the vertices of odd degree.
2. If there are none, the graph has an Eulerian circuit: the answer is the total weight.
3. If there are two, add the shortest route between them to the total weight.
4. If there are four, try all three ways of pairing them and add the pairing with the least total.

Worked example. The street network in the figure has edges $AB = 7$, $AC = 4$, $BC = 3$, $BD = 6$, $BE = 8$, $CD = 5$ and $DE = 2$, total 35. The degrees are A 2, B 4, C 3, D 3, E 2. The odd vertices are C and D, and the shortest route between them is $CD = 5$. The shortest route is $35 + 5 = 40$ km, repeating CD.

The travelling salesman problem

The **travelling salesman problem** 旅行商问题 asks for the shortest cycle that visits every vertex. No quick method is known for large graphs, so we find bounds.

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nearest neighbour from A:
A–C–B–D–A $= 2+3+5+7 = 17$
upper bound
delete A: MST of B, C, D $= 3+5 = 8$
plus AC + AB $= 2+4$: lower bound 14
$14 \leq$ optimal tour ≤ 17

Nearest neighbour gives an upper bound, the deleted-vertex MST a lower bound: the optimal tour lies between 14 and 17

- An **upper bound** 上界, from the **nearest neighbour algorithm**: from a start vertex, always go to the nearest unvisited vertex, then return. From A: $A \to C$ (4) $\to B$ (3) $\to D$ (6) $\to E$ (2) $\to A$ (11), total 26.
- A **lower bound** 下界, from the **deleted vertex algorithm**: delete a vertex, find a minimum spanning tree of the rest, then add the two shortest edges from the deleted vertex. Delete A: the tree DE, BC, CD has weight 10, and the two shortest edges from A are 4 and 7, so the lower bound is 21.

So the shortest tour T satisfies $21 \leq T \leq 26$. Deleting a different vertex can give a better (higher) lower bound: deleting C gives 22. Always state which algorithm and which start vertex you used.

Common mistakes.

- Using the slant height of a cone or pyramid where the formula needs the perpendicular height.
- Leaving the calculator in the wrong angle mode.
- Forgetting the fraction of the circle: a sector of $\theta°$ has area $\dfrac{\theta}{360}\pi r^2$.
- Measuring a bearing anticlockwise or from east. Bearings are clockwise from north, with three figures.
- Drawing a Voronoi edge anywhere between two sites. Each edge lies on the perpendicular bisector of the two sites.
- Adding a site without redrawing every cell it takes area from.
- **HL:** applying matrix transformations in the wrong order: **BA** means apply **A** first.
- **HL:** forgetting the area scale factor is $|\det \mathbf{M}|$.
- **HL:** in Kruskal's algorithm, adding an edge that makes a cycle.
- **HL:** treating the nearest-neighbour route as optimal. It gives an **upper** bound; the deleted-vertex method gives a **lower** bound.
- **HL:** in the Chinese postman problem, repeating any edges. Repeat the shortest pairing of the odd vertices.

Practice

1. Find the distance between $(1, 2, 3)$ and $(4, 6, 15)$, and their midpoint.
2. A pyramid has a square base of side 6 cm and height 4 cm. Find its volume and total surface area.
3. A silo is a cylinder of radius 3 m and height 10 m with a hemisphere on top. Find its volume.
4. A cuboid measures 8 cm by 6 cm by 5 cm. Find the angle between a space diagonal and the base.
5. The angle of elevation of the top of a tower from 40 m away is $32°$. Find its height.
6. From a cliff 60 m high, the angle of depression of a boat is $18°$. How far is the boat from the foot of the cliff?
7. In triangle ABC, $A = 48°$, $B = 75°$ and $a = 12$ cm. Find b and the area.
8. Find the largest angle of a triangle with sides 7, 8 and 10 m.
9. A ship sails 12 km on a bearing of $050°$, then 20 km on $160°$. Find its distance and bearing from the start.
10. A slice of pizza of radius 15 cm has angle $40°$. Find its arc length, area and perimeter.
11. Find the perpendicular bisector of $P(2, 1)$ and $Q(6, 5)$.
12. Wells are at $A(1, 1)$, $B(7, 1)$ and $C(1, 7)$. Find the edges between them and the Voronoi vertex.
13. A fourth well is at $D(9, 9)$. Find the bisector of B and D, and the vertex where it meets $y = x$.
14. Explain why the vertex in question 13 is a good place for a storage tank that must be as far as possible from all four wells, in the region $0 \le x, y \le 10$.
15. **AHL.** Convert $150°$ to radians, and find the arc and area of a sector of radius 8 m and angle 1.2 radians.
16. **AHL.** θ is obtuse and $\sin\theta = 0.6$. Find $\cos\theta$ and $\tan\theta$.
17. **AHL.** Solve $3\sin x + 1 = 2.5$ for $0 \le x \le 2\pi$.
18. **AHL.** Find the single matrix for a rotation of $90°$ anticlockwise followed by an enlargement with scale factor 2, and the area of the image of a triangle of area 3.
19. **AHL.** Write the matrix for a reflection in $y = x$, and find its determinant. What does the sign tell you?
20. **AHL.** Find the angle between $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$, and the area of the triangle they make.
21. **AHL.** A boat moves by $\mathbf{r} = t\begin{pmatrix} 6 \\ 8 \end{pmatrix}$ km. Find its speed, and its closest distance to a lighthouse at $(20, 10)$.

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22. **AHL.** A ball moves by $\mathbf{r} = \begin{pmatrix} 15t \\ 20t - 4.9t^2 \end{pmatrix}$ m. When and how far away does it land?
23. **AHL.** A graph has edges AB, AC, BC and CD. Write its adjacency matrix, and find the number of walks of length 2 from A to D.
24. **AHL.** Use Kruskal's algorithm on the table of village distances. State the order of the edges and the total weight.
25. **AHL.** A network has odd vertices P, Q, R and S, with shortest routes $PQ = 4$, $RS = 6$, $PR = 5$, $QS = 7$, $PS = 8$ and $QR = 3$. The edges total 52. Find the length of the shortest Chinese postman route.
26. **AHL.** For the village table, find an upper bound for the travelling salesman problem by nearest neighbour from A, and a lower bound by deleting C.

Answers.

1. 13; $(2.5, 4, 9)$.
2. 48 cm^3; 96 cm^2.
3. $108\pi \approx 339$ m^3.
4. $\tan\theta = \frac{5}{10}$, $\theta \approx 26.6°$.
5. $40\tan 32° \approx 25.0$ m.
6. $\frac{60}{\tan 18°} \approx 185$ m.
7. $b \approx 15.6$ cm; area ≈ 78.5 cm^2.
8. $\cos C = \frac{13}{112}$, $C \approx 83.3°$.
9. 19.5 km on a bearing of about 125°.
10. 10.5 cm; 78.5 cm^2; 40.5 cm.
11. $y = -x + 7$.
12. $x = 4$ for $y \le 4$, $y = 4$ for $x \le 4$, and $y = x$ for $x \ge 4$, meeting at the vertex $(4, 4)$.
13. $y = -0.25x + 7$; vertex $(5.6, 5.6)$.
14. It is about 4.81 km from its three nearest wells, further than the other vertex (4.24 km) and any point on the boundary (at most about 4.61 km).
15. **AHL.** $\frac{5\pi}{6} \approx 2.62$; arc 9.6 m; area 38.4 m^2.
16. **AHL.** $\cos\theta = -0.8$; $\tan\theta = -0.75$.
17. **AHL.** $x = \frac{\pi}{6}$ or $\frac{5\pi}{6}$.
18. **AHL.** $\begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$; determinant 4, so area 12.
19. **AHL.** $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; determinant -1: area is kept but the shape is flipped over.
20. **AHL.** $\theta \approx 53.3°$; area $\frac{1}{2}\sqrt{45} \approx 3.35$.
21. **AHL.** 10 km/h; 10 km, at $t = 2$.
22. **AHL.** At $t \approx 4.08$ s, about 61.2 m away.

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23. **AHL.** $M = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$; the (A, D) entry of M^2 is 1 (the walk ACD).
24. **AHL.** DE 2, BC 3, AC 4, CD 5 (BD rejected); total 14.
25. **AHL.** Pairings: $4+6=10$, $5+7=12$, $8+3=11$. Use PQ and RS: $52+10=62$.
26. **AHL.** Upper bound 26 ($ACBDEA$); deleting C: tree DE, BD, AB has weight 15, plus $CB=3$ and $CA=4$, so lower bound 22.

Exam tips

- Draw the right-angled triangle you are using on its own, flat on the page.
- Draw a north line at every point where a bearing is measured, and give bearings with three figures.
- Check whether the question wants the curved surface or the total surface area.
- In degrees, a sector is the fraction $\frac{\theta}{360}$ of the circle.
- A Voronoi edge is part of a perpendicular bisector: find the midpoint and the negative reciprocal gradient.
- For the largest empty circle, compare the vertices with the boundary and the corners.
- **HL:** set the GDC to radians when using $l = r\theta$ or $A = \frac{1}{2}r^2\theta$.
- **HL:** for "A then B", the matrix is BA; area scales by $|\det|$.
- **HL:** for closest approach, minimise the distance squared, then check that the connecting vector is perpendicular to the velocity.
- **HL:** name the algorithm, show each step, and state whether a result is an upper or a lower bound.

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Statistics and probability

International Baccalaureate · IB Diploma · Mathematics: Applications and Interpretation · HL

Scope and prerequisites

This handout teaches topic 4, Statistics and probability, of Mathematics: Applications and Interpretation, the course first assessed in 2021: sampling, presenting and summarising data, correlation and regression, Spearman's rank correlation, probability, discrete random variables, the binomial and normal distributions, and hypothesis testing with the chi-squared and t-tests, and, for HL, reliability and validity, non-linear regression, linear combinations of random variables, the central limit theorem, confidence intervals, the Poisson distribution, critical regions and errors, and Markov chains. It is written for both SL and HL students. HL students also apply the further results in the sections marked AHL.

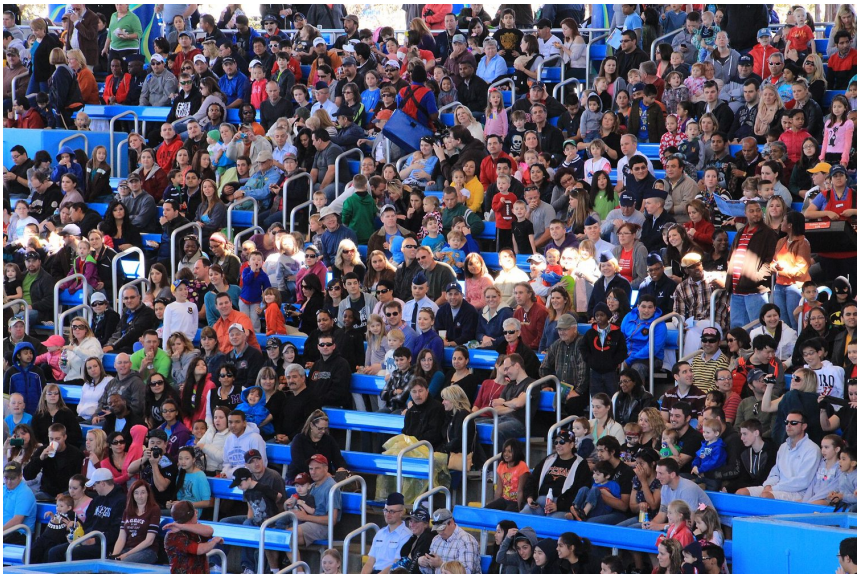
Prerequisites: fractions, decimals and percentages, substitution into formulae, and matrices (topic 1) for the HL Markov chains.

Sampling and data

The lady tasting tea

In the 1920s, at an agricultural research station north of London, the biologist Muriel Bristol said she could taste whether the milk or the tea had been poured into a cup first. The statistician Ronald Fisher designed a test: eight cups, four of each kind, in a random order. If she were only guessing, the chance of picking the right four is $\frac{1}{70} \approx 0.014$, so a perfect score would be strong evidence that she was not guessing. According to one of those present, she identified them all. Fisher used the story to open his book on the design of experiments in 1935. Every hypothesis test in this topic asks his question: could the result have happened by chance alone?

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A crowd is a population: a well-chosen sample tells you about it without asking everyone



M: milk first, T: tea first; 4 of each in random order
choosing the right 4 by guessing: $1/\binom{8}{4} = 1/70 \approx 0.014$

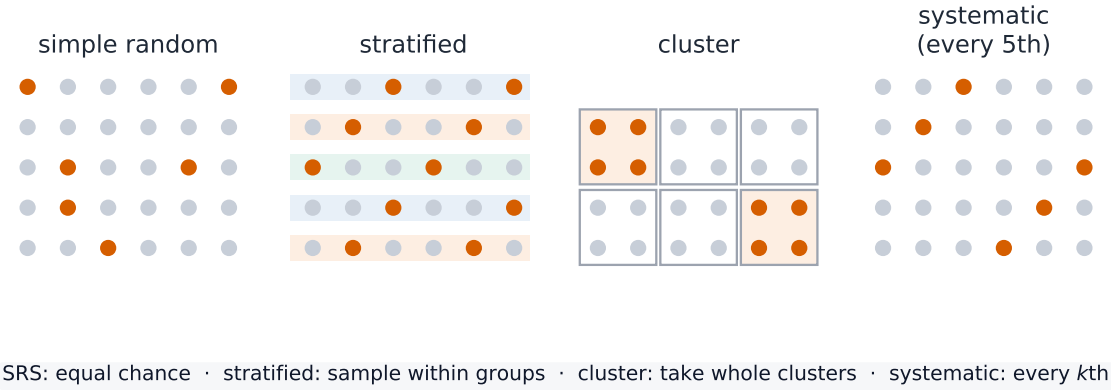
Fisher's eight cups

Populations and samples

- The **population** 总体 is every member of the group being studied; a **sample** 样本 is the part actually measured.
- A sample is **biased** when some members of the population are more likely to be included, so the sample does not represent the population.
- **Discrete data** 离散数据 can take only separate values (number of siblings); **continuous data** 连续数据 can take any value in a range (height, time).

Sampling techniques

Technique	How	Strength	Weakness
Simple random	Every member equally likely, such as random numbers	No selection bias	Needs a full list
Systematic	Every kth member from a list	Quick, spread out	Biased if the list has a pattern
Stratified	Random samples from each group in proportion	Represents every group	Needs group sizes
Quota	Fixed numbers from each group, not random	Quick, cheap	Interviewer chooses: bias
Convenience	Whoever is easy to reach	Easiest	Rarely representative



Four random sampling designs: who gets selected, and how

> **Worked example.** A school has 300 students in Year 10 and 200 in Year 11. A stratified sample of 50 takes $\frac{300}{500} \times 50 = 30$ from Year 10 and 20 from Year 11.

An **outlier** 异常值 is a value far from the rest: more than $1.5 \times$ IQR below Q_1 or above Q_3. Check an outlier before removing it: a recording error can be corrected, but a genuine extreme value is part of the data.

Summarising data

■ Measures of centre and spread

For the data $3, 5, 7, 7, 8, 10, 12, 20$ ($n = 8$):



median $= 3$

mean $= \frac{2+3+3+5+7}{5} = 4$

mode $= 3$ (most common)

range $= 7 - 2 = 5$

Mean, median, mode and range of a small data set

- **mean** 平均数 $\bar{x} = \dfrac{\sum x}{n} = \dfrac{72}{8} = 9$;
- **median** 中位数 $= \dfrac{7+8}{2} = 7.5$, the middle value of the ordered data;
- **mode** 众数 $= 7$, the most frequent value;
- **quartiles** 四分位数 $Q_1 = 6$ and $Q_3 = 11$, so the **interquartile range** 四分位距 is IQR $= 5$. Since $11 + 1.5 \times 5 = 18.5$, the value 20 is an outlier;
- **standard deviation** 标准差 $\sigma \approx 4.90$, from the GDC.

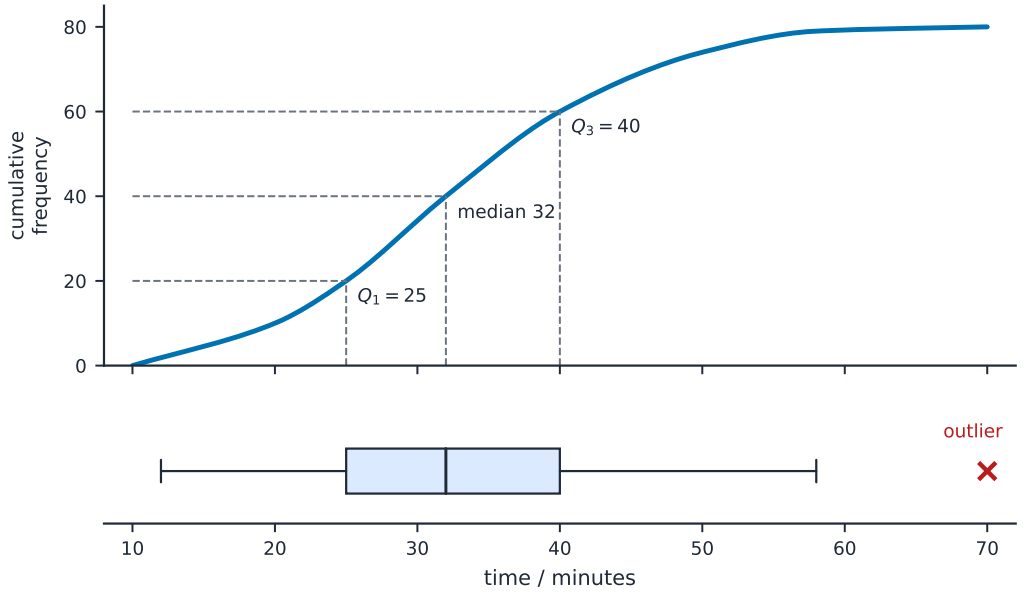
The median and IQR resist outliers; the mean and standard deviation are pulled by them. If every value is multiplied by a and then b is added, the mean and median become $a(\cdot) + b$, while the standard deviation and IQR are multiplied by $|a|$ and not changed by b. For $a \geq 0$, quartiles also become $a(\cdot) + b$; for $a < 0$, their order reverses, so the new Q_1 is $aQ_3 + b$ and the new Q_3 is $aQ_1 + b$.

For grouped data, use the mid-interval values: $\bar{x} \approx \dfrac{\sum fx}{\sum f}$. The result is an estimate because the exact data are not known.

■ Cumulative frequency and box plots

A **cumulative frequency** 累积频率 graph plots the running total against the upper class boundary. Reading across at $\frac{n}{4}$, $\frac{n}{2}$ and $\frac{3n}{4}$ gives estimates of Q_1, the median and Q_3. A **box plot** 箱线图 shows the minimum, Q_1, median, Q_3 and maximum, with outliers marked separately.

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Quartiles from a cumulative frequency curve, and the matching box plot

For the 80 students, the upper fence is $40 + 1.5 \times 15 = 62.5$ minutes, so the 70-minute time is an outlier and the whisker stops at the largest value inside the fence, 58 minutes. When comparing two box plots, compare a measure of centre (median) **and** a measure of spread (IQR) in context.

Correlation and regression

■ Pearson's correlation coefficient

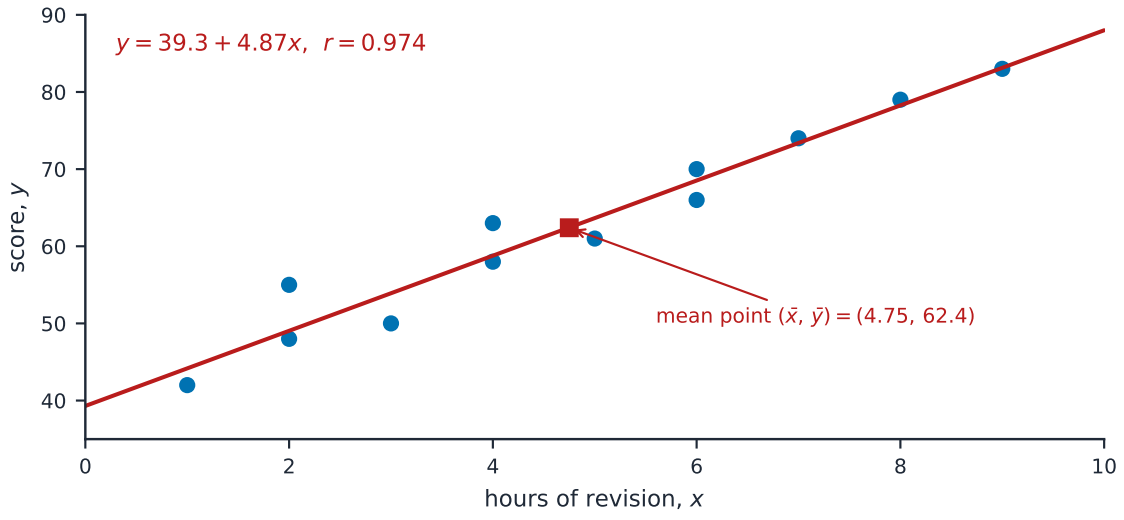
A **scatter diagram** 散点图 shows paired data (x, y). **Pearson's product-moment correlation coefficient** r, found with technology, measures the strength of the **linear** relationship: $-1 \le r \le 1$. A common guide to describing it:

$\lvert r \rvert$	**Strength of linear correlation**
$0 < \lvert r \rvert \le 0.25$	very weak
$0.25 < \lvert r \rvert \le 0.5$	weak
$0.5 < \lvert r \rvert \le 0.75$	moderate
$0.75 < \lvert r \rvert \le 1$	strong

Correlation is not causation: ice-cream sales and drownings both rise in hot weather, because a third variable, temperature, drives both.

The regression line

The **regression line** 回归直线 of y on x, found with technology, minimises the squared vertical distances and passes through the mean point $(\bar{x}, \bar{y})$.



Regression line of y on x through the mean point

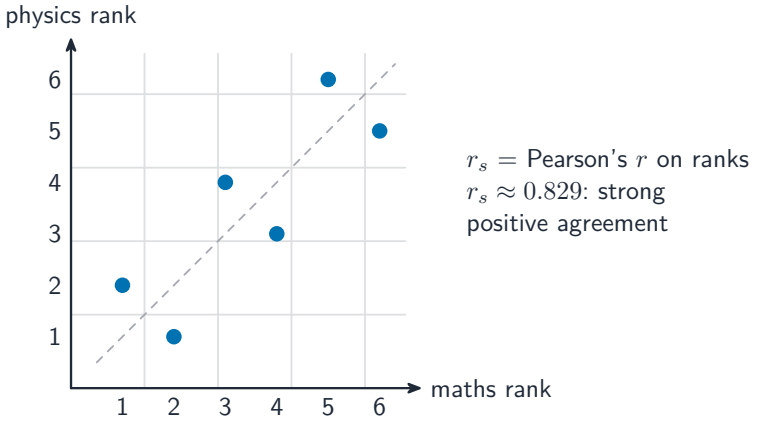
Worked example. For 12 students, hours of revision x and score y give $r = 0.974$ and $y = 39.3 + 4.87x$.

- Prediction for 5.5 hours: $y = 39.3 + 4.87 \times 5.5 \approx 66$. Reliable: r is strong and 5.5 is within the data (**interpolation** 内插).
- The gradient means each extra hour of revision is associated with about 4.9 more marks.
- A prediction for 15 hours gives 112, more than the maximum mark: **extrapolation** 外推 is unreliable.
- Use this line only to predict y from x.

Spearman's rank correlation

Spearman's rank correlation coefficient 斯皮尔曼等级相关系数 r_s is Pearson's r applied to the ranks of the data. It measures how well the relationship is **monotonic**: whether y always increases (or always decreases) as x increases, even if not in a straight line. Equal values share the mean of their ranks.

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Two rankings in strong agreement

Worked example. Six students are ranked in mathematics (1 to 6) and in physics as 2, 1, 4, 3, 6, 5. Pearson's r on the ranks, by GDC, gives $r_s \approx 0.829$: a strong positive agreement between the two rankings.

Use r_s when the data are ranks, when there are outliers, or when the pattern is a curve that keeps rising. For data that rise and then fall, neither coefficient describes the relationship.

Probability

Events and Venn diagrams

For equally likely outcomes, the **probability** 概率 of event A is $P(A) = \dfrac{n(A)}{n(U)}$, where U is the **sample space** 样本空间. $P(A') = 1 - P(A)$, and the **expected number** of times A happens in n trials is $nP(A)$.

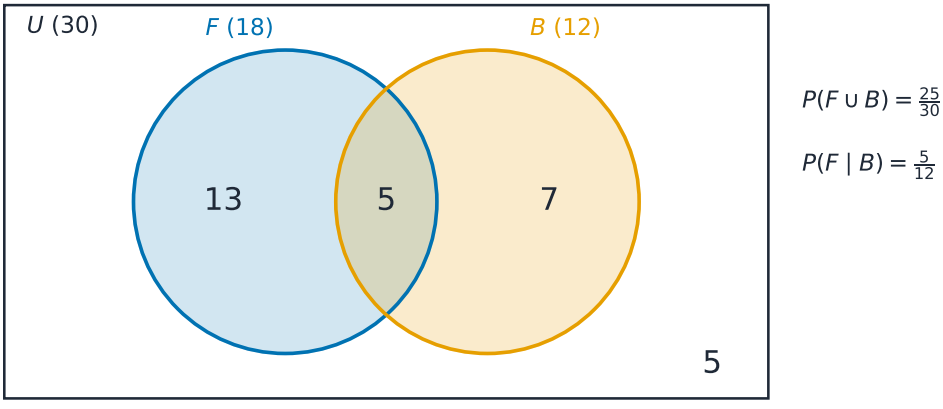
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Fair dice: each face is equally likely, the simplest probability model

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Mutually exclusive 互斥 events cannot both happen: $P(A \cap B) = 0$.



Fill a Venn diagram from the overlap outwards

Worked example. In a class of 30, 18 play football (F), 12 basketball (B) and 5 both. Then F only $= 13$, B only $= 7$, neither $= 5$, and $P(F \cup B) = \frac{25}{30} = \frac{5}{6}$.

■ Conditional probability and independence

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Given that a student plays basketball, the chance they also play football is $P(F \mid B) = \frac{5}{12}$: the sample space shrinks to the 12 basketball players.

Events are **independent** 独立 if $P(A \cap B) = P(A)P(B)$. Here $0.6 \times 0.4 = 0.24 \neq \frac{5}{30}$, so the two sports are not independent.

> **Worked example.** Two counters are taken without replacement from a bag of 3 red and 2 blue. A **tree diagram** 树状图 gives $P(RR) = \frac{3}{5} \times \frac{2}{4} = 0.3$ and $P(BB) = \frac{2}{5} \times \frac{1}{4} = 0.1$, so $P(\text{same colour}) = 0.4$. Multiply along branches, add between them.

Discrete random variables and the binomial distribution

■ Expected value

A **discrete random variable** 离散随机变量 X takes separate values with probabilities that add to 1. Its **expected value** 期望值 is $E(X) = \sum x\,P(X = x)$.

> **Worked example.** A game costs ¥3. The prize is ¥0, ¥2 or ¥10 with probabilities 0.5, 0.3 and 0.2. $E(X) = 0 + 0.6 + 2 = 2.6$, so the expected gain is -0.4 yuan per game. A game is **fair** if the expected gain is 0.

■ The binomial distribution

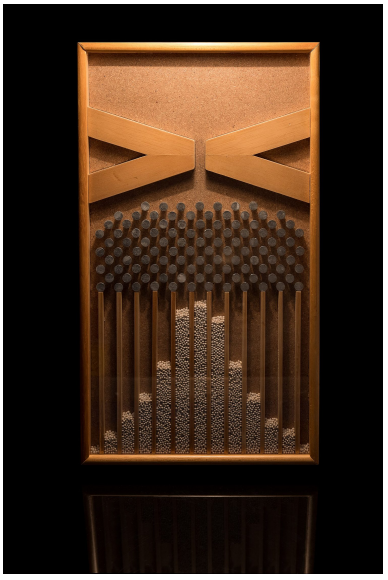
$X \sim B(n, p)$ counts the successes in n independent trials, each with the same probability of success p. $E(X) = np$ and $\mathrm{Var}(X) = np(1-p)$. Use the GDC's binomial pdf for $P(X = r)$ and cdf for $P(X \leq r)$.

> **Worked example.** 20 % of seeds fail to grow. Five are planted, $X \sim B(5, 0.2)$. $P(X = 1) = 0.4096$, $P(X \geq 1) = 1 - 0.8^5 \approx 0.672$, and $E(X) = 1$.

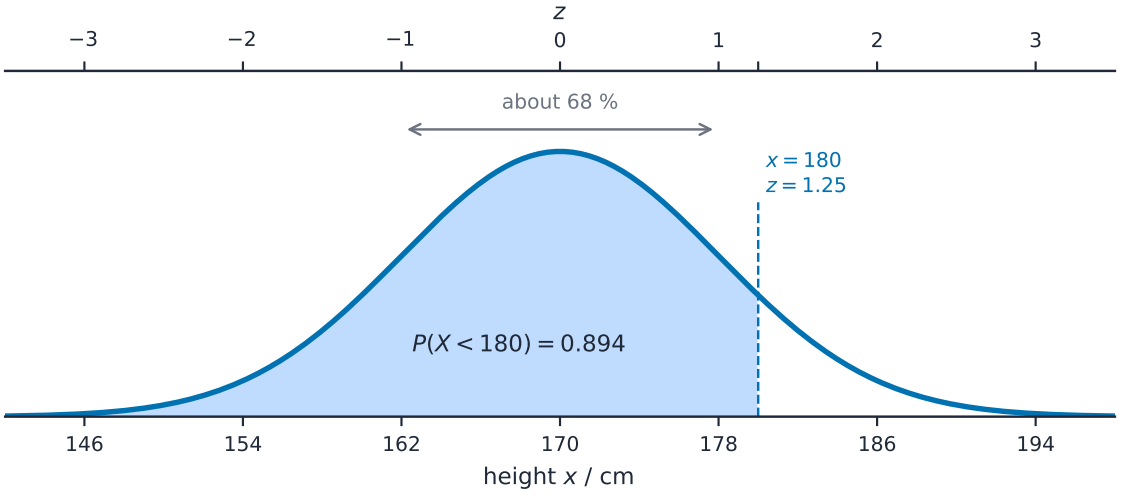
The normal distribution

Many measurements follow a **normal distribution** 正态分布 $X \sim N(\mu, \sigma^2)$: a symmetric bell-shaped curve centred on the mean μ. About 68 % of values lie within 1σ of the mean, 95 % within 2σ and 99.7 % within 3σ.

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A Galton board: balls falling through pins pile up into the bell-shaped normal curve



Probability is area under the curve; z counts standard deviations from the mean

Worked example. Heights $X \sim N(170, 8^2)$ cm.

- $P(X < 180) = 0.894$ (by GDC; $z = \frac{180-170}{8} = 1.25$).
- $P(160 < X < 180) = 0.789$.
- The height exceeded by only 10 % of people: inverse normal, $P(X < h) = 0.9$, gives $h \approx 180.3$ cm.

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Hypothesis testing

The language of a test

1. The **null hypothesis** 原假设 H_0 says there is no effect or no association. The **alternative hypothesis** 备择假设 H_1 says there is.
2. Choose a **significance level** 显著性水平, usually 5 %.
3. Calculate the **p-value**: the probability, if H_0 were true, of a result at least as extreme as the one observed.
4. If the p-value is less than the significance level, reject H_0. Otherwise there is not enough evidence to reject it. A test never proves H_0.

The chi-squared test for independence

The **chi-squared test** 卡方检验 for independence uses a **contingency table** 列联表 of observed frequencies. The expected frequency of each cell, if the variables were independent, is

$$E = \frac{\text{row total} \times \text{column total}}{\text{grand total}}, \qquad \chi^2_{\text{calc}} = \sum \frac{(O-E)^2}{E}$$

with **degrees of freedom** 自由度 $(\text{rows}-1)(\text{columns}-1)$. All expected frequencies must be at least 5.

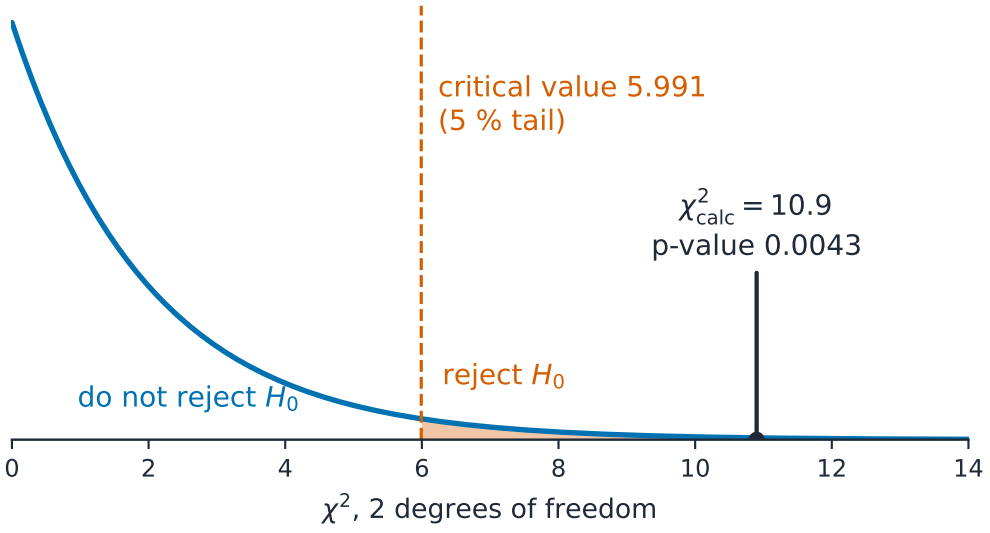
Worked example. 100 people at a school were asked their favourite drink.

Observed	Tea	Coffee	Juice	Total
Students	20	18	22	60
Teachers	20	17	3	40
Total	40	35	25	100

- H_0: favourite drink is independent of being a student or a teacher.
- Expected: students 24, 21, 15; teachers 16, 14, 10. All are at least 5.
- GDC: $\chi^2_{\text{calc}} \approx 10.9$, with $(2-1)(3-1) = 2$ degrees of freedom, p-value ≈ 0.0043.
- $0.0043 < 0.05$, so reject H_0: there is evidence that favourite drink depends on the group. The largest contribution comes from teachers and juice: 3 observed, 10 expected.

You may instead compare χ^2_{calc} with the **critical value** 临界值, given in the question: here 5.991. Since $10.9 > 5.991$, reject H_0.

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Reject H_0 when the statistic falls in the 5 % tail

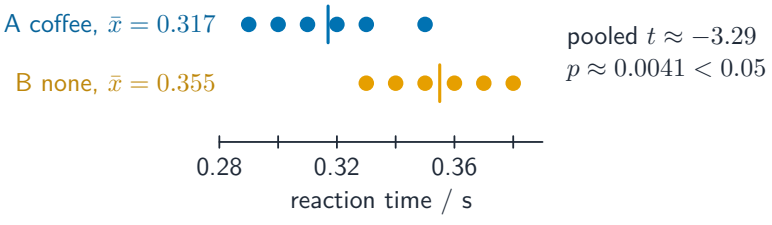
The chi-squared goodness of fit test

A **goodness of fit test** 拟合优度检验 asks whether data follow a given distribution. The degrees of freedom are the number of classes minus 1.

> **Worked example.** A die is rolled 60 times, giving 8, 9, 12, 7, 10 and 14 for the faces 1 to 6. H_0: the die is fair, so each expected frequency is 10. $\chi^2_{\text{calc}} = \frac{4+1+4+9+0+16}{10} = 3.4$, with 5 degrees of freedom and p-value 0.639. Since $0.639 > 0.05$, there is not enough evidence that the die is biased.

The t-test

The **t-test** compares the means of two independent samples from normal populations with equal variances (a **pooled** two-sample t-test). It can be **one-tailed** 单尾 ($H_1 : \mu_1 < \mu_2$) or **two-tailed** 双尾 ($H_1 : \mu_1 \neq \mu_2$). Decide which before looking at the data.



The two samples in the t-test

> **Worked example.** Reaction times in seconds for six students who drank coffee, A, and six who did not, B:

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A	**0.32**	**0.29**	**0.35**	**0.30**	**0.33**	**0.31**
B	0.36	0.34	0.38	0.33	0.37	0.35

- $H_0 : \mu_A = \mu_B$; $H_1 : \mu_A < \mu_B$, at the 5 % level.
- GDC (pooled): $\bar{x}_A \approx 0.317$, $\bar{x}_B = 0.355$, $t \approx -3.29$, p-value ≈ 0.0041.
- $0.0041 < 0.05$, so reject H_0: there is evidence that the coffee group reacts faster. With only six in each group, a larger study would be needed before drawing firm conclusions.

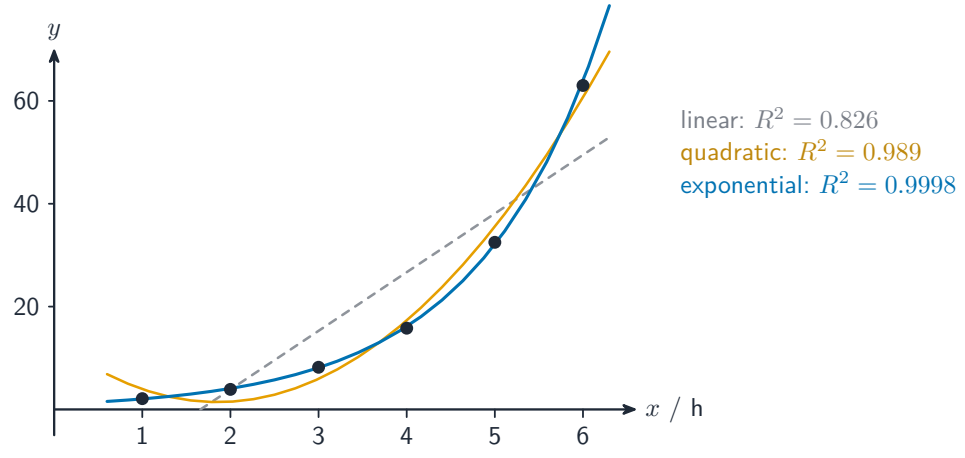
AHL: Designing studies and non-linear regression

Reliability and validity

- A measurement is **reliable** 可靠 if it gives consistent results. **Test–retest** reliability repeats the same test on the same people; **parallel forms** gives two equivalent versions of a test. Compare the results with a correlation coefficient.
- A measurement is **valid** 有效 if it measures what it claims to. **Content validity** asks whether the test covers what it should; **criterion validity** asks whether it agrees with an accepted measure.
- In a chi-squared test, combine neighbouring classes if an expected frequency is below 5, and reduce the degrees of freedom by one for each parameter estimated from the data.

Non-linear regression and R^2

For any model, the **residual** 残差 of a point is observed − predicted. The **sum of square residuals** SS_{res} measures how far the data are from the model, and the **coefficient of determination** 决定系数



Three models for the bacteria count

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$$R^2 = 1 - \frac{SS_{\text{res}}}{SS_{\text{tot}}}$$

is the proportion of the variation in y that the model explains. For a straight line, $R^2 = r^2$.

> **Worked example.** A bacteria count is $y = 2.1, 3.9, 8.2, 15.8, 32.5, 63.0$ at $x = 1, 2, \ldots, 6$ hours.

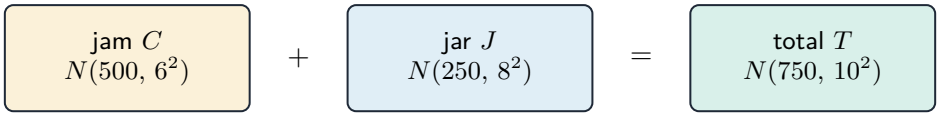
Model (GDC)	SS_{res}	R^2
linear $y = 11.4x - 18.9$	475	0.826
quadratic $y = 3.45x^2 - 12.8x + 13.3$	31.0	0.989
exponential $y = 1.03 \times 1.99^x$	0.596	0.9998

The exponential model has the highest R^2 and also makes sense: the count roughly doubles every hour. The quadratic also has a high R^2 but predicts a minimum near $x = 1.85$ and falling counts before that, which fits no real process. Choose a model by R^2 **and** by context.

AHL: Linear combinations and the central limit theorem

■ Expectation and variance of combinations

For a random variable X and constants a and b:



variances add: $36 + 64 = 100$ (standard deviations do not: $6 + 8 \neq 10$)
$P(T > 765) \approx 0.0668$

Variances add; standard deviations do not

$$E(aX + b) = aE(X) + b, \qquad \text{Var}(aX + b) = a^2\text{Var}(X)$$

For independent $X_1, X_2, \ldots$:

$$E(a_1X_1 \pm a_2X_2) = a_1E(X_1) \pm a_2E(X_2), \qquad \text{Var}(a_1X_1 \pm a_2X_2) = a_1^2\text{Var}(X_1) + a_2^2\text{Var}(X_2)$$

Variances always **add**, even for a difference. A linear combination of independent normal variables is normal.

> **Worked example.** Jam $C \sim N(500, 6^2)$ g is put in a jar $J \sim N(250, 8^2)$ g, independently of the jam mass. The total $T = C + J \sim N(750, 6^2 + 8^2) = N(750, 10^2)$, so $P(T > 765) \approx 0.0668$.

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Note the difference between $3X$ and $X_1 + X_2 + X_3$: one item weighed three times as much has variance $9\sigma^2$, but three separate items have variance $3\sigma^2$.

Unbiased estimates

The sample mean $\bar{x}$ is an **unbiased estimate** 无偏估计 of the population mean μ. For the variance, divide by $n-1$, not n:

$$s_{n-1}^2 = \frac{n}{n-1} s_n^2$$

> **Worked example.** Eight batteries last 47.2, 49.5, 48.1, 50.3, 46.8, 48.9, 47.5 and 49.0 hours. $\bar{x} \approx 48.4$, $s_n^2 \approx 1.29$, and the unbiased estimate of the population variance is $s_{n-1}^2 = \frac{8}{7} \times 1.29 \approx 1.48$, so $s_{n-1} \approx 1.21$.

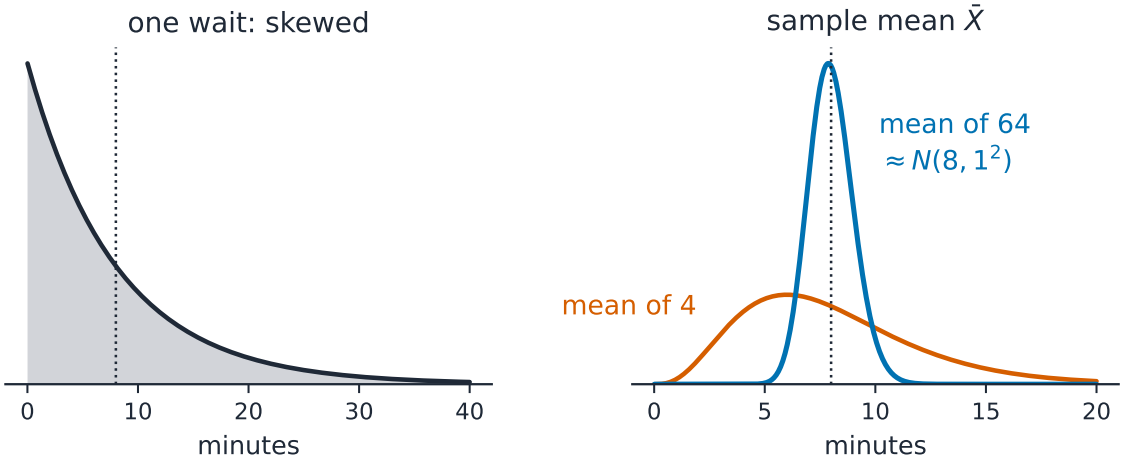
The central limit theorem

The **central limit theorem** 中心极限定理 says that the mean $\bar{X}$ of a random sample of size n from any population with mean μ and variance σ^2 is approximately

$$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

when n is large, usually $n \geq 30$. If the population is itself normal, this is exact for any n.

> **Worked example.** Waiting times at a bank have mean 8 minutes and standard deviation 8 minutes: most waits are short, a few are long, so the distribution is far from normal. For 64 customers, $\bar{X} \sim N\left(8, \frac{64}{64}\right) = N(8, 1^2)$ approximately, so $P(\bar{X} > 10) \approx 0.0228$.

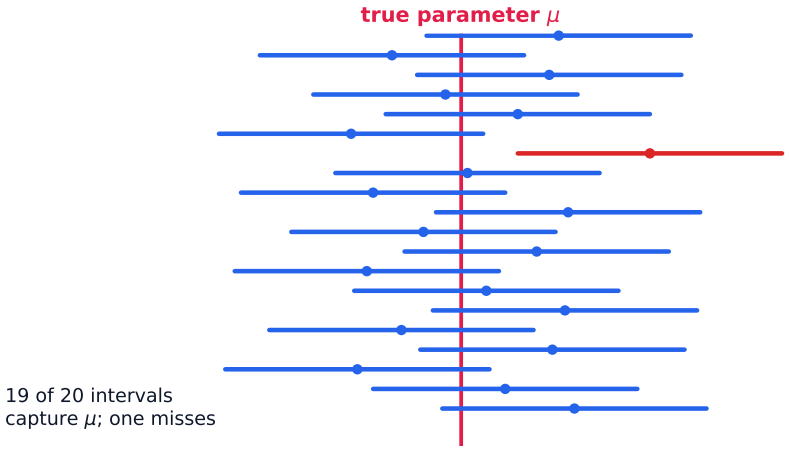


Means of larger samples are closer to normal and less spread out

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Confidence intervals

A **confidence interval** 置信区间 gives a range of likely values for the population mean.



CI form: $\bar{x} \pm z^* \frac{\sigma}{\sqrt{n}}$ · level = long-run capture rate of the method

"95% confident" describes the method, not one interval: over many samples about 95% of the intervals contain μ and about 5% miss it.

- If σ is known: $\bar{x} \pm z\dfrac{\sigma}{\sqrt{n}}$, with $z = 1.96$ for 95 %. For $n = 25$, $\bar{x} = 172.4$ and $\sigma = 8$: $172.4 \pm 1.96 \times 1.6$, that is $169.3 < \mu < 175.5$.
- If σ is unknown, use s_{n-1} and the t-distribution, with $n - 1$ degrees of freedom (GDC: TInterval). For the eight batteries, the 95 % interval is $47.4 < \mu < 49.4$ hours.

A 95 % interval means: if the sampling were repeated many times, about 95 % of the intervals made this way would contain μ. The claimed mean of 50 hours lies outside the interval.

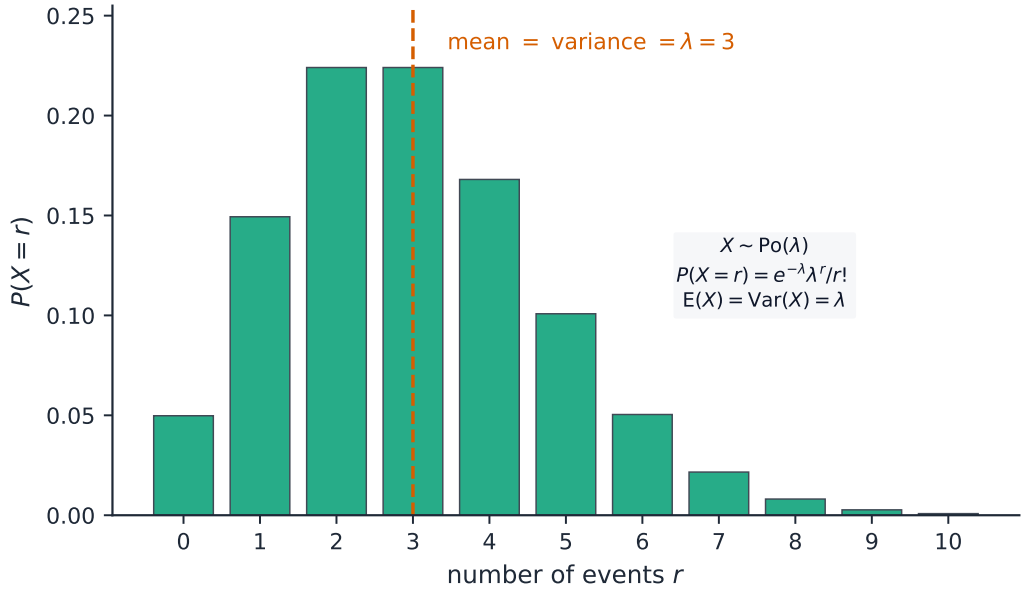
AHL: The Poisson distribution

The **Poisson distribution** 泊松分布 $X \sim Po(m)$ counts random events in a fixed interval of time or space, when events happen independently and at a constant average rate m.

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People arriving at random at a steady average rate: the number per minute follows a Poisson distribution



The Poisson distribution Po(3)*: for a Poisson variable the mean and variance both equal* λ*.*

$$P(X = x) = \frac{m^x e^{-m}}{x!}, \qquad E(X) = \mathrm{Var}(X) = m$$

Worked example. A help desk receives on average 3 calls an hour, $X \sim Po(3)$.

- $P(X = 0) = e^{-3} \approx 0.0498$.
- $P(X \leq 2) \approx 0.423$ (GDC Poisson cdf).

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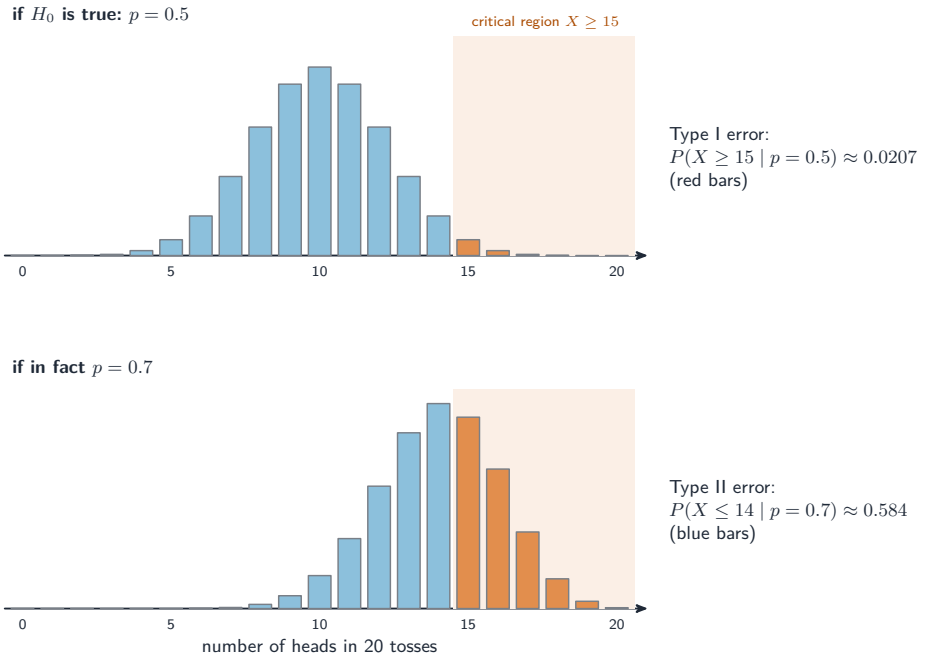
- In two hours the mean is 6, so $Y \sim Po(6)$ and $P(Y \geq 10) = 1 - P(Y \leq 9) \approx 0.0839$.

The sum of independent Poisson variables is Poisson: $Po(m_1) + Po(m_2) = Po(m_1 + m_2)$. A quick check that data might be Poisson: the sample mean and variance should be close.

AHL: Critical regions and errors

■ Critical regions and the two types of error

The **critical region** 拒绝域 is the set of results that lead to rejecting H_0.



The critical region $X \geq 15$ keeps the Type I error at 0.0207, but with $p = 0.7$ the test still misses the bias with probability 0.584

- A **Type I error** 第一类错误 rejects H_0 when it is true. Its probability is the actual significance level.
- A **Type II error** 第二类错误 fails to reject H_0 when it is false.

Worked example. A coin is tossed 20 times to test $H_0 : p = 0.5$ against $H_1 : p > 0.5$ at the 5 % level. $X \sim B(20, 0.5)$.

- $P(X \geq 14) \approx 0.0577 > 0.05$, but $P(X \geq 15) \approx 0.0207 < 0.05$, so the critical region is $X \geq 15$.
- $P(\text{Type I error}) = P(X \geq 15 \mid p = 0.5) \approx 0.0207$.
- If in fact $p = 0.7$, $P(\text{Type II error}) = P(X \leq 14 \mid p = 0.7) \approx 0.584$: with only 20 tosses, a biased coin is missed more often than not.

More tests

- **One-sample t-test** for a mean: eight batteries with $\bar{x} \approx 48.4$ test the claim $\mu = 50$ against $\mu < 50$. GDC: $t \approx -3.70$, p-value ≈ 0.0038, so reject the claim at the 5 % level.
- A **paired** t-test compares two measurements on the same subjects, such as before and after training: test the mean of the differences.
- A test for correlation uses $H_0 : \rho = 0$, where ρ is the population correlation. For the 12 students, $r = 0.974$ gives a p-value far below 0.01, strong evidence of a positive correlation.
- Binomial and Poisson tests compare an observed count with the distribution under H_0, as in the coin example.

AHL: Markov chains

A **Markov chain** 马尔可夫链 moves between states, and the probability of the next state depends only on the current state. The **transition matrix** 转移矩阵 T has, in each column, the probabilities of moving from that state; each column adds to 1. If $\mathbf{s}_0$ is the starting state vector, then after n steps

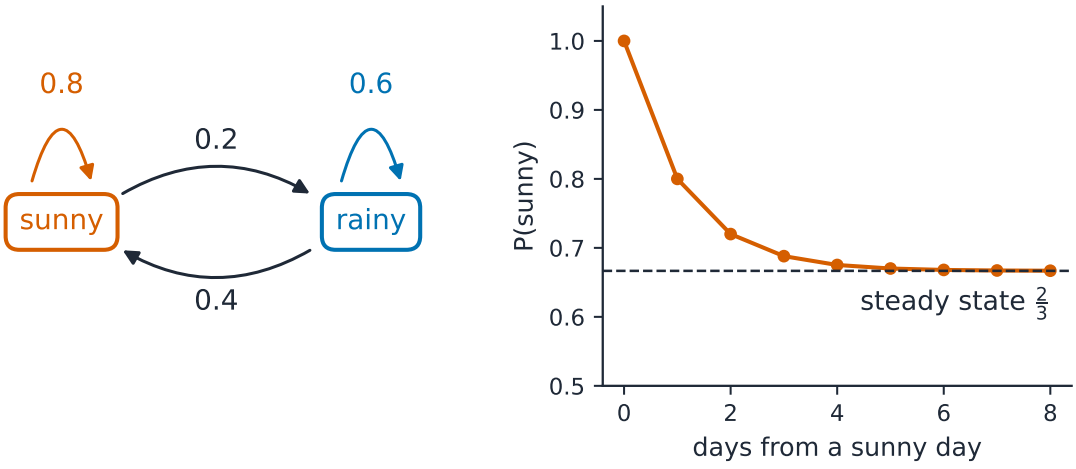
$$\mathbf{s}_n = T^n \mathbf{s}_0$$

Worked example. If today is sunny, tomorrow is sunny with probability 0.8; if today is rainy, tomorrow is sunny with probability 0.4. With states (sunny, rainy):

$$T = \begin{pmatrix} 0.8 & 0.4 \\ 0.2 & 0.6 \end{pmatrix}$$

- Today is sunny, $\mathbf{s}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Tomorrow: $T\mathbf{s}_0 = \begin{pmatrix} 0.8 \\ 0.2 \end{pmatrix}$. In two days: $T^2\mathbf{s}_0 = \begin{pmatrix} 0.72 \\ 0.28 \end{pmatrix}$.
- The **steady state** 稳态 $\mathbf{s}$ satisfies $T\mathbf{s} = \mathbf{s}$: $0.2s_1 = 0.4s_2$, so $s_1 = 2s_2$, and with $s_1 + s_2 = 1$, $\mathbf{s} = \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$.
- In the long run, two days in three are sunny, whatever the weather today. It is the eigenvector of T with eigenvalue 1 (topic 1).

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The chain forgets its start and settles at the steady state

Common mistakes.

- Reading a strong correlation as proof of cause.
- Predicting x from the regression line of y on x, or extrapolating far beyond the data.
- Writing *accept* H_0 or H_0 *is true*. Say: there is not enough evidence to reject H_0.
- Comparing the p-value the wrong way: reject H_0 when p is **less than** the significance level.
- Using the wrong degrees of freedom: a χ^2 test for independence on an $r \times c$ table has $(r-1)(c-1)$.
- Writing $P(X \geq 3) = 1 - P(X \leq 3)$ for a binomial variable. It is $1 - P(X \leq 2)$.
- Reading the variance as the standard deviation: $X \sim N(50, 16)$ has $\sigma = 4$.
- **HL:** subtracting variances: for independent X and Y, $\operatorname{Var}(X - Y) = \operatorname{Var}(X) + \operatorname{Var}(Y)$.
- **HL:** dividing by n for the unbiased estimate of the variance. It uses $n - 1$.
- **HL:** forgetting to scale a Poisson mean: 3 calls per hour means a mean of 1.5 per half hour.
- **HL:** mixing up the errors: a Type I error rejects a true H_0; a Type II error keeps a false one.

Practice

1. A school has 300 students in Year 10 and 200 in Year 11. How many from each year are in a stratified sample of 50?
2. For $3, 5, 7, 7, 8, 10, 12, 20$, find the median and IQR, and show that 20 is an outlier.
3. Each value in a data set with mean 9 and standard deviation 4.90 is doubled and then 3 is added. Find the new mean and standard deviation.

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4. The regression line of score on hours is $y = 39.3 + 4.87x$ with $r = 0.974$, for 0 to 9 hours. Predict the score for 5.5 hours, and comment on a prediction for 15 hours.
5. Six students are ranked $1, 2, 3, 4, 5, 6$ in mathematics and $2, 1, 4, 3, 6, 5$ in physics. Find Spearman's rank correlation coefficient and interpret it.
6. In a class of 30, 18 play football, 12 basketball and 5 both. Find $P(F \cup B)$ and $P(F \mid B)$.
7. Show that playing football and playing basketball are not independent in question 6.
8. Two counters are taken without replacement from 3 red and 2 blue. Find the probability that they are the same colour.
9. A game costs ¥3, with prizes ¥0, ¥2, ¥10 with probabilities 0.5, 0.3, 0.2. Find the expected gain.
10. $X \sim B(5, 0.2)$. Find $P(X = 1)$, $P(X \geq 1)$ and $E(X)$.
11. Heights are $N(170, 8^2)$ cm. Find $P(160 < X < 180)$ and the height exceeded by 10 % of people.
12. Find the expected frequencies for the drinks table, and the degrees of freedom.
13. Test at the 5 % level whether the die with frequencies 8, 9, 12, 7, 10, 14 is fair.
14. For the reaction-time data, state the hypotheses for a one-tailed t-test, and interpret the p-value 0.0041.
15. **AHL.** Explain the difference between reliability and validity, with an example of each.
16. **AHL.** Linear, quadratic and exponential models have R^2 0.826, 0.989 and 0.9998 for the bacteria data. Which model would you choose, and why?
17. **AHL.** $C \sim N(500, 6^2)$ and $J \sim N(250, 8^2)$ are independent. Find the distribution of $C + J$ and $P(C + J > 765)$.
18. **AHL.** X has variance 4. Find $\operatorname{Var}(3X)$ and $\operatorname{Var}(X_1 + X_2 + X_3)$ for three independent copies of X.
19. **AHL.** Find the unbiased estimate of the population variance from $n = 8$ and $s_n^2 = 1.29$.
20. **AHL.** Waiting times have mean 8 and standard deviation 8 minutes. Find the approximate probability that the mean of 64 waits exceeds 10 minutes.
21. **AHL.** Find a 95 % confidence interval for μ when $n = 25$, $\bar{x} = 172.4$ and $\sigma = 8$.
22. **AHL.** Calls arrive at a mean rate of 3 an hour. Find the probability of no calls in an hour, and of at least 10 calls in two hours.
23. **AHL.** For 20 tosses of a coin, testing $H_0 : p = 0.5$ against $H_1 : p > 0.5$ at 5 %, find the critical region and the probability of a Type I error.
24. **AHL.** In question 23, find the probability of a Type II error if $p = 0.7$.
25. **AHL.** For the weather chain $T = \begin{pmatrix} 0.8 & 0.4 \\ 0.2 & 0.6 \end{pmatrix}$, find the probability of sun in two days if today is sunny, and the steady state.

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26. **AHL.** Explain why the steady state does not depend on today's weather.

Answers.

1. 30 from Year 10 and 20 from Year 11.
2. Median 7.5; IQR $11 - 6 = 5$; $20 > 11 + 7.5 = 18.5$, so 20 is an outlier.
3. Mean $2(9) + 3 = 21$; standard deviation $2 \times 4.90 = 9.80$.
4. About 66 marks, reliable (strong r, interpolation); 15 hours gives 112, an unreliable extrapolation beyond the data and the maximum mark.
5. $r_s \approx 0.829$: strong agreement between the two subjects' rankings.
6. $\frac{5}{6}$; $\frac{5}{12}$.
7. $P(F)P(B) = 0.6 \times 0.4 = 0.24$, but $P(F \cap B) = \frac{5}{30} \approx 0.167$.
8. $0.3 + 0.1 = 0.4$.
9. $2.6 - 3 = -0.4$ yuan.
10. 0.4096; 0.672; 1.
11. 0.789; about 180.3 cm.
12. Students 24, 21, 15; teachers 16, 14, 10; 2 degrees of freedom.
13. $\chi^2 = 3.4$, 5 degrees of freedom, p-value $0.639 > 0.05$: not enough evidence that the die is biased.
14. $H_0 : \mu_A = \mu_B$, $H_1 : \mu_A < \mu_B$. $0.0041 < 0.05$, so reject H_0: evidence that the coffee group reacts faster.
15. **AHL.** Reliable: consistent results, such as the same score on a retest. Valid: measures what it claims, such as a test that covers the whole syllabus.
16. **AHL.** Exponential: highest R^2, and the count roughly doubles each hour, which suits growth of bacteria.
17. **AHL.** $N(750, 10^2)$; about 0.0668.
18. **AHL.** 36; 12.
19. **AHL.** $\frac{8}{7} \times 1.29 \approx 1.48$.
20. **AHL.** $\bar{X} \sim N(8, 1^2)$, so about 0.0228.
21. **AHL.** $169.3 < \mu < 175.5$.
22. **AHL.** $e^{-3} \approx 0.0498$; $1 - P(Y \leq 9)$ for $Y \sim Po(6)$, about 0.0839.
23. **AHL.** $X \geq 15$; about 0.0207.
24. **AHL.** $P(X \leq 14 \mid p = 0.7) \approx 0.584$.
25. **AHL.** 0.72; $\left(\frac{2}{3}, \frac{1}{3}\right)$.
26. **AHL.** $T^n \mathbf{s}_0$ approaches the eigenvector with eigenvalue 1 for any starting vector, because the other eigenvalue, 0.4, has powers that tend to 0.

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Exam tips

- Name the sampling method and say why it might be biased in context.
- Compare data sets with a centre **and** a spread, in context.
- Interpret r by strength and direction, and remember that correlation is not causation.
- Write H_0 and H_1 in words about the context, and decide one- or two-tailed first.
- Compare the p-value with the significance level, then write the conclusion in context: "there is (not) enough evidence that …".
- Check that every expected frequency in a chi-squared test is at least 5.
- **HL:** variances add for sums **and** differences of independent variables.
- **HL:** for a critical region, list the probabilities either side of the boundary.
- **HL:** a Markov transition matrix has columns that add to 1.

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Calculus

International Baccalaureate · IB Diploma · Mathematics: Applications and Interpretation · HL

Scope and prerequisites

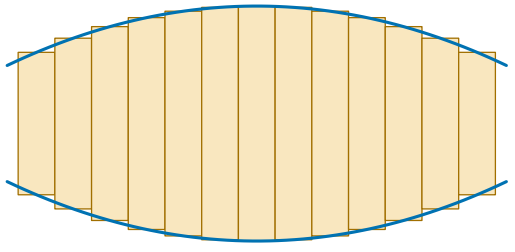
This handout teaches topic 5, Calculus, of Mathematics: Applications and Interpretation, the course first assessed in 2021: gradients and derivatives of ax^n, increasing and decreasing functions, tangents and normals, stationary points and optimisation, integration of ax^n and areas, and the trapezoidal rule, and, for HL, further derivatives and the chain, product and quotient rules, related rates and the second derivative, further integration and volumes of revolution, kinematics, differential equations with slope fields and Euler's method, and coupled systems with phase portraits. It is written for both SL and HL students. HL students also apply the further results in the sections marked AHL.

Prerequisites: gradients of lines and functions (topic 2), indices, and, for HL, eigenvalues and eigenvectors (topic 1).

Gradients and derivatives

Kepler's wine casks

In 1613 the astronomer Johannes Kepler bought wine for his wedding in Linz. He watched the merchant measure each barrel by pushing a rod through the hole in its side to the far corner and reading off a length, whatever the shape of the barrel. Kepler doubted that one length could give a volume, and he wrote a whole book about it. He found volumes by cutting a solid into very thin slices and adding them up, an early form of integration. He also showed that the Austrian barrel shape gave almost the largest volume for its rod length, and that near a maximum, small changes in shape hardly change the volume, so the merchants' rule was nearly fair. That is the key idea of optimisation: at a maximum the rate of change is zero.

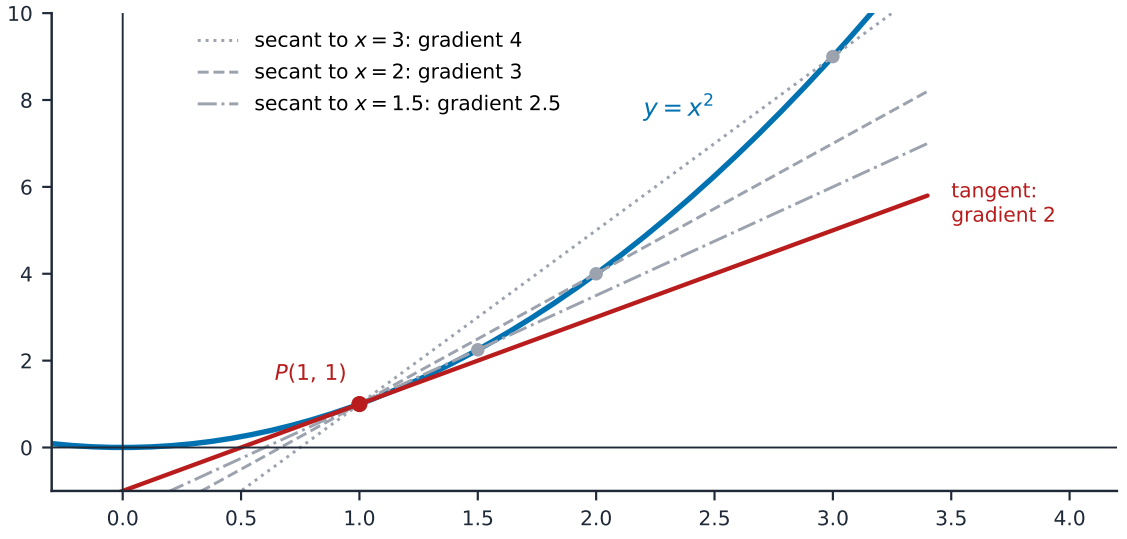


Kepler's idea: slice and add

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Secants and the tangent

The gradient of a chord, or **secant** 割线, between $P(x, f(x))$ and a nearby point $Q(x+h, f(x+h))$ is $\dfrac{f(x+h)-f(x)}{h}$. As $h \to 0$ the secant approaches the **tangent** 切线 at P. The limit of the secant gradients is the **derivative** 导数 $f'(x)$, also written $\dfrac{dy}{dx}$.



Secant gradients approach the tangent gradient

The derivative has two meanings:

- the **gradient** of the curve $y = f(x)$ at a point;
- the **rate of change** of y with respect to x: if V is a volume in litres and t is time in seconds, $\dfrac{dV}{dt}$ is in litres per second.

The power rule

For any integer n, the derivative of ax^n is anx^{n-1}, and the derivative of a constant is 0. Differentiate term by term.

> **Worked example.** $y = 4x^3 - 5x + \dfrac{2}{x} + 7$. Write $\dfrac{2}{x} = 2x^{-1}$. Then $\dfrac{dy}{dx} = 12x^2 - 5 - 2x^{-2} = 12x^2 - 5 - \dfrac{2}{x^2}$.

f is **increasing** 递增 where $f'(x) > 0$ and **decreasing** 递减 where $f'(x) < 0$. For $f(x) = x^3 - 3x$, $f'(x) = 3x^2 - 3$, which is negative for $-1 < x < 1$: f decreases there and increases elsewhere.

Tangents and normals

At $x = a$ the tangent has gradient $m = f'(a)$ and passes through $(a, f(a))$. The **normal** 法线 is perpendicular to the tangent, with gradient $-\dfrac{1}{m}$.

> **Worked example.** For $y = x^2$ at $x = 2$: $y = 4$ and $\dfrac{dy}{dx} = 2x = 4$.

- Tangent: $y - 4 = 4(x - 2)$, so $y = 4x - 4$.
- Normal: gradient $-\frac{1}{4}$, so $y - 4 = -\frac{1}{4}(x - 2)$, or $x + 4y - 18 = 0$.

On a calculator paper, the GDC finds the gradient at a point directly (numerical derivative); write down the value and what it means.

Stationary points and optimisation

Local maximum and minimum points

A **stationary point** 驻点 is where $f'(x) = 0$: the tangent is horizontal. Look at the sign of $f'(x)$ on each side:



Fenced enclosures: optimisation finds the dimensions that give the largest area for a fixed length of fence

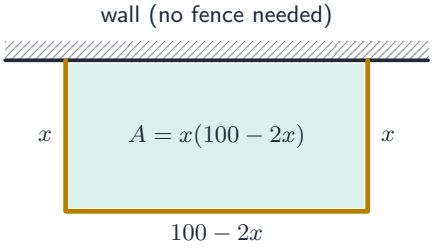
f' before	f' after	Point
$+$	$-$	**local maximum** 局部最大值
$-$	$+$	**local minimum** 局部最小值

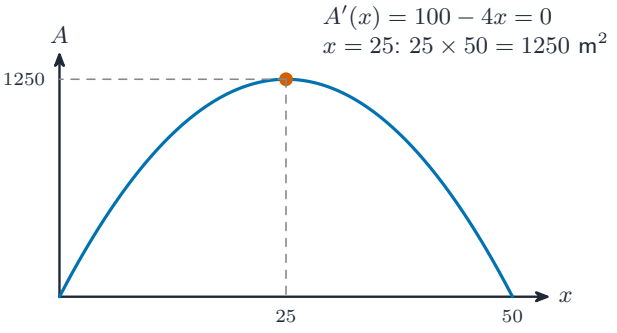
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Worked example. $f(x) = x^3 - 3x$. $f'(x) = 3x^2 - 3 = 0$ gives $x = \pm 1$. Since f' changes from $+$ to $-$ at $x = -1$, $(-1, 2)$ is a local maximum; it changes from $-$ to $+$ at $x = 1$, so $(1, -2)$ is a local minimum.

Optimisation

To make a quantity as large or as small as possible:





Use the fixed fence length to write A in one variable; $A'(x) = 0$ at $x = 25$ gives the largest pen

1. Write it as a function of **one** variable, using any fixed condition to remove the others.
2. Differentiate and solve the derivative $= 0$, using the GDC if needed.
3. Check that it is a maximum or a minimum, and check the ends of the domain.
4. Answer the question asked, with units.

Worked example. A drinks can must hold 500 cm^3. Find the radius that uses the least metal.

- Volume: $\pi r^2 h = 500$, so $h = \dfrac{500}{\pi r^2}$.
- Surface area: $A = 2\pi r^2 + 2\pi r h = 2\pi r^2 + \dfrac{1000}{r}$, for $r > 0$.
- $\dfrac{dA}{dr} = 4\pi r - \dfrac{1000}{r^2} = 0$ gives $r^3 = \dfrac{250}{\pi}$, so $r \approx 4.30$ cm, and then $h \approx 8.60$ cm.
- $\dfrac{dA}{dr}$ changes from negative to positive, so this is a minimum: $A \approx 349$ cm^2.

The best can has its height equal to its diameter. Real cans are taller and thinner, because the ends are made of thicker metal and the shape must suit a hand and a fridge: the model leaves these out.

Worked example. A farmer has 100 m of fence to make a rectangular pen against a long wall, so only three sides need fencing. With the two sides at right angles to the wall x m long, the third side is $100-2x$ and the area is $A = x(100-2x) = 100x-2x^2$, for $0 < x < 50$. $\dfrac{dA}{dx} = 100 - 4x = 0$ at $x = 25$, and $\dfrac{dA}{dx}$ changes from positive to negative there, so the largest area is $25 \times 50 = 1250$ m^2.

Worked example. A company's profit is $P(x) = -2x^2 + 120x - 1000$ yuan when it sells x items a day. $P'(x) = -4x + 120 = 0$ at $x = 30$, and $P(30) = ¥800$ is the maximum daily profit.

Integration

Antiderivatives

Integration 积分 reverses differentiation. For $n \neq -1$:

$$\int ax^n \, dx = \frac{a}{n+1}x^{n+1} + c$$

The **constant of integration** 积分常数 c appears because every constant differentiates to 0. A **boundary condition**, a known point on the curve, finds c.

Worked example. A curve has gradient $f'(x) = 6x^2 - 4x + 5$ and passes through $(1, 6)$. Then $f(x) = 2x^3 - 2x^2 + 5x + c$, and $f(1) = 5 + c = 6$, so $c = 1$ and $f(x) = 2x^3 - 2x^2 + 5x + 1$.

Definite integrals and area

The **definite integral** 定积分 $\displaystyle\int_a^b f(x)\,dx$ is the area between the curve and the x-axis from $x = a$ to $x = b$, when $f(x) \geq 0$. On a calculator paper, find it with the GDC and write down the integral you used.

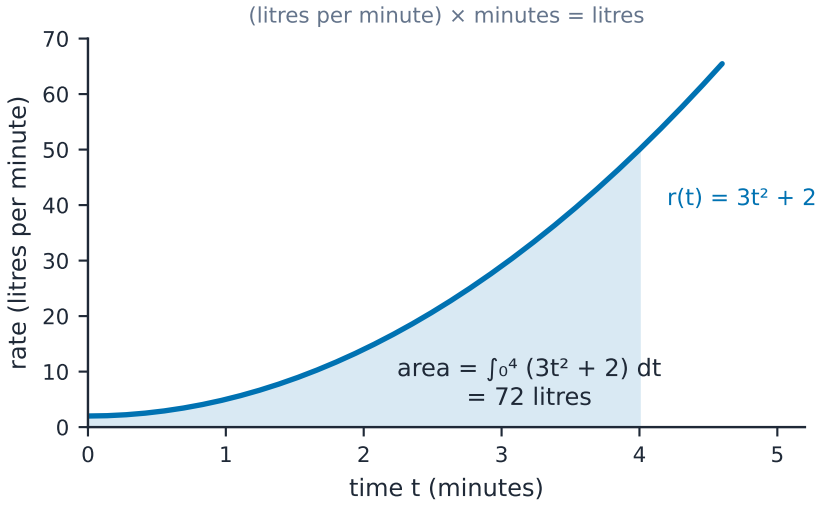
Worked example. The area under $y = 4 - x^2$ from $x = 0$ to $x = 2$ is $\displaystyle\int_0^2 (4 - x^2)\,dx = \left[4x - \frac{x^3}{3}\right]_0^2 = 8 - \frac{8}{3} = \frac{16}{3} \approx 5.33$.

From a rate to a total

If $r(t)$ is a rate of change, then $\displaystyle\int_a^b r(t)\,dt$ is the total change from $t = a$ to $t = b$: the area under a rate graph is an amount.

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Worked example. Water flows into a tank at $r(t) = 3t^2 + 2$ litres per minute. In the first 4 minutes the tank gains $\displaystyle\int_0^4 (3t^2 + 2)\,dt = \Big[t^3 + 2t\Big]_0^4 = 64 + 8 = 72$ litres. The units of the area are (litres per minute) $\times$ minutes $=$ litres.



The area under a rate graph is an amount: 72 litres in 4 minutes

The trapezoidal rule

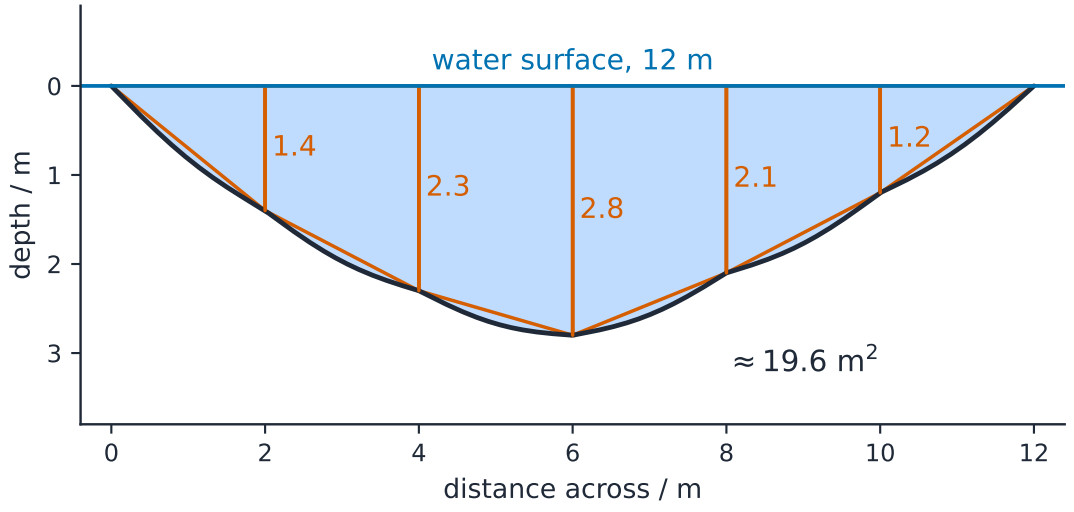
When only data are known, or the function cannot be integrated, estimate the area with trapezoids. With n strips of equal width $h = \dfrac{b-a}{n}$:

$$\int_a^b y\,dx \approx \frac{h}{2}\Big(y_0 + y_n + 2(y_1 + y_2 + \cdots + y_{n-1})\Big)$$

Worked example. A river is 12 m wide. Its depth, measured every 2 m across, is 0, 1.4, 2.3, 2.8, 2.1, 1.2 and 0 m. The cross-section area is about

$$\frac{2}{2}\Big(0 + 0 + 2(1.4 + 2.3 + 2.8 + 2.1 + 1.2)\Big) = 19.6 \text{ m}^2$$

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Trapezoids under a curve that bends downwards give an underestimate

If the curve bends downwards (concave down), as here, the straight tops of the trapezoids lie below it, so the rule **underestimates** the area. If the curve bends upwards, it overestimates. More, narrower strips give a better estimate.

AHL: Further differentiation

■ Standard derivatives and the chain rule

$f(x)$	$f'(x)$
x^n (n rational)	nx^{n-1}
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\dfrac{1}{\cos^2 x}$
e^x	e^x
$\ln x$	$\dfrac{1}{x}$

Angles are in radians. For a function of a function, use the **chain rule** 链式法则: if $y = g(u)$ and $u = f(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Worked example. $y = e^{3x^2}$: with $u = 3x^2$, $\dfrac{dy}{dx} = e^u \times 6x = 6xe^{3x^2}$. Similarly $\dfrac{d}{dx}\sin(5x) = 5\cos(5x)$.

■ Product and quotient rules

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}, \qquad \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

Worked example. $\dfrac{d}{dx}(x^2\sin x) = 2x\sin x + x^2\cos x$, by the **product rule** 乘积法则.

Worked example. $\dfrac{d}{dx}\left(\dfrac{\ln x}{x}\right) = \dfrac{x\cdot\frac{1}{x} - \ln x\cdot 1}{x^2} = \dfrac{1-\ln x}{x^2}$, by the **quotient rule** 商法则. It is zero at $x = e$, the maximum of $\frac{\ln x}{x}$.

■ Related rates

When two quantities both change with time, the chain rule links their rates.

Worked example. Air is pumped into a spherical balloon at 50 cm^3/s. How fast is the radius growing when $r = 10$ cm? $V = \frac{4}{3}\pi r^3$, so $\dfrac{dV}{dt} = 4\pi r^2\dfrac{dr}{dt}$. Then $50 = 4\pi(100)\dfrac{dr}{dt}$, so $\dfrac{dr}{dt} \approx 0.0398$ cm/s. The radius grows ever more slowly as the balloon gets bigger.

■ The second derivative

The **second derivative** 二阶导数 $f''(x)$ is the rate of change of the gradient.

- $f''(x) > 0$: the curve is **concave up**, and a stationary point there is a minimum.
- $f''(x) < 0$: the curve is **concave down**, and a stationary point there is a maximum.
- A **point of inflection** 拐点 is where f'' changes sign.

For $f(x) = x^3 - 3x$, $f''(x) = 6x$: $f''(-1) = -6$ confirms the maximum, $f''(1) = 6$ the minimum, and $(0, 0)$ is a point of inflection.

AHL: Further integration and volumes

■ Standard integrals and integration by inspection

$$\int \frac{1}{x}\,dx = \ln|x| + c, \qquad \int e^x\,dx = e^x + c, \qquad \int \sin x\,dx = -\cos x + c, \qquad \int \cos x\,dx = \sin x + c$$

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A potter's vessel is a solid of revolution: rotating a curve about an axis gives its volume

With a linear inside, divide by its coefficient: $\int e^{2x}\,dx = \frac{1}{2}e^{2x} + c$ and $\int \cos(3x)\,dx = \frac{1}{3}\sin(3x) + c$.

If the integrand contains a function and (a multiple of) its derivative, reverse the chain rule. This is **integration by inspection**, or by substitution.

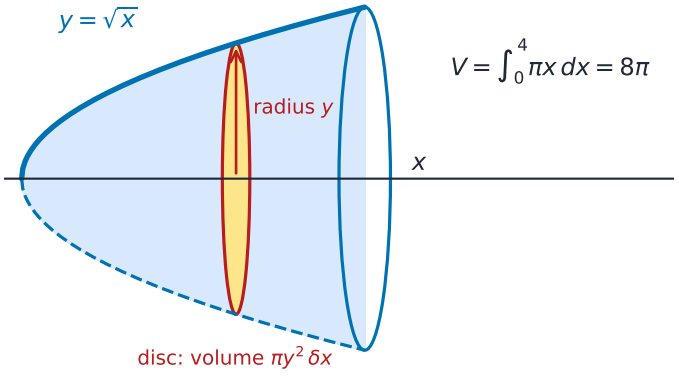
> **Worked example.** $\int 2x(x^2+1)^4\,dx = \frac{(x^2+1)^5}{5} + c$, because $2x$ is the derivative of x^2+1. Similarly $\int xe^{x^2}\,dx = \frac{1}{2}e^{x^2} + c$. Check by differentiating.

■ Areas and volumes of revolution

The area between a curve and the y-axis is $\int_c^d x\,dy$. When a region is rotated 2π about an axis, the **volume of revolution** 旋转体体积 is a stack of thin discs:

$$V = \pi\int_a^b y^2\,dx \text{ (about the } x\text{-axis)}, \qquad V = \pi\int_c^d x^2\,dy \text{ (about the } y\text{-axis)}$$

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A volume of revolution is a stack of thin discs

Worked example. The region under $y = \sqrt{x}$ for $0 \le x \le 4$ is rotated about the x-axis: $V = \pi \displaystyle\int_0^4 x\,dx = 8\pi \approx 25.1$. Kepler's thin slices, made exact.

AHL: Kinematics

For motion in a straight line, with **displacement** 位移 s at time t:

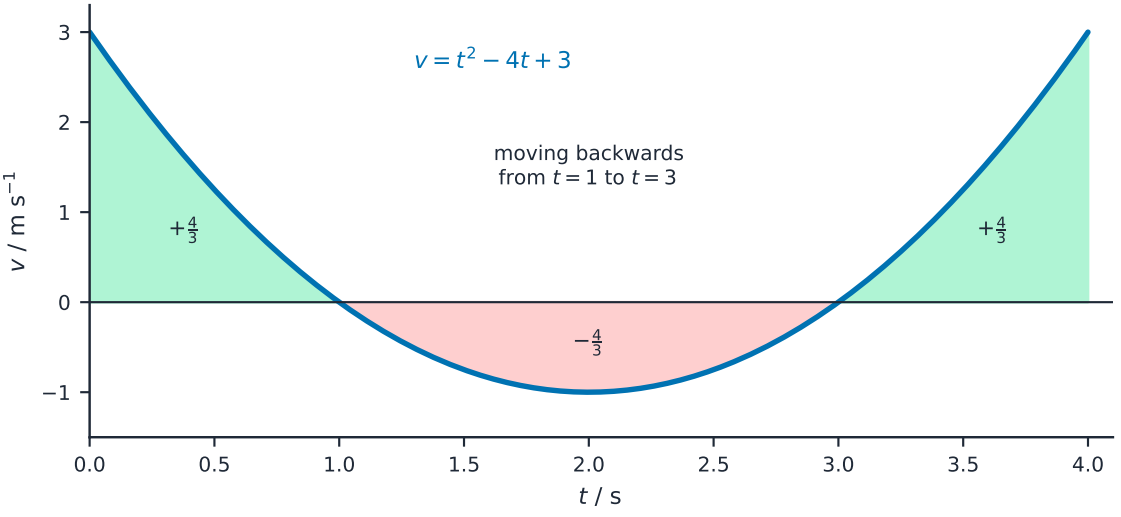


A high-speed train: differentiate displacement for velocity, and velocity for acceleration

$$v = \frac{ds}{dt}, \qquad a = \frac{dv}{dt} = \frac{d^2s}{dt^2}, \qquad s = \int v\,dt$$

The **velocity** 速度 v has a sign that shows direction; speed is $|v|$. The **acceleration**

加速度 is a. Displacement is $\int_{t_1}^{t_2} v\,dt$, but the **distance travelled** is $\int_{t_1}^{t_2} |v|\,dt$, which the GDC finds directly.



Signed areas give displacement; total area gives distance

Worked example. $v = t^2 - 4t + 3$ m/s for $0 \le t \le 4$.

- $v = (t-1)(t-3) = 0$ at $t = 1$ and $t = 3$: the particle changes direction twice.
- $a = 2t - 4$; at $t = 0$, $a = -4$ m/s^2.
- Displacement: $\int_0^4 v\,dt = \frac{4}{3}$ m. Distance: $\int_0^4 |v|\,dt = 4$ m.

AHL: Differential equations

Setting up and separating variables

A **differential equation** 微分方程 links a quantity with its rate of change. "The rate of growth is proportional to the size" is $\frac{dN}{dt} = kN$, with solution $N = Ae^{kt}$.

If the equation has the form $\frac{dy}{dx} = f(x)g(y)$, use **separation of variables** 分离变量法:

$$\int \frac{1}{g(y)}\,dy = \int f(x)\,dx$$

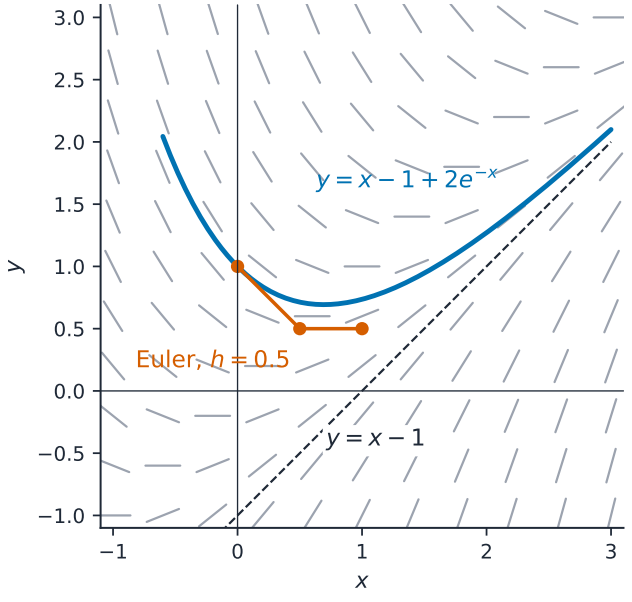
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Worked example. $\dfrac{dy}{dx} = xy$ with $y = 2$ when $x = 0$. Then $\displaystyle\int \frac{1}{y}\,dy = \int x\,dx$ gives $\ln y = \dfrac{x^2}{2} + c$. At $x = 0$, $c = \ln 2$, so $y = 2e^{x^2/2}$. At $x = 1$, $y \approx 3.30$.

Newton's law of cooling from topic 2 comes from $\dfrac{dT}{dt} = -k(T - 20)$: separating gives $T = 20 + Ae^{-kt}$.

Slope fields

A **slope field** 斜率场 draws, at many points, a short line with the gradient the differential equation gives there. Solution curves follow the lines, like leaves floating on a stream.



The solution follows the slope field; Euler's steps follow it roughly

For $\dfrac{dy}{dx} = x - y$, the slope is 0 along the line $y = x$, positive below it and negative above it. Every solution approaches the line $y = x - 1$, along which the slope is exactly 1.

Euler's method

Euler's method 欧拉法 follows the slope field in small steps of width h:

$$x_{n+1} = x_n + h, \qquad y_{n+1} = y_n + h \times \left.\frac{dy}{dx}\right|_{(x_n, y_n)}$$

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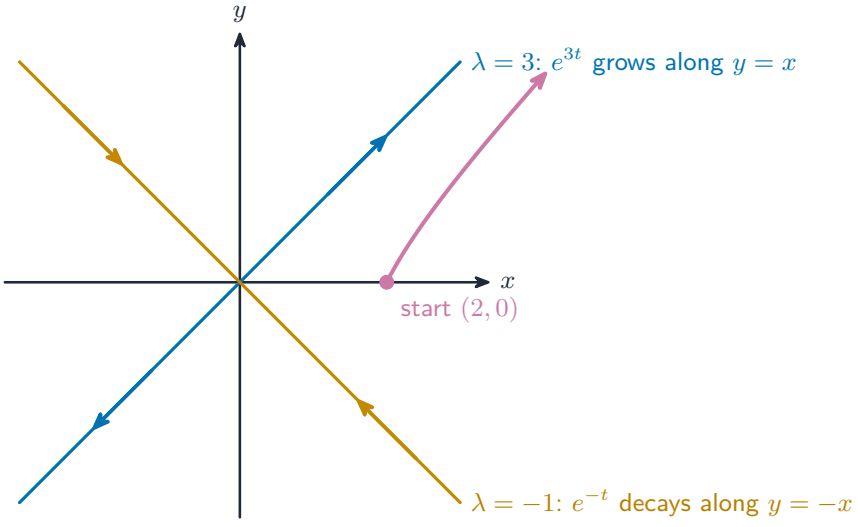
Worked example. $\dfrac{dy}{dx} = x - y$, $y(0) = 1$, $h = 0.5$.

- At $(0, 1)$ the gradient is -1, so $y_1 = 1 + 0.5(-1) = 0.5$ at $x = 0.5$.
- At $(0.5, 0.5)$ the gradient is 0, so $y_2 = 0.5$ at $x = 1$.
- The exact value is $y(1) = 2e^{-1} \approx 0.736$. With $h = 0.1$ (a spreadsheet or GDC program), Euler gives 0.697: smaller steps are more accurate.

AHL: Coupled systems and phase portraits

Coupled linear systems

Two quantities that affect each other's rates form a **coupled system** 耦合系统:



The growing eigenvalue term dominates, so the solution from (2, 0) heads out along $y = x$

$$\frac{dx}{dt} = ax + by, \qquad \frac{dy}{dt} = cx + dy, \qquad \text{that is} \quad \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = M\begin{pmatrix} x \\ y \end{pmatrix}$$

If M has eigenvalues λ_1, λ_2 with two linearly independent eigenvectors $\mathbf{p}_1$, $\mathbf{p}_2$ (topic 1), the solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = Ae^{\lambda_1 t}\mathbf{p}_1 + Be^{\lambda_2 t}\mathbf{p}_2$$

Worked example. $\dfrac{dx}{dt} = x + 2y$, $\dfrac{dy}{dt} = 2x + y$, with $x = 2$ and $y = 0$ at $t = 0$.

- $M = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ has eigenvalues 3 and -1, with eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

- $\begin{pmatrix}2\\0\end{pmatrix} = 1\begin{pmatrix}1\\1\end{pmatrix} + 1\begin{pmatrix}1\\-1\end{pmatrix}$, so $A = B = 1$.
- $x = e^{3t} + e^{-t}$ and $y = e^{3t} - e^{-t}$. In the long run the e^{3t} term wins, and the point moves out along the line $y = x$.

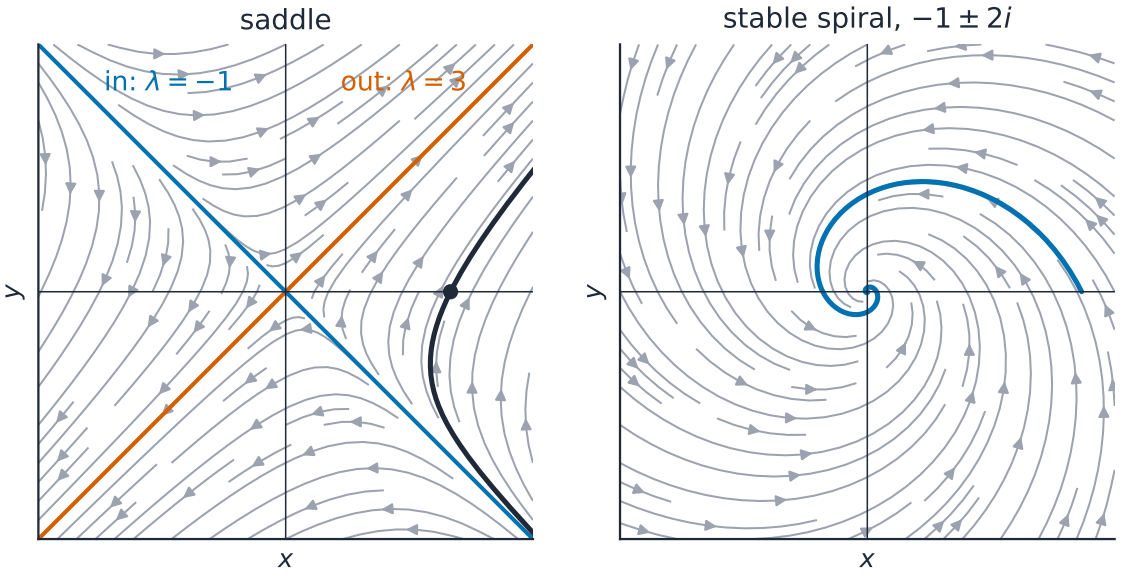
Phase portraits

A **phase portrait** 相图 shows the paths $(x(t), y(t))$ in the xy-plane. The eigenvalues decide its shape; the origin is the **equilibrium point** 平衡点.

Eigenvalues	Phase portrait	Equilibrium
real, both negative	paths approach the origin, generally along curves	stable
real, both positive	paths move away from the origin, generally along curves	unstable
real, opposite signs	**saddle point** 鞍点: in along one eigenvector, out along the other	unstable
complex, $\alpha \pm \beta i$ with $\alpha < 0$	spiral inwards	stable
complex, $\alpha > 0$	spiral outwards	unstable
imaginary, $\pm\beta i$	closed loops round the origin (a centre)	stable, but paths do not approach the origin

For a spiral, find the direction of rotation by working out $\frac{dx}{dt}$ and $\frac{dy}{dt}$ at one point, such as $(1, 0)$.

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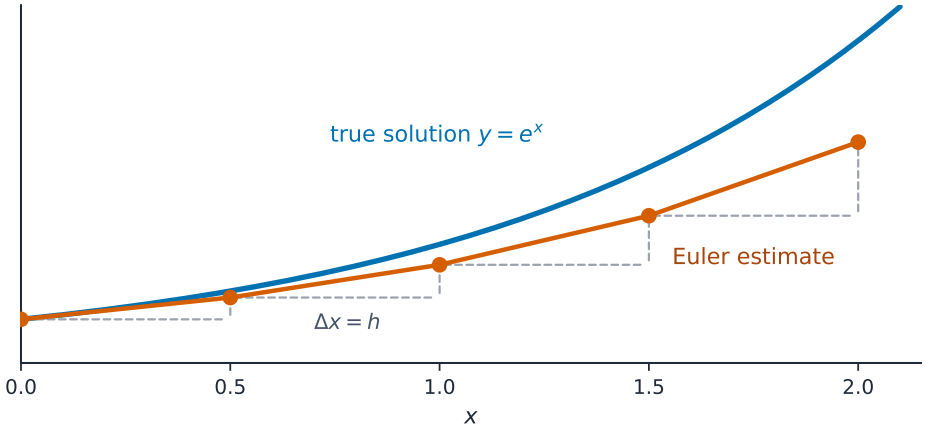


Real eigenvalues of opposite sign give a saddle; complex ones with negative real part give a stable spiral

Worked example. $\frac{dx}{dt} = -x - 2y$, $\frac{dy}{dt} = 2x - y$. The matrix $\begin{pmatrix} -1 & -2 \\ 2 & -1 \end{pmatrix}$ has eigenvalues $-1 \pm 2i$: the real part is negative, so the paths spiral into the origin. At $(1, 0)$, $\frac{dx}{dt} = -1$ and $\frac{dy}{dt} = 2$: the point moves up, so the spiral turns anticlockwise.

■ Euler's method for systems and second-order equations

Euler's method works for a coupled system, one step at a time for both variables:



$$\frac{dy}{dx} = y,\ y(0) = 1: \quad y_{n+1} = y_n + h\,y_n$$

Euler's method steps along the tangent; the estimate drifts from the true solution

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$$x_{n+1} = x_n + h\frac{dx}{dt}, \qquad y_{n+1} = y_n + h\frac{dy}{dt}, \qquad t_{n+1} = t_n + h$$

For $\dfrac{dx}{dt} = x + 2y$, $\dfrac{dy}{dt} = 2x + y$ from $(2, 0)$ with $h = 0.1$: the rates are 2 and 4, so the next point is $(2.2, 0.4)$. The exact values at $t = 0.1$ are $(2.25, 0.445)$.

A **second-order** equation such as $\dfrac{d^2x}{dt^2} + 2\dfrac{dx}{dt} + 5x = 0$, a damped spring, becomes a coupled system with $y = \dfrac{dx}{dt}$:

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = -5x - 2y$$

The matrix $\begin{pmatrix} 0 & 1 \\ -5 & -2 \end{pmatrix}$ has eigenvalues $-1 \pm 2i$: the spring oscillates with decreasing size. From $x = 1$, $y = 0$ with $h = 0.1$, one Euler step gives $x = 1$, $y = -0.5$.

Common mistakes.

- Differentiating $\dfrac{1}{x^2}$ without rewriting it: write x^{-2}, then the derivative is $-2x^{-3}$.
- Forgetting $+C$ on an indefinite integral.
- Using the tangent gradient m for a normal. The normal has gradient $-\dfrac{1}{m}$.
- Counting strips wrongly in the trapezoidal rule: n strips have width $h = \dfrac{b-a}{n}$ and $n + 1$ ordinates.
- Integrating straight across a root. Area below the x-axis gives a negative integral, so split at the roots.
- Not checking the ends of the domain in an optimisation question.
- **HL:** missing the inner derivative in the chain rule: $\dfrac{d}{dx}e^{3x} = 3e^{3x}$.
- **HL:** dropping the units or a rate in a related-rates question; write the chain such as $\dfrac{dV}{dt} = \dfrac{dV}{dr} \cdot \dfrac{dr}{dt}$ first.
- **HL:** separating variables and forgetting the constant, then being unable to use the initial condition.
- **HL:** using $\pi \int (R - r)^2\, dx$ for a washer. The volume is $\pi \int (R^2 - r^2)\, dx$.
- **HL:** in Euler's method, updating x and y with different step sizes, or using the new gradient in the same step.

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Practice

1. Differentiate $y = 4x^3 - 5x + \dfrac{2}{x} + 7$.
2. Find where $f(x) = x^3 - 3x$ is decreasing.
3. Find the equations of the tangent and normal to $y = x^2$ at $x = 2$.
4. Find and classify the stationary points of $f(x) = x^3 - 3x$.
5. A profit is $P(x) = -2x^2 + 120x - 1000$. Find the number of items that gives the greatest profit, and the profit.
6. A can holds 500 cm^3. Show that $A = 2\pi r^2 + \dfrac{1000}{r}$, and find the radius that minimises A.
7. Find $\displaystyle\int (6x^2 - 4x + 5)\,dx$.
8. A curve has gradient $6x^2 - 4x + 5$ and passes through $(1, 6)$. Find its equation.
9. Find the area under $y = 4 - x^2$ from $x = 0$ to $x = 2$.
10. Use the trapezoidal rule with the river depths 0, 1.4, 2.3, 2.8, 2.1, 1.2, 0 m every 2 m to estimate the cross-section area.
11. Is the estimate in question 10 too large or too small? Explain.
12. The volume of water in a tank is $V = 2t^3 - 3t^2 + 50$ litres after t hours. Find $\dfrac{dV}{dt}$ at $t = 4$ and say what it means.
13. Find the gradient of $y = x^2 - \dfrac{8}{x}$ at $x = 2$.
14. Explain why the derivative is zero at the maximum volume of Kepler's barrel.
15. **AHL.** Differentiate e^{3x^2}, $x^2 \sin x$ and $\dfrac{\ln x}{x}$.
16. **AHL.** A spherical balloon gains 50 cm^3 of air per second. Find $\dfrac{dr}{dt}$ when $r = 10$ cm.
17. **AHL.** Use the second derivative to classify the stationary points of $x^3 - 3x$, and find its point of inflection.
18. **AHL.** Find $\displaystyle\int 2x(x^2 + 1)^4\,dx$ and $\displaystyle\int e^{2x}\,dx$.
19. **AHL.** The region under $y = \sqrt{x}$ for $0 \le x \le 4$ is rotated about the x-axis. Find the volume.
20. **AHL.** $v = t^2 - 4t + 3$ m/s for $0 \le t \le 4$. Find the displacement and the distance travelled.
21. **AHL.** Solve $\dfrac{dy}{dx} = xy$ with $y(0) = 2$.
22. **AHL.** Use Euler's method with $h = 0.5$ to estimate $y(1)$ for $\dfrac{dy}{dx} = x - y$, $y(0) = 1$.
23. **AHL.** Solve $\dfrac{dx}{dt} = x + 2y$, $\dfrac{dy}{dt} = 2x + y$ with $x(0) = 2$, $y(0) = 0$.

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24. **AHL.** Classify the equilibrium of $\dfrac{dx}{dt} = -x - 2y$, $\dfrac{dy}{dt} = 2x - y$, and give the direction of rotation.
25. **AHL.** Write $\dfrac{d^2x}{dt^2} + 2\dfrac{dx}{dt} + 5x = 0$ as a coupled system, and describe the motion.
26. **AHL.** Take one Euler step with $h = 0.1$ for the system in question 23, from $(2, 0)$.

Answers.

1. $12x^2 - 5 - \dfrac{2}{x^2}$.
2. $-1 < x < 1$.
3. Tangent $y = 4x - 4$; normal $x + 4y - 18 = 0$.
4. Maximum $(-1, 2)$; minimum $(1, -2)$.
5. 30 items; ¥800.
6. $h = \frac{500}{\pi r^2}$, so $A = 2\pi r^2 + 2\pi r \cdot \frac{500}{\pi r^2}$; $r = \sqrt[3]{250/\pi} \approx 4.30$ cm.
7. $2x^3 - 2x^2 + 5x + c$.
8. $y = 2x^3 - 2x^2 + 5x + 1$.
9. $\frac{16}{3} \approx 5.33$.
10. 19.6 m^2.
11. Too small: the river bed curves downwards between the measurements, so the trapezoids lie inside the true cross-section.
12. $\frac{dV}{dt} = 6t^2 - 6t = 72$: at $t = 4$ the volume is increasing at 72 litres per hour.
13. $2x + \frac{8}{x^2} = 4 + 2 = 6$.
14. At a maximum the tangent is horizontal, so the rate of change of volume with shape is zero: small changes in shape hardly change the volume.
15. **AHL.** $6xe^{3x^2}$; $2x\sin x + x^2\cos x$; $\frac{1-\ln x}{x^2}$.
16. **AHL.** $\frac{50}{400\pi} \approx 0.0398$ cm/s.
17. **AHL.** $f''(-1) = -6$, maximum; $f''(1) = 6$, minimum; inflection at $(0, 0)$.
18. **AHL.** $\frac{(x^2+1)^5}{5} + c$; $\frac{1}{2}e^{2x} + c$.
19. **AHL.** $8\pi \approx 25.1$.
20. **AHL.** Displacement $\frac{4}{3}$ m; distance 4 m.
21. **AHL.** $y = 2e^{x^2/2}$.
22. **AHL.** $y(0.5) \approx 0.5$, $y(1) \approx 0.5$ (exact 0.736).
23. **AHL.** $x = e^{3t} + e^{-t}$, $y = e^{3t} - e^{-t}$.
24. **AHL.** Eigenvalues $-1 \pm 2i$: stable spiral, anticlockwise.
25. **AHL.** $\frac{dx}{dt} = y$, $\frac{dy}{dt} = -5x - 2y$; eigenvalues $-1 \pm 2i$, so a damped oscillation.
26. **AHL.** $(2.2, 0.4)$.

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Exam tips

- Rewrite $\frac{a}{x^n}$ as ax^{-n} before differentiating or integrating.
- Give the meaning of a derivative with units: "litres per second".
- In optimisation, write the quantity in one variable, justify the maximum or minimum, and answer in context.
- Do not forget $+c$ in an indefinite integral; use a boundary condition to find it.
- For the trapezoidal rule, count strips, not points: 7 measurements make 6 strips.
- **HL:** set the GDC to radians for derivatives and integrals of trigonometric functions.
- **HL:** distance travelled is $\int |v|\,dt$; displacement is $\int v\,dt$.
- **HL:** in a phase portrait, find the eigenvalues first, then the eigenvector lines, then the direction at one point.

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School-supervised mathematical exploration

International Baccalaureate · IB Diploma · Mathematics: Applications and Interpretation · HL

Scope and prerequisites

This handout teaches the internal assessment of Mathematics: Applications and Interpretation, the course first assessed in 2021: what the mathematical exploration is, how to choose a topic and a research question, the five assessment criteria, how to communicate mathematics in writing, how to collect data, build a model, test it and reflect on it, the working process and academic integrity, and, for HL, what criterion E expects at HL. It is written for both SL and HL students. HL students also apply the further results in the sections marked AHL.

Prerequisites: none beyond the course itself. The exploration uses whatever mathematics fits your question, from any topic.

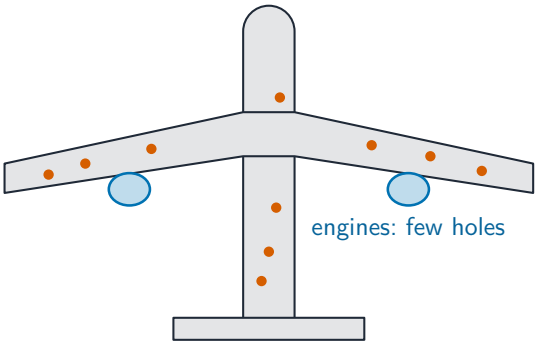
The worked example in this handout uses original, fictional data. It is not a real student's exploration and must not be copied as one.

What the exploration is

The holes that were not there

During the Second World War the mathematician Abraham Wald worked with a group of statisticians advising the United States military. One question, in the version of the story usually told, was where to add armour to bomber aircraft. The planes that came back had many bullet holes in the wings and the body, and few in the engines, so the obvious answer was to armour the wings and body. Wald pointed out that the data came only from planes that had **survived**. Planes hit in the engines were missing from the data because they had not come back. The armour should go where the returning planes had no holes. Wald used ordinary mathematics; what mattered was asking how the data had been collected and what that did to the conclusion. That kind of thinking is what the exploration rewards.

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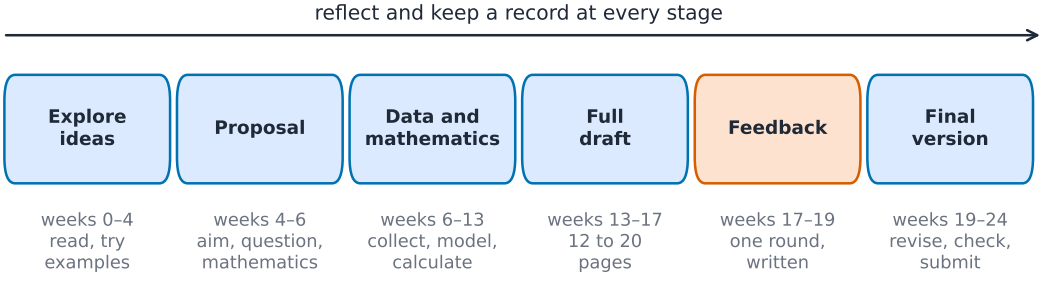


Survivorship bias

The task

The **mathematical exploration** 数学探究 is a piece of written work, about 12 to 20 pages long, in which you investigate an area of mathematics that interests you. It is the course's **internal assessment** 内部评估: your teacher marks it, and the IB checks a sample of marking from each school (**moderation** 评分审核). It is worth 20 % of the final grade at both SL and HL.

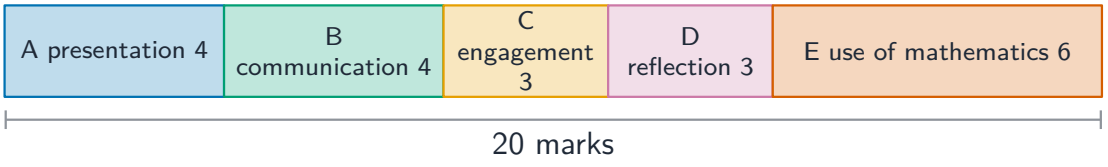
In Applications and Interpretation the exploration often starts from a real situation: data you collect, a model you build, or a decision you analyse. Pure topics are allowed too. Either way it is your own investigation, written for a reader who is one of your classmates.



Start early: collecting good data takes longer than you expect

The five criteria

Criterion	Marks	What it rewards
A **Presentation** 表达	4	Organised, coherent and concise: an introduction with an aim, a clear structure and a conclusion
B **Mathematical communication** 数学交流	4	Correct notation, symbols and terms; defined variables; labelled graphs and tables
C **Personal engagement** 个人投入	3	Your own thinking and approach, shown throughout the work
D **Reflection** 反思	3	Critical comments on your results, methods, assumptions and limitations
E **Use of mathematics** 数学运用	6	Relevant, correct mathematics at the level of the course, with clear understanding



How the 20 marks are shared

The total is 20 marks. Criterion E carries the most marks, but the other four together carry more.

Choosing a topic and a research question

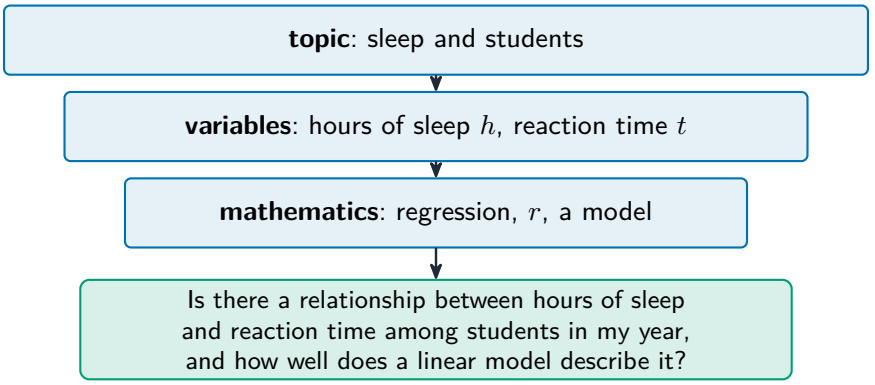
What makes a good topic

- **Interest.** You will spend months on it. Choose something you genuinely want to know.
- **Mathematics.** The question must need mathematics from the course, used and explained by you. Collecting data and drawing one bar chart is not enough.
- **Scope.** "Statistics of sport" is too broad; one calculation is too narrow.
- **Feasibility.** You must be able to get enough good data, safely and with permission, in the time available.

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From topic to research question

A **research question** 研究问题 names the variables and the mathematics.



Narrowing a topic to a research question

Weak	Better
Sleep and students	Is there a relationship between hours of sleep and reaction time among students in my year, and how well does a linear model describe it?
Bus delays	Do the delays of the number 12 bus follow a normal distribution, and what time should I leave home to be on time 95 % of the time?
Shops in my city	How well does a Voronoi diagram of the supermarkets in my district predict which supermarket people actually use?
Phone prices	Which model, linear, exponential or power, best describes how the price of a phone falls with its age, judged by R^2 and by context?

Good questions are specific and leave you decisions to make: which data, which model, which test.

Collecting data

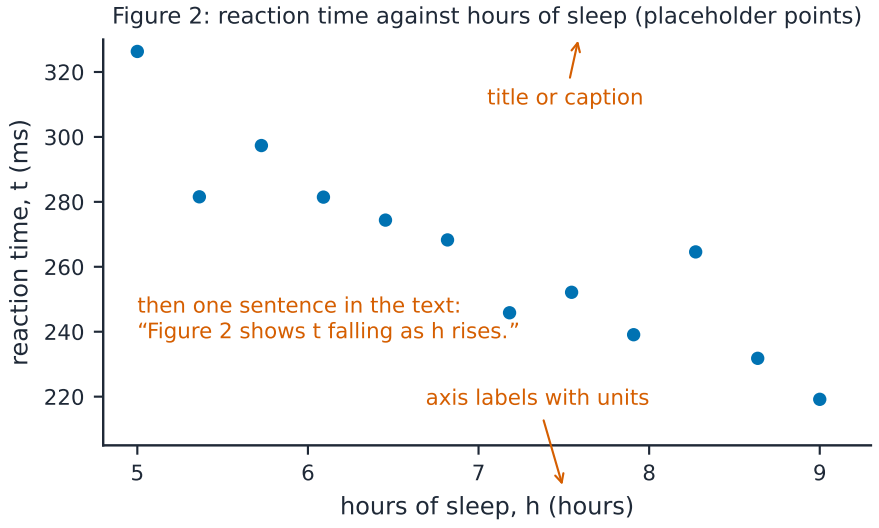
The quality of an exploration is limited by the quality of its data.

- **Primary data** 一手数据 you collect yourself show personal engagement, but plan the method: sample size, sampling technique (topic 4), and how you measure.
- **Secondary data** 二手数据 from a published source must be cited, and you must explain why it is reliable.
- Think like Wald: who is **missing** from your sample, and how could that bias the result?
- People's data need their consent, and should be anonymous.

Communicating mathematics

Writing for a classmate

- State the aim at the start and answer it at the end.
- Explain why each method is chosen before you use it.
- Put long tables of raw data in an appendix and keep a summary in the text.
- Every graph and table needs a title or caption, labelled axes and units, and a sentence in the text that says what it shows.
- Cite every source, including data sets and software, in a bibliography.



A graph the reader can use: caption, labelled axes with units, and a sentence about it

Correct notation

Calculator and spreadsheet notation is not acceptable in the written work.

Do not write	Write
`y = 2.3x^2 + 1.2E-3`	$y = 2.3x^2 + 1.2 \times 10^{-3}$
`sqrt(x)`, `x*y`	$\sqrt{x}$, xy
`r = -.989`, so it fits	$r = -0.989$, and explain what it means
`p = 3.9E-4`	p-value $= 0.00039$, less than the 5 % significance level

Define every variable when it first appears, with its units: “Let h be the hours of sleep the night before, and t the reaction time in milliseconds.”

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Modelling and testing: a worked case

The question and the data

A student, whom we will call Mei, wonders whether sleep affects reaction time. Her research question: *Is there a relationship between hours of sleep and reaction time among students in my year, and can a model predict one from the other?*

Mei asks 12 volunteers, with their consent, to report their hours of sleep and to take an online reaction test the next morning; each time is the mean of five attempts. These are fictional data for this handout:

h **(hours)**	**5.5**	**6.0**	**6.5**	**6.5**	**7.0**	**7.0**	**7.5**	**8.0**	**8.0**	**8.5**	**9.0**	**9.0**
t (ms)	312	305	298	301	290	294	285	279	283	276	270	273

A linear model

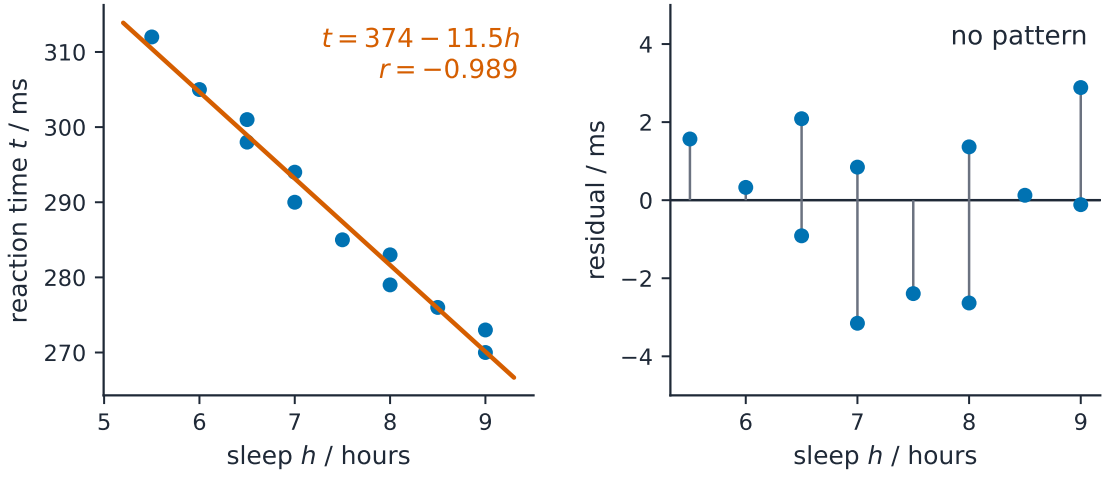
The scatter diagram looks linear, so Mei fits a **regression line** 回归直线 (topic 4):

$$t = 374 - 11.5h, \qquad r = -0.989$$

The gradient means each extra hour of sleep is associated with a reaction time about 11.5 ms shorter. The intercept, 374 ms for no sleep at all, is far outside the data and should not be interpreted.

Testing the model

Mei checks the **residuals** 残差, observed minus predicted, using the unrounded fitted line $t = 373.78826896 - 11.51931330h$. They lie between -3.2 and 2.9 ms, with no pattern, so a straight line is a reasonable model within the data.



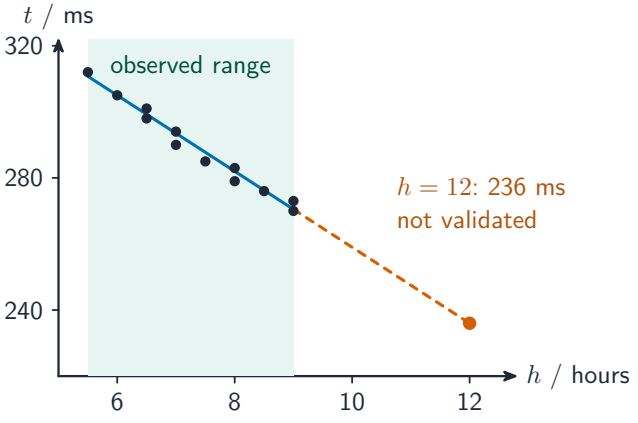
A strong correlation, and residuals with no pattern

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Mei also splits the students into two groups, less than 8 hours (7 students, mean 297.9 ms) and at least 8 hours (5 students, mean 276.2 ms), and runs a one-tailed t-test. The p-value is about 0.0004, so there is strong evidence that the mean reaction time is lower in the group that slept more.

Reflection

Reflection is critical thinking about the results and the method, written wherever it is relevant.



Extrapolation beyond the data is not validated

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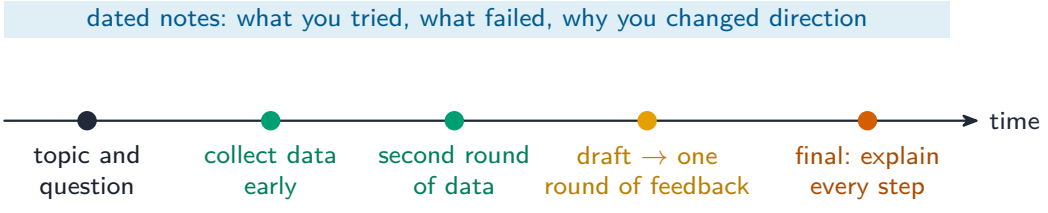
Descriptive (weak)	Critical (strong)
The model is accurate.	Within $5.5 \leq h \leq 9$ the observed residuals are all within 3.2 ms; this describes fit to the sample, not guaranteed prediction accuracy. At $h = 12$ it predicts 236 ms, outside the observed sleep range: this extrapolation has not been validated.
Sleep makes you faster.	The data show correlation, not causation. Students who sleep more may also drink less coffee or be more relaxed; an experiment that controlled sleep would be needed.
My sample was small.	The 12 volunteers were from one class, so the sample is not random and may not represent my year: people who volunteer may care more about sleep.
The sleep data were good.	Hours of sleep were self-reported, so their validity is uncertain. A sleep-tracking watch for a few students would test how accurate the reports were.
I did two tests.	Splitting h into two groups threw away information; the regression uses every value, so it is the stronger method here.

Good reflection names an **assumption** 假设, says what effect it has, and suggests what would test it.

Integrity and the working process

Working steadily

- Keep dated notes of what you tried, what failed and why you changed direction.
- Collect the data early: a second round of collection is often needed.
- Your teacher gives written feedback on **one** draft. Use it fully.
- Check the length: far too long is not concise, far too short rarely has enough mathematics.



Steady work: early data, one feedback draft, notes all the way

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Academic integrity

- Never invent data, change results to make them look better, or copy a sample exploration.
- Cite sources of ideas, data, images and software, in the text and in the bibliography.
- If you use an AI tool, cite it as you would any other source, following your school's policy. Presenting its output as your own work is academic misconduct.
- Collect data about people only with their consent, and keep it anonymous.

Expect to be asked to explain any part of your work. If you cannot explain a step, it should not be in the exploration.

AHL: What criterion E expects at HL

Sophisticated and rigorous

Criterion E is judged against the level of your course. An SL student who uses SL mathematics thoroughly and correctly can reach the top level. At HL, the top levels also expect the mathematics to be **sophisticated and rigorous**:

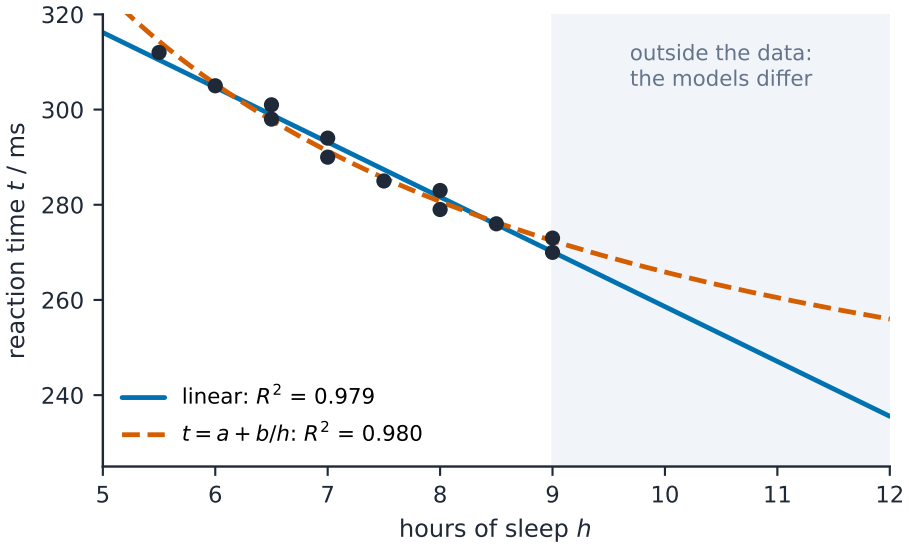
- **Sophisticated**: well chosen and beyond a routine procedure, such as testing a hypothesis about a parameter, comparing non-linear models, or building a model from a differential equation.
- **Rigorous**: logically complete. Assumptions and domains are stated, the conditions of each test are checked, and technology outputs are explained, not only reported.

Raising Mei's exploration to HL

An HL version could:

- test $H_0 : \rho = 0$ against $H_1 : \rho < 0$: with $r = -0.989$ and $n = 12$, the p-value is far below 0.01, strong evidence of a negative correlation in the population;
- check the conditions of the t-test (roughly normal data, similar spreads) instead of assuming them, and give a confidence interval for each group's mean;
- compare the line with a non-linear model, such as $t = a + \dfrac{b}{h}$, by R^2 and by what each predicts for long sleep, and explain which is more believable.

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Two models that fit the same data can disagree outside it

Using HL mathematics only for decoration does not help. Criterion E rewards mathematics that answers the research question.

Common mistakes.

- Choosing a topic so wide that no mathematics is done. Narrow it to one question you can answer with your data.
- Collecting too little data, or data you cannot cite.
- Using calculator notation in the report: write 2.4×10^{-5}, not 2.4E-05.
- Graphs without labelled axes, units or a source for the data.
- Fitting a model without testing it, for example with residuals or a new data point.
- A reflection that only repeats the results. Say what they mean, how reliable they are and what limits them.
- **HL:** using a technique you cannot explain. Rigour at HL means every step is justified.

Practice

1. Rewrite the topic "Statistics in basketball" as a focused research question that names the mathematics.
2. Rewrite in correct notation: `y = 4.1x^3 - 2E-4`, `sqrt(2)*pi`, `r = .87`.
3. Using $t = 374 - 11.5h$, predict the reaction time for $h = 7.5$, and find the residual for the student with $t = 285$.
4. Explain why Mei should not interpret the intercept 374 ms.
5. Explain how Wald's insight about the bombers applies to a survey of students who reply to an online questionnaire.

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6. Which criterion, A to E, does each feature mainly affect? (a) Axes with no units; (b) no conclusion; (c) a wrong p-value; (d) a comment on self-reported data; (e) designing your own data collection.
7. Explain why "sleep makes you faster" is not supported by Mei's data, and what would test it.
8. Turn this into critical reflection: "I could have used more data."
9. A student downloads a data set and uses an AI tool to explain a test. What must they do?
10. Explain why splitting a continuous variable into two groups can make a test weaker.
11. **AHL.** State the hypotheses for a test of correlation between sleep and reaction time, and interpret a p-value below 0.01.
12. **AHL.** Suggest one non-linear model for Mei's data, and explain how to compare it with the line.

Answers.

1. For example: "Is there a relationship between a player's height and their rebounds per game in my school league, and is it linear?" Any answer must name variables and a method.
2. $y = 4.1x^3 - 2 \times 10^{-4}$; $\sqrt{2}\,\pi$; $r = 0.87$.
3. $374 - 11.5(7.5) \approx 288$ ms, so the residual is $285 - 288 \approx -3$ ms (about -2.4 ms with the unrounded line).
4. $h = 0$ is far outside the data ($5.5 \leq h \leq 9$); predicting there is extrapolation, and a person with no sleep may not behave like the model at all.
5. Only students who chose to reply are in the data; those who did not reply may differ, for example in interest in the topic, so the results may be biased. It is the missing data that matter.
6. (a) B; (b) A; (c) E; (d) D; (e) C.
7. The data show correlation only: another variable, such as caffeine or stress, may affect both. An experiment in which sleep is controlled, with other variables kept the same, would test cause and effect.
8. For example: "With 12 students from one class, one unusual student changes r noticeably; a random sample of about 40 from the whole year would make the model more reliable and representative."
9. Cite both, in the text and in the bibliography, and explain why the data are reliable. The AI tool's output must not be presented as their own work.
10. Grouping replaces each value by its group, so differences within a group are lost; a method that uses every value, such as regression, uses all the information.
11. **AHL.** $H_0 : \rho = 0$, $H_1 : \rho < 0$. A p-value below 0.01 is strong evidence that hours of sleep and reaction time are negatively correlated in the population.

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12. **AHL.** For example $t = a + \dfrac{b}{h}$; fit both with technology, compare R^2 and the residual patterns, and compare what each predicts for long sleep, choosing the one that also makes sense in context.

Exam tips

- Choose a question you care about, and check that it needs mathematics from the course.
- Plan data collection: sample size, sampling method, consent and who might be missing.
- Write for a classmate: define every variable, label every graph, and explain every method.
- Never use calculator notation in the written work.
- Look at residuals, not only r, before trusting a model; never interpret far outside the data.
- Reflect critically throughout: name assumptions, their effects and how to test them.
- Cite every source, including software and AI tools; never invent or alter data.
- **HL:** check the conditions of every test and justify every model, not only fit and report.

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