

5.2 Root finding and numerical integration

Name: _____ Class: _____ Date: _____ Total: 13 marks

KEY WORDS iteration

Objective

Root finding and numerical integration. Supported HL concept preparation; original classroom practice, not an official IB paper or assessed learner work.

1 Key knowledge and worked examples

■ How reliable is an approximation?

- A nonlinear equation has no convenient exact solution. An approximation still needs evidence that it is reliable.
- This lesson studies **iteration** : A repeated update in which each new approximation is calculated from the previous one.

■ Choose the mathematical structure

- A continuous function with opposite signs at two endpoints has a root between them. Newton's method uses $x_{\text{next}} = x - f(x)/f'(x)$, with a nonzero derivative. The trapezium rule approximates a definite integral using endpoint heights.
- State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.

■ Work through a checked case

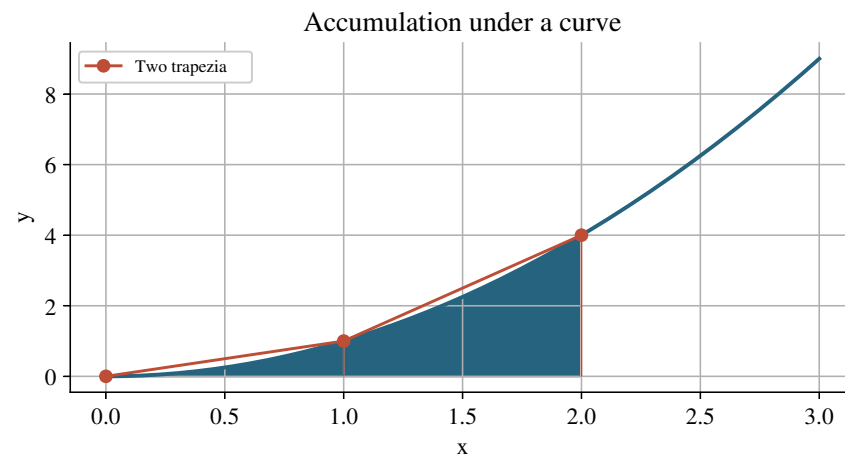
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Example:

For $f(x) = x^2 - 2$ and $x_0 = 1.5$, Newton gives $x_1 = 1.5 - (2.25 - 2)/3 = 1.4166667$. With $y = x^2$ on $[0, 2]$ and two equal strips, $h = 1$ and trapezium area $= (1/2)(0 + 2 \times 1 + 4) = 3$; the exact area

is $8/3$.



■ Test a tempting shortcut

- A sign change across a discontinuity does not prove a root. Iteration can diverge or cycle. The trapezium rule's overestimate or underestimate depends on curvature, not just whether the function increases.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Warn:

Every sign change proves a root, including one across a discontinuity. This claim is false. Explain which definition or assumption it violates.

■ Interpret a new situation

- Give a stopping criterion and report sensible accuracy. Confirm the approximated root by a sign bracket around the stated rounded answer; explain any failure of the chosen iteration.
- A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.

■ Use this in your course

- Current first-assessment-2021 Applications and Interpretation HL. This is authored concept support; the full guide is needed to certify every objective.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key:

A repeated update in which each new approximation is calculated from the previous one. Choose the relationship, show the method, check its assumptions and interpret the result.

2 Practice

2.1 After one Newton step for x^2-2 from $x_0=1.5$, find x_1 . Show the relation, substitution and units. [2]

2.2 Use two strips for the trapezium estimate of integral x^2 from 0 to 2. Show the relation, substitution and units. [2]

2.3 Find the exact integral of x^2 from 0 to 2. Show the relation, substitution and units.[2]

2.4 Decide whether this claim is true and justify: Every sign change proves a root, including one across a discontinuity. [2]

2.5 Which description correctly defines iteration? Choose one answer.

- **A** A repeated update in which each new approximation is calculated from the previous one.
 - **B** A numerical answer that needs no assumptions or interpretation
 - **C** A rule that remains valid even when its required conditions fail
- [1]

3 Exam-style questions

3.1 Apply one Newton step to $f(x)=x^3-2x+2$ from $x_0=0$ and then one more. What happens, and what does it teach about convergence? [4]

Solutions

2.1 $x_1 = 1.5 - (1.5^2 - 2)/(2 \times 1.5) = 17/12 \approx 1.416667$.

2.2 $h=1$ with heights 0,1,4 gives $(1/2)(0+2 \times 1+4)=3$.

2.3 The exact integral is $[x^3/3]$ from 0 to 2 $= 8/3$.

2.4 False. A sign change across a discontinuity does not prove a root. Iteration can diverge or cycle. The trapezium rule's overestimate or underestimate depends on curvature, not just whether the function increases.

2.5 A A repeated update in which each new approximation is calculated from the previous one.

3.1 $x_1 = 0 - 2/(-2) = 1$; $x_2 = 1 - 1/1 = 0$. It cycles 0, 1 rather than converging. Newton requires suitable starting values/derivatives; use a bracketed method or inspect the function rather than assuming every iteration converges.