

Calculus

International Baccalaureate · IB Diploma · Mathematics:
Applications and Interpretation · HL

Scope and prerequisites

This handout covers selected HL topics for the course first assessed in 2021; it is not the whole syllabus.

Prerequisites: read the stated quantities and units, and use arithmetic and the model conditions below.

The examples are original and fictional: they are not real measurements or real student investigations.

Derivatives and stationary points

■ What is the slope at one point?

- A curved road has different slopes at different positions. An average gradient cannot describe every point.
- This lesson studies **derivative** : The instantaneous rate of change, also the gradient of a tangent.

■ The method

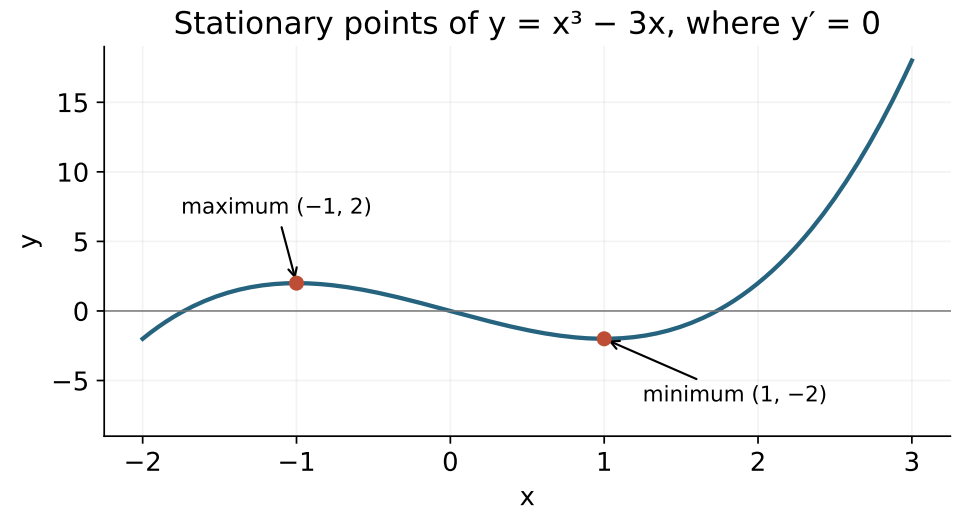
- For $y = ax^n$, $dy/dx = anx^{n-1}$.
- A stationary point satisfies $dy/dx = 0$.
- Check the sign change of the derivative, or the second derivative when it is nonzero, to classify it.

■ A worked case

$$\frac{d}{dx}(ax^n) = anx^{n-1}, \quad f'(x) = 0$$

Example:

For $y = x^3 - 3x$, $dy/dx = 3x^2 - 3$. At $x = 1$, the gradient is 0 and $y = -2$. The second derivative is $6x$, positive at $x = 1$, so this is a local minimum. At $x = -1$, $y = 2$ and the second derivative is negative, giving a local maximum.



$$y' = 3x^2 - 3 = 0 \text{ at } x = \pm 1: \text{ maximum } (-1, 2), \text{ minimum } (1, -2)$$

■ Where it goes wrong

- A zero derivative does not always mean a maximum or minimum: $y = x^3$ is stationary at 0 but continues increasing.
- An endpoint can also produce an extreme value on a restricted domain.

Not true: Every point with zero derivative is a local maximum or minimum. A zero derivative can also be a point of inflection: $y = x^3$ at $x = 0$ is neither a maximum nor a minimum.

■ Applying it

- At GCSE/IGCSE use only the polynomial scope allowed by the tier; do not add chain, product or quotient rules there.
- At advanced level, connect the derivative to rates and optimization with a valid domain.

■ In your course

- Applications and Interpretation HL, the course first assessed in 2021.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

Key:

- **Derivative:** The instantaneous rate of change, also the gradient of a tangent.
- For $y = ax^n$, $dy/dx = anx^{n-1}$.
- Watch for: A zero derivative does not always mean a maximum or minimum: $y = x^3$ is stationary at 0 but continues increasing.

■ Write from a derivative of 10

- For $y = x^2 - 2x$, find the derivative at $x = 6$.
- Find the stationary value. A derivative of 0 is not always a turning point.

Root finding and numerical integration

■ How reliable is an approximation?

- A nonlinear equation has no convenient exact solution. An approximation still needs evidence that it is reliable.
- This lesson studies **iteration**: A repeated update in which each new approximation is calculated from the previous one.

■ The method

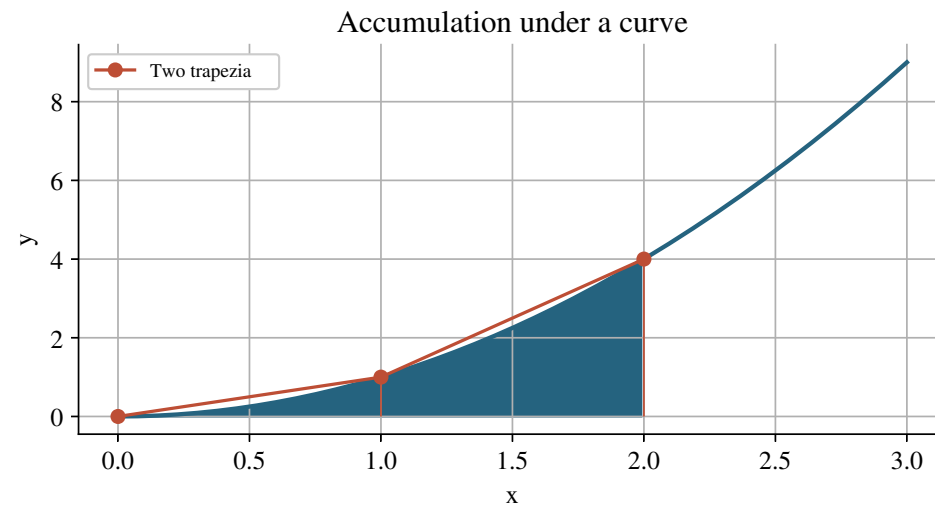
- A continuous function with opposite signs at two endpoints has a root between them.
- Newton's method uses $x_{\text{next}} = x - f(x)/f'(x)$, with a nonzero derivative.
- The trapezium rule approximates a definite integral using endpoint heights.

■ A worked case

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Example:

For $f(x) = x^2 - 2$ and $x_0 = 1.5$, Newton gives $x_1 = 1.5 - (2.25 - 2)/3 = 1.4166667$. With $y = x^2$ on $[0, 2]$ and two equal strips, $h = 1$ and trapezium area $= (1/2)(0 + 2 \times 1 + 4) = 3$; the exact area is $8/3$.



Two trapezia give $0.5 + 2.5 = 3$; the exact area is $8/3$, so the rule overestimates this convex curve

■ Where it goes wrong

- A sign change across a discontinuity does not prove a root.
- Iteration can diverge or cycle.
- The trapezium rule's overestimate or underestimate depends on curvature, not just whether the function increases.

Not true: Every sign change proves a root, including one across a discontinuity. A sign change shows a root only if the function is continuous there: $1/x$ changes sign across $x = 0$ but has no root.

■ Applying it

- Give a stopping criterion and report sensible accuracy.
- Confirm the approximated root by a sign bracket around the stated rounded answer; explain any failure of the chosen iteration.

■ In your course

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Key:

- **Iteration:** A repeated update in which each new approximation is calculated from the previous one.
- A continuous function with opposite signs at two endpoints has a root between them.
- Watch for: A sign change across a discontinuity does not prove a root.

■ Write from a Newton step to 1.25

- For x squared minus 1, take one Newton step from 2.
- A sign change across a discontinuity does not prove a root.

Chain, product, quotient and implicit differentiation

■ Does the inside expression change too?

- A cost curve is a power of a changing expression. Differentiating the outer power alone misses the rate of its input.
- This lesson studies **chain rule** : The rule that multiplies the outer derivative by the inner derivative for a composite function.

■ The method

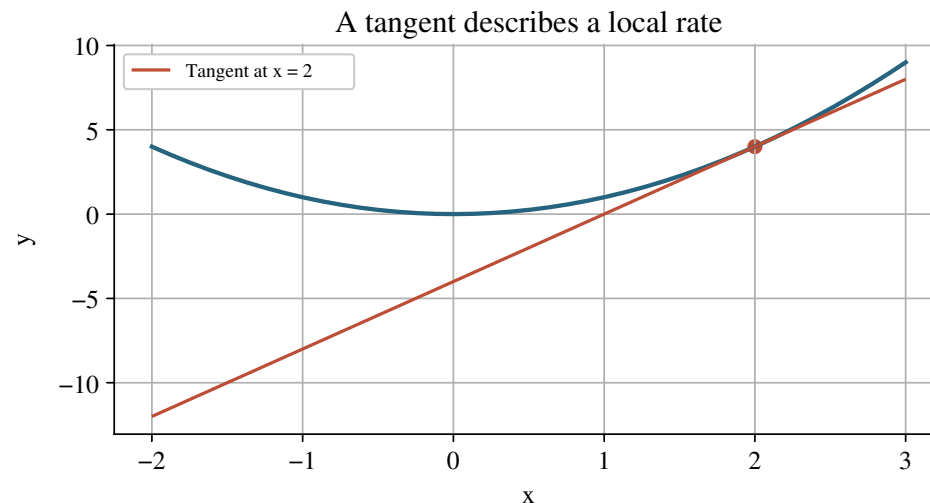
- For $y=f(g(x))$, $y' = f'(g(x))g'(x)$.
- For uv , differentiate to $u'v + uv'$.
- For u/v , use $(u'v - uv')/v^2$, where $v \neq 0$.

■ A worked case

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$$

Example:

For $y=(3x+1)^4$, $y' = 4(3x+1)^3 \times 3 = 12(3x+1)^3$. At $x=0$, the gradient is 12.
 For $x^2+y^2=25$, differentiate implicitly: $2x+2y y' = 0$, so $y' = -x/y$ when $y \neq 0$.



The tangent to $y = x^2$ at $x = 2$ has gradient $2x = 4$: $y = 4x - 4$

■ Where it goes wrong

- A derivative of a product is not the product of the derivatives.
- In implicit differentiation, every differentiated function of y brings a dy/dx factor.
- A quotient's denominator is squared.

Not true: The derivative of $u(x)v(x)$ is always u' times v prime. The product rule is $(uv)' = u'v + uv'$: for $x \cdot x$ the derivative is $2x$, not $1 \times 1 = 1$.

■ Applying it

- Choose a useful form before differentiating: expanding a short polynomial may be simpler.
- For related rates, write the relation in symbols, differentiate with respect to time, then substitute measured values.

■ In your course

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Key:

- **Chain rule:** The rule that multiplies the outer derivative by the inner derivative for a composite function.
- For $y=f(g(x))$, $y' = f'(g(x))g'(x)$.
- Watch for: A derivative of a product is not the product of the derivatives.

■ Write from a product derivative of 17

- For $y = (3x + 2)(x + 1)$, find the derivative at $x = 2$.
- The derivative of a product is not the product of the derivatives.

Substitution, parts and partial fractions

■ Can we integrate the two factors separately?

- A simple-looking product such as xe^x cannot be integrated by separately integrating each factor.
- This lesson studies **integration by parts** : An integration method based on the derivative of a product.

■ The method

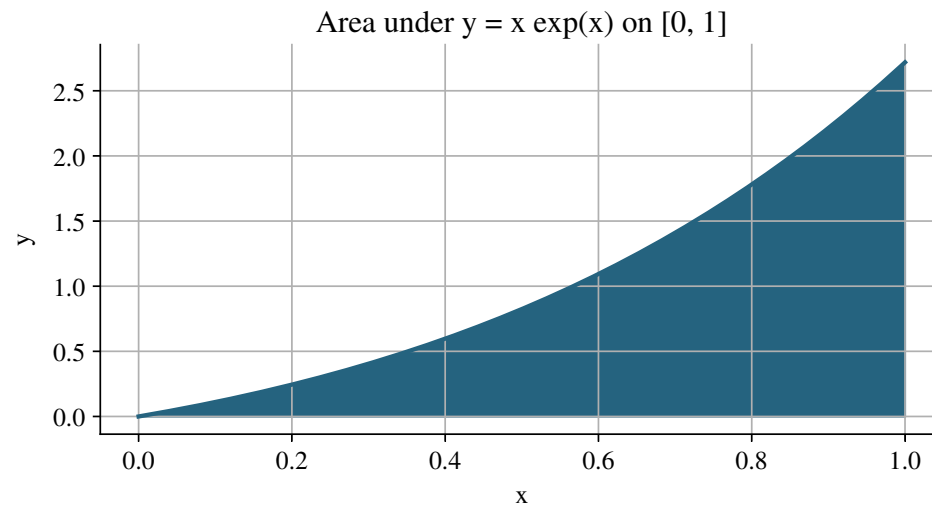
- Use substitution when an inner derivative appears as a factor.
- By parts, $\int u v' = uv - \int u' v$.
- For a rational function, divide first if needed, then decompose into partial fractions.

■ A worked case

$$\int uv' dx = uv - \int u'v dx$$

Example:

For $\int xe^x dx$, take $u=x$ and $v'=e^x$. Then the integral is $xe^x - e^x + C$.
 For $\int 2x/(x^2+1) dx$, substitute $w=x^2+1$, $dw=2x dx$; the answer is $\ln(x^2+1) + C$.



$$\text{Integration by parts: } \int_0^1 x e^x dx = [x e^x - e^x]_0^1 = 1$$

■ Where it goes wrong

- Choosing u and v' well matters: the remaining integral should become simpler.
- In a definite substitution, either change the limits or return to x before applying the original limits.

Not true: The integral of a product equals the product of its separate integrals. A product needs a method such as integration by parts or substitution: $\int x \cdot x dx = x^3/3$, not $(x^2/2)(x^2/2)$.

■ Applying it

- Check by differentiating.
- For volumes of revolution around the x -axis use $V = \pi \int y^2 dx$; do not confuse the square of a function with the integral of the function.

■ In your course

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Key:

- **Integration by parts:** An integration method based on the derivative of a product.
- Use substitution when an inner derivative appears as a factor.
- Watch for: choosing u and v prime well matters: the remaining integral should become simpler.

■ Write from a volume factor of 20

- The line $y = 2$ is revolved about the x-axis from 0 to 5. Find the volume divided by pi.
- The integral of a product is not the product of the integrals.

Differential equations and numerical solutions

■ What does the starting value decide?

- A growing population's rate is proportional to its current size. The rate equation describes a whole family until an initial population is supplied.
- This lesson studies **initial condition** : A specified value that selects a particular solution from a family.

■ The method

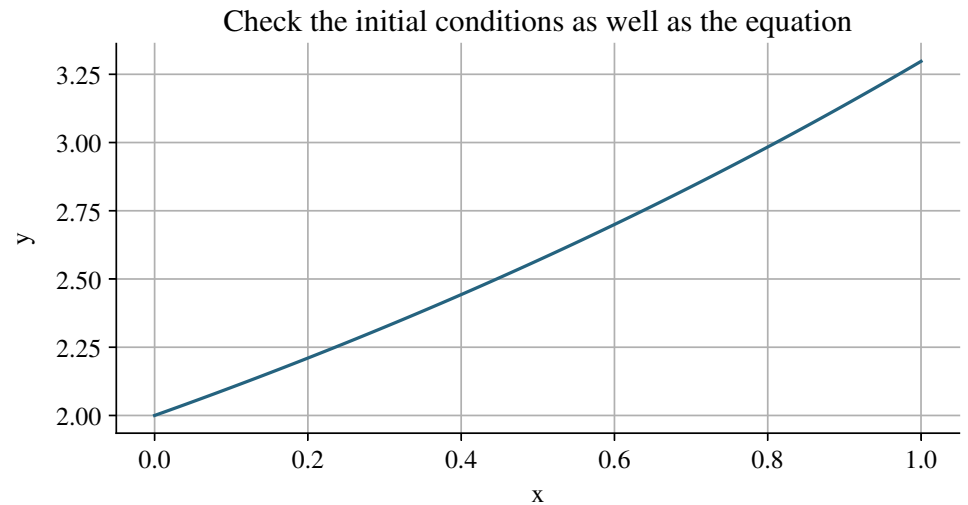
- For $dy/dx=ky$, separate variables: $dy/y=k dx$, giving $y=Ae^{(kx)}$.
- Use an initial condition to find A.
- Euler's method takes $y_{next}=y+h f(x,y)$, with a chosen step h.

■ A worked case

$$\frac{dy}{dx} = ky, \quad y = Ae^{kx}$$

Example:

With $dy/dx=0.5y$ and $y(0)=2$, $y=2e^{(0.5x)}$. Euler with $h=0.2$ gives $y(0.2)\approx 2+0.2\times 1=2.2$ and $y(0.4)\approx 2.2+0.2\times 1.1=2.42$. The exact second value is about 2.4428.



$$dy/dx = 0.5y \text{ with } y(0) = 2 \text{ gives } y = 2e^{(0.5x)}; y(1) \approx 3.30$$

■ Where it goes wrong

- An initial condition determines the integration constant; it is not a replacement for integrating.
- Dividing by y can omit an equilibrium solution $y=0$.
- Euler accuracy depends on step size and the equation.

Not true: Every step size gives exactly the same Euler approximation. Euler's method depends on the step size; smaller steps usually give a more accurate value.

■ Applying it

- For second-order linear equations, combine the complementary function with an appropriate particular integral, then apply the required initial conditions.
- This is Further Pure scope; check the named unit.

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Key:

- **Initial condition:** A specified value that selects a particular solution from a family.
- For $dy/dx=ky$, separate variables: $dy/y=k dx$, giving $y=Ae^{(kx)}$.
- Watch for: an initial condition determines the integration constant; it is not a replacement for integrating.

■ **Write from a first Euler value of 6.6**

- Use Euler with step 0.2 for $y' = 0.5y$, starting at $y = 6$.
- Find the first two approximations. A different step size gives a different approximation.

■ **Write from a second Euler value of 7.26**

- Continue the same computation one more step.
- Each step multiplies by 1.1: 6, 6.6, 7.26.

■ **Write from the exact value 7.33**

- Solve $y' = 0.5y$ exactly and evaluate at $x = 0.4$.
- $y = 6e^{0.5x}$ gives about 7.328 – Euler's 7.26 undershoots the upward-bending curve.