Number and algebra

International Baccalaureate · IB Diploma · Mathematics: Applications and Interpretation · HL

Scope and prerequisites

This handout covers selected HL topics for the course first assessed in 2021; it is not the whole syllabus.

Prerequisites: read the stated quantities and units, and use arithmetic and the model conditions below.

The examples are original and fictional: they are not real measurements or real student investigations.

Exact arithmetic and estimation

Can we pack without leftovers?

- A supplier packs 72 pencils and 90 pens into identical gift bags. How can we avoid leftovers?
- This lesson studies **prime factor** : A prime number that divides the integer exactly.

The method

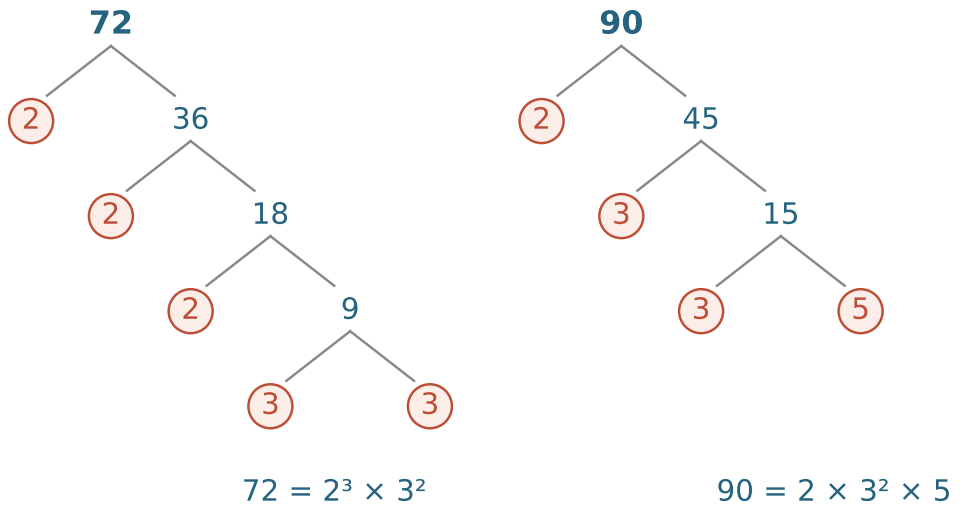
- Prime factors reveal shared structure.
- Use the smallest common prime powers for the HCF and the largest for the LCM.
- Estimate before calculating; use brackets to preserve the order of operations.

A worked case

$$a = \prod p_i^{\alpha_i}, \quad b = \prod p_i^{\beta_i}, \quad \mathrm{HCF}(a,b) = \prod p_i^{\min(\alpha_i,\beta_i)}$$

Example:

72=2^3×3^2 and 90=2×3^2×5. Their HCF is 2×9=18. Make 18 bags with 4 pencils and 5 pens each. Their LCM is 2^3×3^2×5=360.



Factor trees: $72 = 2^3 \times 3^2$ and $90 = 2 \times 3^2 \times 5$, so HCF $= 2 \times 3^2 = 18$

Where it goes wrong

- The HCF divides both numbers; the LCM is a multiple of both.
- They answer different questions.
- A decimal estimate is not an exact fraction.

Not true: The HCF of two positive integers is always larger than either integer. The HCF divides both numbers, so it can never be larger than the smaller one: the HCF of 72 and 90 is 18.

Applying it

- For a non-calculator paper, keep fractions exact and show cancellation.
- For a calculator paper, enter the full expression and compare with your estimate.

In your course

- Applications and Interpretation HL, the course first assessed in 2021.

Key:

- **Prime factor**: A prime number that divides the integer exactly.
- Prime factors reveal shared structure.
- Watch for: the HCF divides both numbers; the LCM is a multiple of both.

■ Write from an HCF of 4

- Find the HCF and LCM of 20 and 28.
- The HCF is smaller than both integers.

Percentages, ratio and proportional reasoning

■ Can we recover the original price?

- A coat is reduced by 20% to ¥240. The discount applies to the original price, not to the sale price.
- This lesson studies **multiplier** : A factor that performs a percentage change in one multiplication.

■ The method

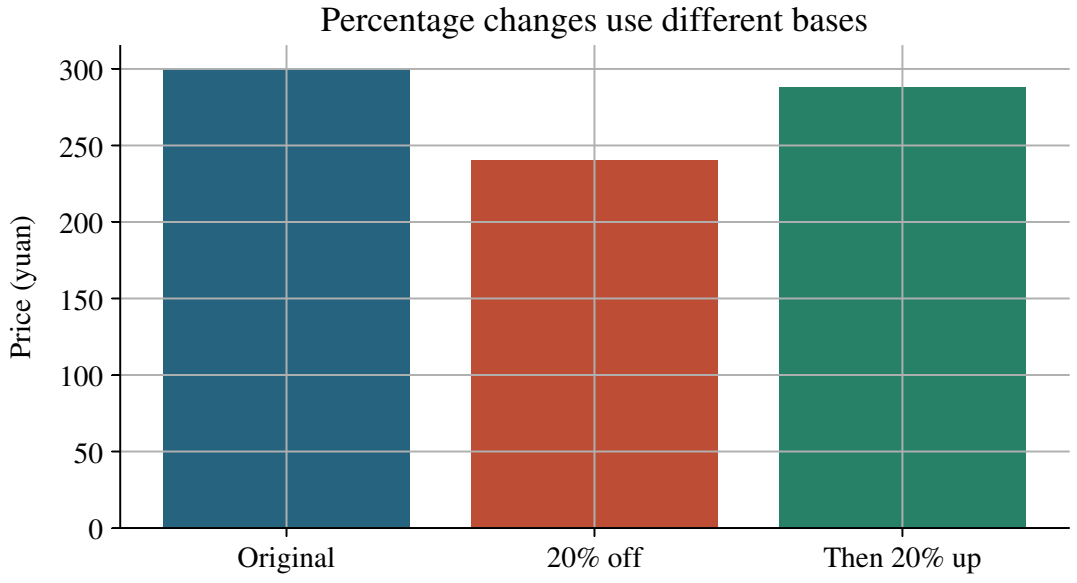
- A p% increase has multiplier 1+p/100; a decrease has multiplier 1-p/100.
- Reverse a percentage by dividing by the multiplier.
- In a ratio, first find the total number of parts.

■ A worked case

$$P_{\text{new}} = P_{\text{old}} \left(1 + \frac{r}{100}\right)$$

Example:

Let the original price be P. The model is sale price=0.8P. Hence P=240/0.8=300. A later 20% increase gives 240×1.2=288, so the two changes do not cancel.



300 → 240 → 288 yuan: 20% off then 20% up does not return to 300, because the second 20% is of 240

Where it goes wrong

- A percentage uses a stated base.
- Subtracting the percentages loses that base.
- For compound change, multiply the multipliers; do not add the percentages.

Not true: A 20% decrease followed by a 20% increase restores the starting price. The 20% increase is taken of a smaller amount: $100\times0.8\times1.2=96$, not 100.

Applying it

- For direct proportion use y=kx; for inverse proportion use y=k/x.
- Calculate k from a known pair before using a new value.
- State what you held constant.

In your course

- Applications and Interpretation HL, the course first assessed in 2021.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

Key:

- **Multiplier**: A factor that performs a percentage change in one multiplication.

- A p% increase has multiplier 1+p/100; a decrease has multiplier 1-p/100.
- Watch for: A percentage uses a stated base.

■ Write from a final price of 396

- A price of 400 falls by 10 percent and then rises by 10 percent.
- Find the final price. The two changes do not cancel.

Sequences, series and recurrence

■ Does the change add or multiply?

- A saving plan adds ¥30 more each week; a population model grows by 5% each year. Equal differences and equal ratios need different models.
- This lesson studies **common ratio** : The constant multiplier between consecutive terms of a geometric sequence.

■ The method

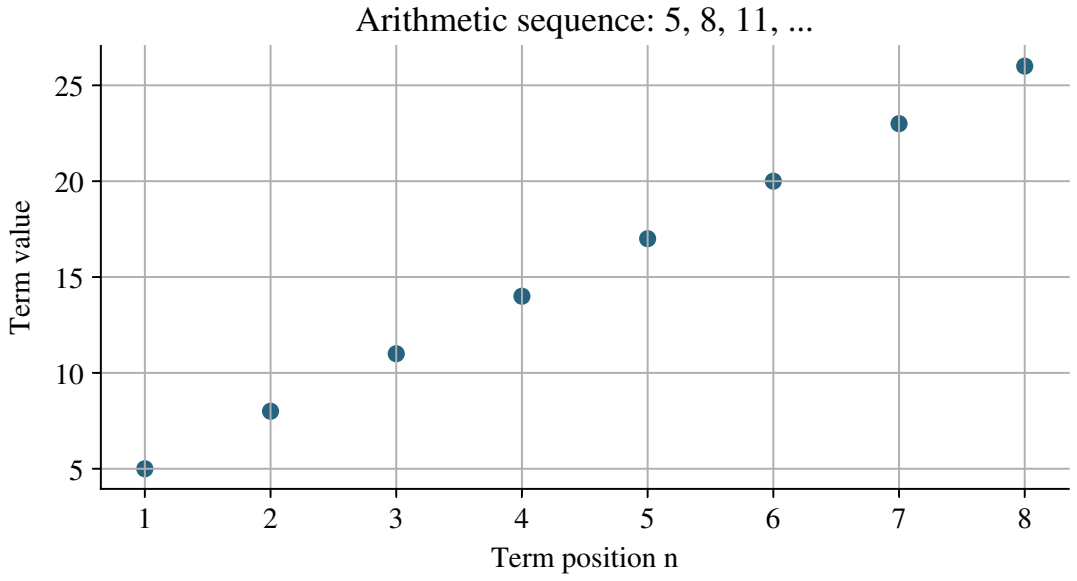
- For an arithmetic progression, u_n=a+(n-1)d and S_n=n(2a+(n-1)d)/2.
- For a geometric progression, u_n=ar^(n-1) and S_n=a(1-r^n)/(1-r).
- An infinite geometric sum exists only if |r|<1.

■ A worked case

$$u_n = a + (n-1)d, \qquad S_n = \frac{n}{2}\left[2a + (n-1)d\right]$$

Example:

For a=5,d=3,n=8, u_8=5+7×3=26 and S_8=8(10+21)/2=124. For a=12,r=1/2, S infinity=12/(1-1/2)=24. For u_(n+1)=2u_n+1 with u_1=1, the next terms are 3,7,15.



$u_n = 5 + 3(n - 1)$: equal steps of 3 put the terms on a straight line

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Where it goes wrong

- The first term has index 1, so the exponent is n-1.
- A sequence is a list; a series is a sum.
- A geometric sequence can alternate in sign and still converge.

Not true: Every geometric series has a finite sum to infinity. A sum to infinity exists only when $-1<r<1$; for r=2 the terms grow without limit.

Applying it

- Explain whether the context justifies additive or multiplicative change.
- In finance, distinguish a single deposit from a stream of deposits before choosing a sum formula.

In your course

- Applications and Interpretation HL, the course first assessed in 2021.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

Key:

- **Common ratio**: The constant multiplier between consecutive terms of a geometric sequence.

- For an arithmetic progression, u_n=a+(n-1)d and S_n=n(2a+(n-1)d)/2.
- Watch for: the first term has index 1, so the exponent is n-1.

Write from the 6th term 19

- An arithmetic sequence has first term 9 and common difference 2.
- Find the 6th term. An infinite geometric series needs a common ratio with size below 1.

Repeated-deposit sequence models and timing

Which timeline does the sum describe?

- Two equal payments can earn different amounts of interest if one is paid earlier.
- This lesson studies **deposit**: An amount added to an account at a stated time.

The method

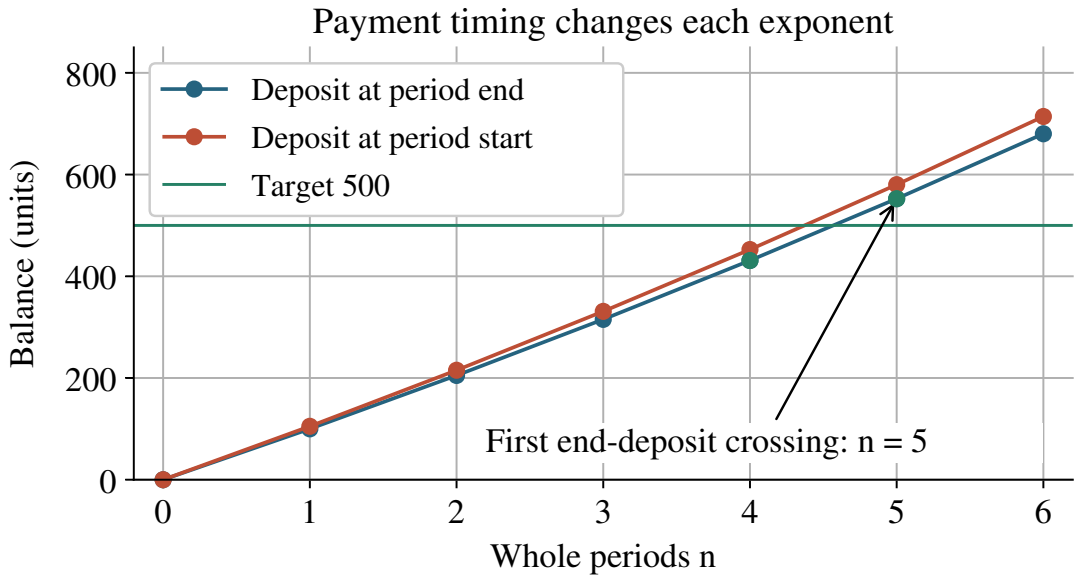
- Define the balance immediately after a stated event and give the initial balance.
- With growth factor r and an end-of-period deposit d, $B_0=0$ and B_n=rB_(n−1)+d.
- The nth balance is d(1+r+…+r^(n−1))=d(r^n−1)/(r−1) for r≠1.
- For r=1, use B_n=nd.
- The period count n is a nonnegative integer.

A worked case

$$B_0 = 0, \quad B_n = rB_{n-1} + d, \quad B_n = d\frac{r^n - 1}{r - 1} \ (r \neq 1)$$

Example:

Use an illustrative growth rate of 5% per period and deposit 100 units at each period end. B_1=100, B_2=1.05×100+100=205 and B_3=1.05×205+100=315.25. The first deposit earns interest for n−1 periods, while the final one earns none at the time B_n is measured. Thus B_n=100(1.05^n−1)/0.05. For a target of at least 500, B_4=431.0125<500 and B_5=552.563125≥500, so 5 periods first suffice. The balance exceeds the five deposits totalling 500 by 52.563125 units. If the same deposits are made at each period start and the balance is measured at the period end, C_n=1.05(B_n); hence C_3=331.0125, not 315.25.



Depositing at the start earns one more period of interest; the end-deposit balance first passes 500 at $n = 5$

Where it goes wrong

- Do not give the final end-of-period deposit an extra interest period.
- A balance includes both deposits and growth; it is not the interest alone.
- An exact target and an at-least target need different integer checks.
- A quoted annual rate cannot simply be used as a per-month factor without an appropriate conversion.

Not true: Beginning- and end-of-period deposits always give the same final balance. A deposit at the start of a period earns one more period of interest, so the final balance is larger.

Applying it

- The illustration assumes a fixed per-period rate, equal deposits, no fees or withdrawals and the stated compounding/payment times.
- If these change, refine the recurrence: B_n=(1+i_n)B_(n−1)+d_n−w_n for end-of-period deposits/withdrawals.
- Keep a separate ledger of payments to identify growth.
- Compare adjacent period balances before claiming the first target crossing; here all balances are nonnegative and B_n−B_(n−1)=0.05B_(n−1)+100>0.

In your course

- Applications and Interpretation HL, the course first assessed in 2021.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

Key:

- **Deposit**: An amount added to an account at a stated time.
- Define the balance immediately after a stated event and give the initial balance.
- Watch for: do not give the final end-of-period deposit an extra interest period.

Matrix transformations, inverses and eigenvalues

Where do the basis vectors go?

- A map stretches one direction and shears another. A matrix records the transformation of each basis vector.
- This lesson studies **determinant** : A scalar that determines whether a square matrix is invertible and, in two dimensions, its signed area factor.

The method

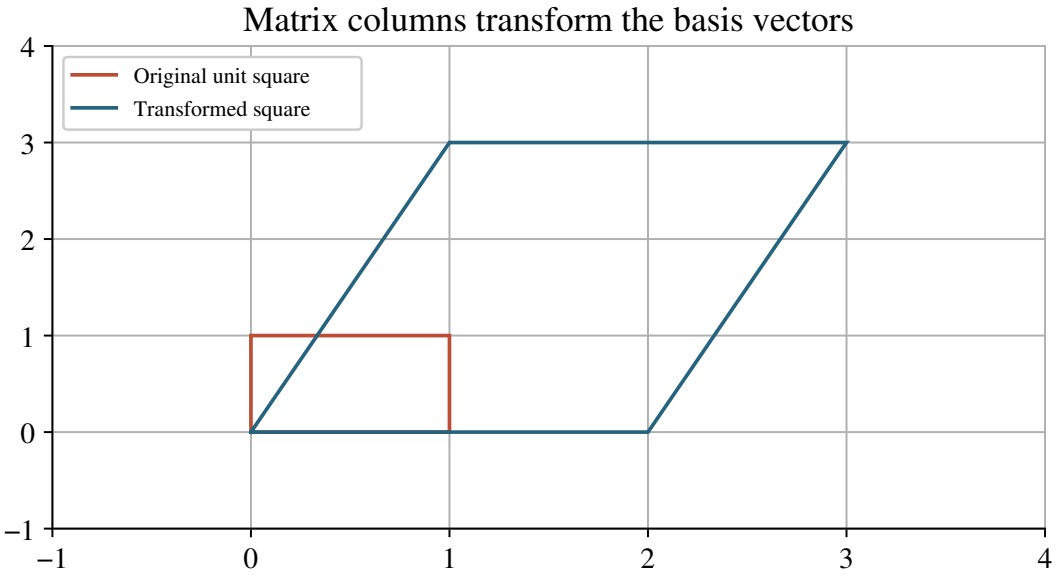
- A 2×2 matrix [[a,b],[c,d]] has determinant ad-bc and inverse [[d,-b],[-c,a]]/(ad-bc) if the determinant is nonzero.
- Transform column vectors by left multiplication.
- The rightmost matrix acts first in a product.

A worked case

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

Example:

For A=[[2,1],[0,3]], det A=6. A(1,2)=(4,6). Its inverse is [[1/2,-1/6],[0,1/3]]. The characteristic equation is (2-λ)(3-λ)=0, so the eigenvalues are 2 and 3.



The columns (2, 0) and (1, 3) are the images of the basis vectors; the area scale factor is the determinant, 6

■ Where it goes wrong

- Matrix multiplication is usually not commutative.
- A determinant of zero means the transformation loses a dimension and has no inverse.
- Eigenvalues and diagonalisation are advanced scope, not ordinary GCSE vector content.

Not true: Every square matrix has an inverse. A square matrix with determinant 0 is singular and has no inverse.

■ Applying it

- Find an eigenvector by solving $(A-\lambda I)v=0$ with $v\neq 0$.
- Explain the geometrical meaning: this vector keeps its line direction under the transformation.

■ In your course

- Applications and Interpretation HL, the course first assessed in 2021.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

Key:

- **Determinant**: A scalar that determines whether a square matrix is invertible and, in two dimensions, its signed area factor.
- A 2×2 matrix [[a,b],[c,d]] has determinant ad-bc and inverse [[d,-b],[-c,a]]/(ad-bc) if the determinant is nonzero.
- Watch for: matrix multiplication is usually not commutative.

Write from a determinant of 8

- The matrix has rows (4, 1) and (0, 2).
- Find its determinant. A square matrix with determinant 0 has no inverse.

Complex numbers and roots

What if the root is not real?

- The equation $x^2+1=0$ has no real roots. Extending the number system lets us represent and calculate with both roots.
- This lesson studies **complex conjugate** : The number obtained by reversing the sign of the imaginary part.

The method

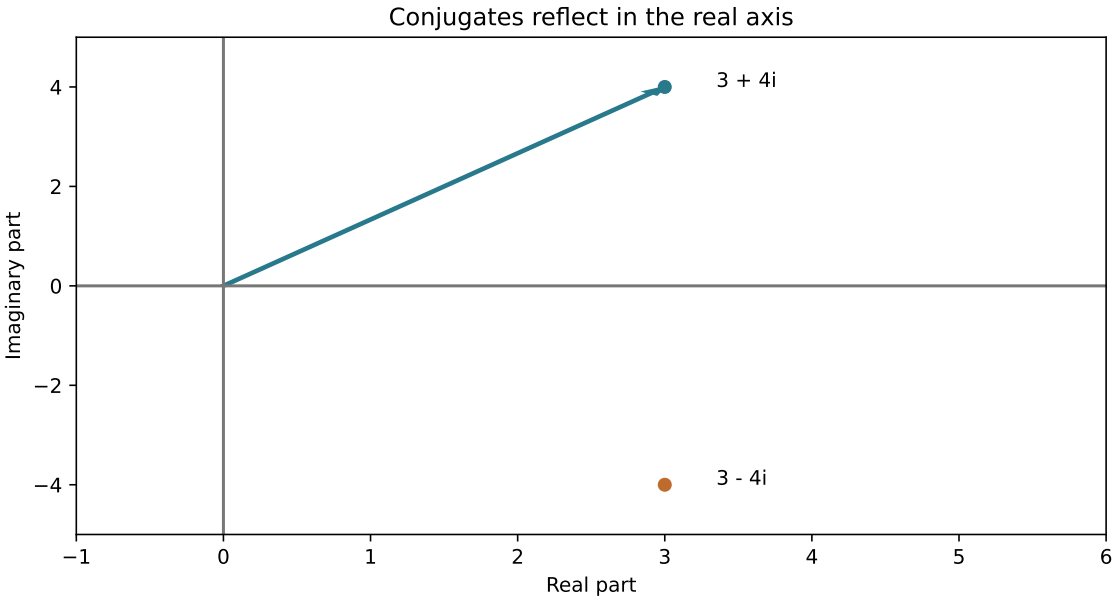
- Write z=a+bi with $i^2=-1$.
- The conjugate is a-bi and z times its conjugate=a^2+b^2.
- The modulus is $\sqrt{(a^2+b^2)}$; the argument needs the correct quadrant.
- Conjugate roots occur for polynomials with real coefficients.

A worked case

$$z = a + bi, \qquad |z| = \sqrt{a^2 + b^2}, \qquad z\overline{z} = a^2 + b^2$$

Example:

For z=3+4i, |z|=5 and z conjugate=3-4i. The reciprocal is (3-4i)/25. For x^2-6x+13=0, the roots are 3±2i. Multiplying by the conjugate makes a complex denominator real.



The conjugate 3 − 4i is the reflection of 3 + 4i in the real axis; both have modulus 5

Where it goes wrong

- The square root of a sum is not the sum of the square roots.
- An inverse tangent alone may return the wrong quadrant.
- Do not confuse modulus with squared modulus.

Not true: For a+bi, its modulus is always a+b. The modulus is $\sqrt{(a^2+b^2)}$: for 3+4i it is 5, not 7.

Applying it

- Draw the real axis horizontally and imaginary axis vertically.
- For products, moduli multiply and arguments add; powers extend this geometric pattern through de Moivre's theorem.

In your course

- Applications and Interpretation HL, the course first assessed in 2021.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

Key:

- **Complex conjugate**: The number obtained by reversing the sign of the imaginary part.

- Write z=a+bi with i^2=-1.
- Watch for: the square root of a sum is not the sum of the square roots.

Write from a modulus of 17

- Find the modulus of 8 plus 15 i.
- The modulus is not the sum of the two parts.

Write from a modulus of 5

- Find the modulus of 3 plus 4 i.
- Root of 9 plus 16 = 5 – Pythagoras again, in the complex plane.

Write from a product of 25

- Multiply z = 3 plus 4 i by its conjugate.
- The product is 25 – z times its conjugate is always the modulus squared, a real number.