5.2 Root finding and numerical integration

Name: _____ Class: _____ Date: _____ **Total: 13 marks**

KEY WORDS iteration

Objective

Root finding and numerical integration. Original classroom practice for IB HL; not an official IB paper.

1 Key knowledge and worked examples

How reliable is an approximation?

- A nonlinear equation has no convenient exact solution. An approximation still needs evidence that it is reliable.
- This lesson studies **iteration** : A repeated update in which each new approximation is calculated from the previous one.

The method

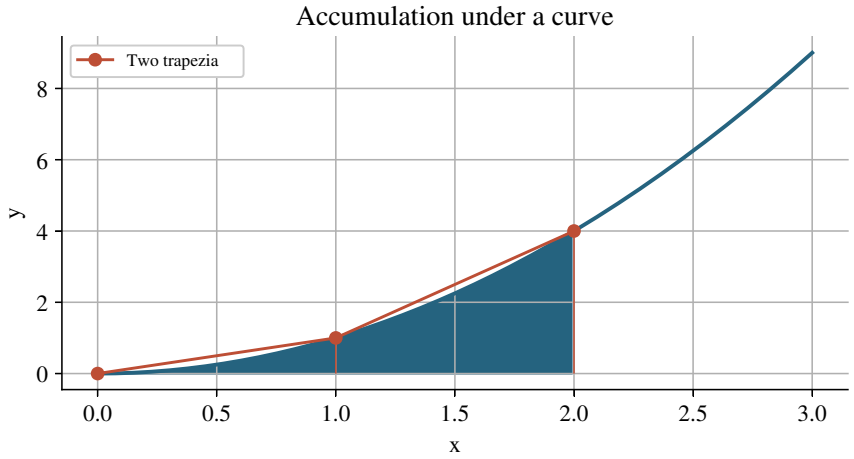
- A continuous function with opposite signs at two endpoints has a root between them.
- Newton's method uses x next=x-f(x)/f prime(x), with a nonzero derivative.
- The trapezium rule approximates a definite integral using endpoint heights.

A worked case

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Example:

For f(x)=x^2-2 and x_0=1.5, Newton gives x_1=1.5-(2.25-2)/3=1.4166667. With y=x^2 on [0,2] and two equal strips, h=1 and trapezium area=(1/2)(0+2×1+4)=3; the exact area is 8/3.



Where it goes wrong

- A sign change across a discontinuity does not prove a root.
- Iteration can diverge or cycle.
- The trapezium rule's overestimate or underestimate depends on curvature, not just whether the function increases.

Not true: Every sign change proves a root, including one across a discontinuity. A sign change shows a root only if the function is continuous there: 1/x changes sign across x=0 but has no root.

Applying it

- Give a stopping criterion and report sensible accuracy.
- Confirm the approximated root by a sign bracket around the stated rounded answer; explain any failure of the chosen iteration.

In your course

- Applications and Interpretation HL, the course first assessed in 2021.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

Key:

- **Iteration**: A repeated update in which each new approximation is calculated from the previous one.
- A continuous function with opposite signs at two endpoints has a root between them.
- Watch for: A sign change across a discontinuity does not prove a root.

■ **Write from a Newton step to 1.25**

- For x squared minus 1, take one Newton step from 2.
- A sign change across a discontinuity does not prove a root.

2 Practice

2.1 After one Newton step for x^2-2 from x_0=1.5, find x_1. Give it to 3 significant figures. Show the relation, substitution and units. [2]

2.2 Use two strips for the trapezium estimate of integral x^2 from 0 to 2. Show the relation, substitution and units. [2]

2.3 Find the exact integral of x^2 from 0 to 2. Give it to 3 significant figures. Show the relation, substitution and units. [2]

2.4 Decide whether this claim is true and justify: Every sign change proves a root, including one across a discontinuity. [2]

2.5 f(x)=1/x changes sign between -1 and 1. Why does this NOT prove a root in [-1,1]? Choose one answer.

- **A** The function is discontinuous at 0, so the sign-change rule does not apply
- **B** The interval does not contain 0, so the sign change happens elsewhere
- **C** A sign change never indicates a root unless the function is a polynomial [1]

3 Exam-style questions

3.1 Apply one Newton step to $f(x)=x^3-2x+2$ from $x_0=0$ and then one more. What happens, and what does it teach about convergence? [4]

Solutions

2.1 $x_1 = 1.5 - (1.5^2 - 2)/(2 \times 1.5) = 17/12 \approx 1.416667$.

2.2 $h = 1$ with heights 0,1,4 gives $(1/2)(0 + 2 \times 1 + 4) = 3$.

2.3 The exact integral is $[x^3/3]$ from 0 to $2 = 8/3$.

2.4 False. A sign change shows a root only if the function is continuous there: $1/x$ changes sign across $x = 0$ but has no root.

2.5 A The intermediate value reasoning requires continuity on the whole interval. At the asymptote $x = 0$ the function jumps from negative to positive with no root existing.

3.1 $x_1 = 0 - 2/(-2) = 1$; $x_2 = 1 - 1/1 = 0$. It cycles 0, 1 rather than converging. Newton requires suitable starting values/derivatives; use a bracketed method or inspect the function rather than assuming every iteration converges.