# 3.5 Graph theory: walks, trees and routes

Name: _____ Class: _____ Date: _____ **Total: 30 marks**

**KEY WORDS** vertex • edge • degree • adjacency matrix • spanning tree • Eulerian circuit

## Objective

Build the skills to answer exam questions on **graph theory** : degrees and adjacency matrices, Eulerian and Hamiltonian routes, minimum spanning trees, the Chinese postman and the travelling salesman problems.

**You must be able to:**

- use vertices, edges, degrees, walks, trails, paths, circuits and cycles
- use adjacency matrices: entries of $\mathbf{M}^k$ count walks of length $k$
- find minimum spanning trees with Kruskal's or Prim's algorithm
- solve Chinese postman problems and find upper and lower bounds for the travelling salesman problem

## 1 Key knowledge and worked examples

### Graph theory

The sum of the degrees of a graph is **twice** the number of edges, so the number of odd vertices is even.

| route | uses every ... once | exists when |
|---|---|---|
| Eulerian circuit | edge | all degrees even (connected graph) |
| Eulerian trail | edge | exactly two odd vertices (start at one, end at the other) |
| Hamiltonian cycle | vertex | no simple test |

**Chinese postman** (travel every edge, back to the start): total weight of all edges + the shortest way to join the odd vertices in pairs.

**Travelling salesman** (visit every vertex, back to the start): **upper bound** by nearest neighbour; **lower bound** by deleting a vertex, finding a minimum spanning tree of the rest and adding the two shortest edges from the deleted vertex.

**Kruskal:** add the shortest edge that does not make a cycle; repeat until all vertices are joined.

**Worked example.** Degrees $3, 3, 2, 2$: sum 10, so the graph has **5** edges.

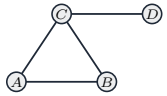
**Answer shapes:** list the edges in the order you choose them, with their weights.

## 2 Practice

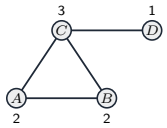
**2.1** A student says a graph has vertices of degrees 3, 3, 2 and 1. Explain why this is impossible. [2]

③ ③ ② ①

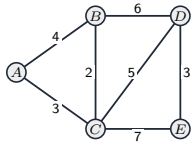
**2.2** The graph has edges $AB$, $AC$, $BC$ and $CD$. Write down its adjacency matrix $\mathbf{M}$ (order $A, B, C, D$) and the number of walks of length 2 from $A$ to $D$. [3]




**2.3** For the graph in the previous part, decide whether it has an Eulerian circuit or an Eulerian trail. [3]




**2.4** Use Kruskal's algorithm to find a minimum spanning tree of the weighted graph, and its total weight. [3]







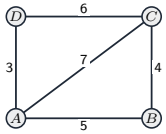

exercises.lol exercises.com.cn

## 3 Exam-style questions

**3.1** A road inspector must drive along every road and return to $A$. The roads are $AB = 5$, $BC = 4$, $CD = 6$, $DA = 3$ and $AC = 7$ (km).

(a) Find the total length of the roads. [1]

(b) Find the vertices of odd degree. [2]




(c) Find the shortest route between these two vertices. [2]

(d) Find the length of the shortest inspection route. [2]

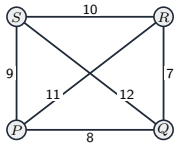
(e) Write down one such route. [2]



**3.2** A salesperson visits towns $P$, $Q$, $R$ and $S$ and returns to the start. The distances (km) are $PQ = 8$, $PR = 11$, $PS = 9$, $QR = 7$, $QS = 12$, $RS = 10$.

(a) Use the nearest-neighbour algorithm from $P$ to find an upper bound. [3]

(b) By deleting $P$, find a lower bound. [5]




(c) What can you conclude? [2]



exercises.lol exercises.com.cn

## Solutions

Marking: **M1** a correct method, **A1** the correct answer, **B1** an independent mark.

Each answer shows the steps, then the **result**.

**2.1**

sum of degrees $= 9$; the sum must be even (twice the number of edges) (R1)

**2.2**

$\mathbf{M} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$; **1** walk ($A$–$C$–$D$)

**2.3**

degrees $A2$, $B2$, $C3$, $D1$; two odd vertices (R1); an Eulerian **trail** from $C$ to $D$, but no circuit

**2.4**

$BC$ 2, $AC$ 3, $DE$ 3; reject $AB$ 4 (cycle); $CD$ 5; total **13**

**3.1**

(a) **25** km

(b) $A$ and $C$, each of degree 3

(c) $AC$ direct $= 7$; via $B$ or $D = 9$; **7** km

(d) $25 + 7 =$ **32** km

(e) e.g. $A$–$B$–$C$–$A$–$C$–$D$–$A$

**3.2**

(a) $P$–$Q$ 8, $Q$–$R$ 7, $R$–$S$ 10, $S$–$P$ 9; upper bound **34**

(b) minimum spanning tree of $Q, R, S$: $QR$ 7, $RS$ 10; 17; two shortest edges from $P$: $8+9$; lower bound **34**

(c) the bounds are equal (R1), so the tour $PQRSP$ of 34 km is optimal



exercises.lol exercises.com.cn