3.5 Graph theory: walks, trees and routes

Name: _____ Class: _____ Date: _____ **Total: 30 marks**

KEY WORDS vertex • edge • degree • adjacency matrix • spanning tree • Eulerian circuit

Objective

Build the skills to answer exam questions on **graph theory** : degrees and adjacency matrices, Eulerian and Hamiltonian routes, minimum spanning trees, the Chinese postman and the travelling salesman problems.

You must be able to:

- use vertices, edges, degrees, walks, trails, paths, circuits and cycles
- use adjacency matrices: entries of $\mathbf{M}^k$ count walks of length k
- find minimum spanning trees with Kruskal's or Prim's algorithm
- solve Chinese postman problems and find upper and lower bounds for the travelling salesman problem

1 Key knowledge and worked examples

■ Graph theory

The sum of the degrees of a graph is **twice** the number of edges, so the number of odd vertices is even.

route	uses every ... once	exists when
Eulerian circuit	edge	all degrees even (connected graph)
Eulerian trail	edge	exactly two odd vertices (start at one, end at the other)
Hamiltonian cycle	vertex	no simple test

Chinese postman (travel every edge, back to the start): total weight of all edges + the shortest way to join the odd vertices in pairs.

Travelling salesman (visit every vertex, back to the start): **upper bound** by

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nearest neighbour; **lower bound** by deleting a vertex, finding a minimum spanning tree of the rest and adding the two shortest edges from the deleted vertex.

Kruskal: add the shortest edge that does not make a cycle; repeat until all vertices are joined.

Worked example. Degrees $3, 3, 2, 2$: sum 10, so the graph has **5** edges.

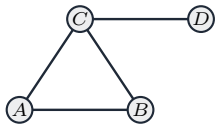
Answer shapes: list the edges in the order you choose them, with their weights.

2 Practice

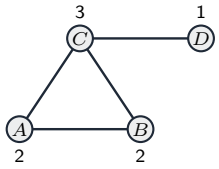
2.1 A student says a graph has vertices of degrees 3, 3, 2 and 1. Explain why this is impossible. [2]

③ ③ ② ①

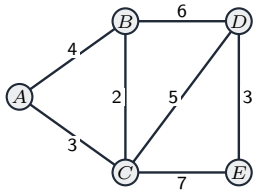
2.2 The graph has edges AB, AC, BC and CD. Write down its adjacency matrix **M** (order A, B, C, D) and the number of walks of length 2 from A to D. [3]



2.3 For the graph in the previous part, decide whether it has an Eulerian circuit or an Eulerian trail. [3]



2.4 Use Kruskal's algorithm to find a minimum spanning tree of the weighted graph, and its total weight. [3]

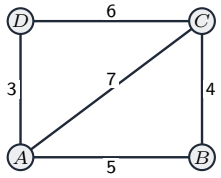


3 Exam-style questions

3.1 A road inspector must drive along every road and return to A. The roads are $AB = 5$, $BC = 4$, $CD = 6$, $DA = 3$ and $AC = 7$ (km).

(a) Find the total length of the roads. [1]

(b) Find the vertices of odd degree. [2]



(c) Find the shortest route between these two vertices. [2]

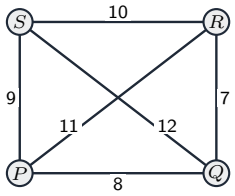
(d) Find the length of the shortest inspection route. [2]

(e) Write down one such route. [2]

3.2 A salesperson visits towns P, Q, R and S and returns to the start. The distances (km) are $PQ = 8$, $PR = 11$, $PS = 9$, $QR = 7$, $QS = 12$, $RS = 10$.

(a) Use the nearest-neighbour algorithm from P to find an upper bound. [3]

(b) By deleting P, find a lower bound. [5]



(c) What can you conclude? [2]

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Solutions

Marking: **M1** a correct method, **A1** the correct answer, **B1** an independent mark.

Each answer shows the steps, then the **result**.

2.1

sum of degrees $= 9$; the sum must be even (twice the number of edges) (R1)

2.2

$\mathbf{M} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$; **1** walk ($A$–$C$–$D$)

2.3

degrees $A2$, $B2$, $C3$, $D1$; two odd vertices (R1); an Eulerian **trail** from C to D, but no circuit

2.4

BC 2, AC 3, DE 3; reject AB 4 (cycle); CD 5; total **13**

3.1

(a) **25** km

(b) A and C, each of degree 3

(c) AC direct $= 7$; via B or $D = 9$; **7** km

(d) $25 + 7 =$ **32** km

(e) e.g. A–B–C–A–C–D–A

3.2

(a) P–Q 8, Q–R 7, R–S 10, S–P 9; upper bound **34**

(b) minimum spanning tree of Q, R, S: QR 7, RS 10; 17; two shortest edges from P: $8+9$; lower bound **34**

(c) the bounds are equal (R1), so the tour $PQRSP$ of 34 km is optimal

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