

Exercise walk-through

5.5

Differential equations and numerical solutions

International Baccalaureate · IB Diploma · Mathematics: Applications and Interpretation

5.5 Differential equations and numerical solutions

Name: _____ Class: _____ Date: _____

Total: 13 marks

KEY WORDS initial condition

Objective

Differential equations and numerical solutions. Original classroom practice for IB HL; not an official IB paper.

1 Key knowledge and worked examples

■ What does the starting value decide?

- A growing population's rate is proportional to its current size. The rate equation describes a whole family until an initial population is supplied.
- This lesson studies **initial condition** : A specified value that selects a

particular solution from a family.

■ The method

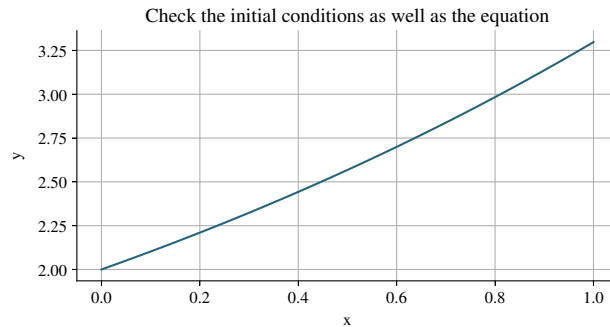
- For $dy/dx=ky$, separate variables: $dy/y=k dx$, giving $y=Ae^{\wedge}(kx)$.
 - Use an initial condition to find A.
 - Euler's method takes $y_{next}=y+h f(x,y)$, with a chosen step h .
-

■ A worked case

$$\frac{dy}{dx} = ky, \quad y = Ae^{kx}$$

Example:

With $dy/dx=0.5y$ and $y(0)=2$, $y=2e^{(0.5x)}$. Euler with $h=0.2$ gives $y(0.2)\approx 2+0.2\times 1=2.2$ and $y(0.4)\approx 2.2+0.2\times 1.1=2.42$. The exact second value is about 2.4428.



■ Where it goes wrong

- An initial condition determines the integration constant; it is not a replacement for integrating.
- Dividing by y can omit an equilibrium solution $y=0$.
- Euler accuracy depends on step size and the equation.

Not true: Every step size gives exactly the same Euler approximation. Euler's method depends on the step size; smaller steps usually give a more accurate value.

■ Applying it

- For second-order linear equations, combine the complementary function with an appropriate particular integral, then apply the required initial conditions.
 - This is Further Pure scope; check the named unit.
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■ In your course

- Applications and Interpretation HL, the course first assessed in 2021.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

Key:

- **Initial condition:** A specified value that selects a particular solution from a family.
 - For $dy/dx=ky$, separate variables: $dy/y=k \, dx$, giving $y=Ae^{(kx)}$.
 - Watch for: an initial condition determines the integration constant; it is not a replacement for integrating.
-

■ Write from a first Euler value of 6.6

- Use Euler with step 0.2 for $y' = 0.5y$, starting at $y = 6$.
- Find the first two approximations. A different step size gives a different approximation.

Exercises 2.1-2.3

2.1 Using Euler with $h=0.2$ for $y' = 0.5y$, $y_0=2$, find y_1 . Show the relation, substitution and units. [2]

2.2 For that approximation, find y_2 . Show the relation, substitution and units. [2]

2.3 For $y = 2e^{(0.5x)}$, find y at $x=0$. Show the relation, substitution and units. [2]

Exercises 2.1-2.3

2.1 Using Euler with $h=0.2$ for $y' = 0.5y$, $y_0=2$, find y_1 . Show the relation, substitution and units. [2]

2.1 Euler uses $y_1 = y_0 + h f(y_0) = 2 + 0.2 \times 0.5 \times 2 = 2.2$.

2.2 For that approximation, find y_2 . Show the relation, substitution and units. [2]

2.2 Repeat from y_1 : $y_2 = 2.2 + 0.2 \times 0.5 \times 2.2 = 2.42$.

2.3 For $y = 2e^{(0.5x)}$, find y at $x=0$. Show the relation, substitution and units. [2]

2.3 At $x=0$ the exponential is 1, so $y=2$.

Exercises 2.4·2.5

2.4 Decide whether this claim is true and justify: Every step size gives exactly the same Euler approximation. [2]

2.5 For $dy/dx=0.5y$ with $y(0)=2$, what does the initial condition determine? Choose one answer.

- **A** The step size for Euler's method, which is set equal to $y(0)=2$
- **B** The growth constant k , which changes from 0.5 to 2 once $y(0)=2$ is used
- **C** The constant A in the family $y=Ae^{(0.5x)}$, giving $y=2e^{(0.5x)}$

[1]

Exercises 2.4·2.5

2.4 Decide whether this claim is true and justify: Every step size gives exactly the same Euler approximation. [2]

2.4 False. Euler's method depends on the step size; smaller steps usually give a more accurate value.

2.5 For $dy/dx=0.5y$ with $y(0)=2$, what does the initial condition determine? Choose one answer.

- **A** The step size for Euler's method, which is set equal to $y(0)=2$
- **B** The growth constant k , which changes from 0.5 to 2 once $y(0)=2$ is used
- **C** The constant A in the family $y=Ae^{(0.5x)}$, giving $y=2e^{(0.5x)}$ [1]

2.5 C Separation gives the family $y=Ae^{(0.5x)}$ with a free constant. Substituting $x=0$, $y=2$ fixes $A=2$, selecting one particular solution from the family.

Exercise 3.1

3.1 Use Euler $h=0.5$ for $y'=-y$, $y(0)=2$ to estimate $y(1)$. Compare with $2e^{-1}$ and state a limitation. [4]

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3.1 Updates $2 \rightarrow 1 \rightarrow 0.5$; $\text{exact} \approx 0.735759$. Euler underestimates here; finite steps introduce error. Halve the step and compare convergence, rather than treating the numerical value as exact.