Exercise walk-through

1.5
Matrix transformations, inverses and eigenvalues

International Baccalaureate · IB Diploma · Mathematics: Applications and Interpret

INTERNATIONAL BACCALAUREATE · IB DIPLOMA · MATHEMATICS: APPLICATIONS AND INTERPRETATION · HL • EXERCISE SHEET

1.5 Matrix transformations, inverses and eigenvalues

Name: _____ Class: _____ Date: _____ **Total: 13 marks**

KEY WORDS determinant

Objective

Matrix transformations, inverses and eigenvalues. Original classroom practice for IB HL; not an official IB paper.

1 Key knowledge and worked examples

Where do the basis vectors go?

- A map stretches one direction and shears another. A matrix records the transformation of each basis vector.
- This lesson studies **determinant** : A scalar that determines whether a square

matrix is invertible and, in two dimensions, its signed area factor.

The method

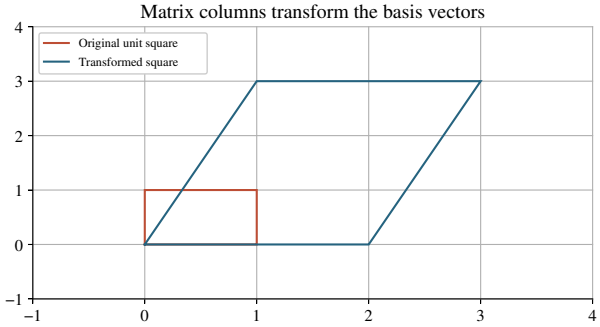
- A 2×2 matrix [[a,b],[c,d]] has determinant ad-bc and inverse [[d,-b],[-c,a]]/(ad-bc) if the determinant is nonzero.
- Transform column vectors by left multiplication.
- The rightmost matrix acts first in a product.

A worked case

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

Example:

For A=[[2,1],[0,3]], det A=6. A(1,2)=(4,6). Its inverse is [[1/2,-1/6],[0,1/3]]. The characteristic equation is (2-λ)(3-λ)=0, so the eigenvalues are 2 and 3.



■ **Where it goes wrong**

- Matrix multiplication is usually not commutative.
- A determinant of zero means the transformation loses a dimension and has no inverse.
- Eigenvalues and diagonalisation are advanced scope, not ordinary GCSE vector content.

Not true: Every square matrix has an inverse. A square matrix with determinant 0 is singular and has no inverse.

■ **Applying it**

- Find an eigenvector by solving $(A-\lambda I)v=0$ with $v\neq 0$.
- Explain the geometrical meaning: this vector keeps its line direction under the transformation.

■ **In your course**

- Applications and Interpretation HL, the course first assessed in 2021.
- Every paper allows a calculator, but method marks still need the expression or equation written before its value.

Key:

- **Determinant**: A scalar that determines whether a square matrix is invertible and, in two dimensions, its signed area factor.
- A 2×2 matrix [[a,b],[c,d]] has determinant ad-bc and inverse [[d,-b],[-c,a]]/(ad-bc) if the determinant is nonzero.
- Watch for: matrix multiplication is usually not commutative.

- **Write from a determinant of 8**
  - The matrix has rows (4, 1) and (0, 2).
  - Find its determinant. A square matrix with determinant 0 has no inverse.

Exercises 2.1·2.3

2.1 Find the determinant of [[2,1],[0,3]]. Show the relation, substitution and units. [2]

2.2 Find the x-component after that matrix transforms (1,2). Show the relation, substitution and units. [2]

2.3 Find the larger eigenvalue of that matrix. Show the relation, substitution and units. [2]

Exercises 2.1·2.3

2.1 Find the determinant of [[2,1],[0,3]]. Show the relation, substitution and units. [2]

2.1 The determinant is $2\times3-1\times0=6$.

2.2 Find the x-component after that matrix transforms (1,2). Show the relation, substitution and units. [2]

2.2 The first transformed coordinate is $2\times1+1\times2=4$.

2.3 Find the larger eigenvalue of that matrix. Show the relation, substitution and units. [2]

2.3 This triangular matrix has eigenvalues given by its diagonal entries 2 and 3, so the larger is 3.

Exercises 2.4·3.1

2.4 Decide whether this claim is true and justify: Every square matrix has an inverse. [2]

2.5 Why does a zero determinant mean a 2×2 matrix has no inverse? Choose one answer.

- **A** Only non-square matrices have inverses, and a zero determinant shows it is square
- **B** The inverse formula divides by ad-bc, and zero would mean squashing the plane onto a line
- **C** A matrix with a zero determinant has negative entries, which cannot be inverted [1]

3.1 For A=[[2, 1], [0, 3]], find an eigenvector for each eigenvalue and check whether(1, 1) is an eigenvector. [4]

Exercises 2.4·3.1

2.4 Decide whether this claim is true and justify: Every square matrix has an inverse.[2]

2.4 False. A square matrix with determinant 0 is singular and has no inverse.

2.5 Why does a zero determinant mean a 2×2 matrix has no inverse? Choose one answer.

- **A** Only non-square matrices have inverses, and a zero determinant shows it is square
- **B** The inverse formula divides by ad-bc, and zero would mean squashing the plane onto a line
- **C** A matrix with a zero determinant has negative entries, which cannot be inverted[1]

2.5 B The determinant is the signed area scale factor. det=0 collapses a dimension - distinct points share an image - so no inverse transformation can separate them, and division by ad-bc fails.

3.1 For A=[[2, 1], [0, 3]], find an eigenvector for each eigenvalue and check whether(1, 1) is an eigenvector. [4]

3.1 λ2:eigenvector(1, 0); λ3:(1, 1). A(1, 1)=(3, 3)=3(1, 1). Nonzero scalar multiples also work; the zero vector is excluded.