Calculus

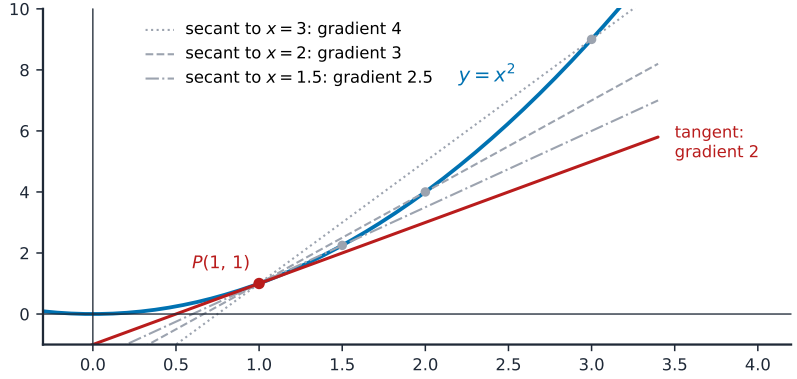
Topic 5 · SL and HL

From average to instant

> An average is a chord's gradient; an instant is a **tangent**'s gradient.

- 36 km in 30 minutes: 72 km/h on average.
- The speedometer shows the speed now.
- This unit: derivatives $\to$ rules $\to$ optimisation $\to$ integrals.

Secants approach the tangent



- **Secant** gradient $\frac{f(x+h)-f(x)}{h}$; as $h \to 0$ it tends to $f'(x)$.

The power rule

1. **Function.** $y = 4x^3 - 5x + 2x^{-1} + 7$
2. **Differentiate.** $12x^2 - 5 - 2x^{-2}$
3. **Tidy.** $12x^2 - 5 - \frac{2}{x^2}$

Decision

$\frac{d}{dx}x^n = nx^{n-1}$

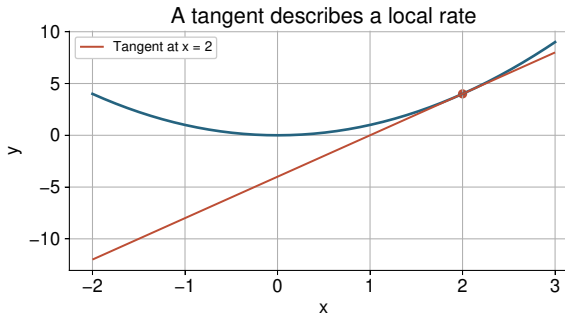
Tangent and normal

1. **Gradient.** $2x = 4$
2. **Tangent.** $y = 4x - 4$
3. **Normal.** $x + 4y - 18 = 0$

Decision

Perpendicular gradients multiply to -1

The tangent touches



- $y = x^2$ has gradient 4 at $(2, 4)$.
- **Tangent**: $y = 4x - 4$.

Standard derivatives

$f(x)$	$f'(x)$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
e^x	e^x
$\ln x$	$\frac{1}{x}$

Chain, product, quotient

Rule	**Example**
Chain	$(3x^2+1)^5 \to 30x(3x^2+1)^4$
Product	$x^2e^{3x} \to xe^{3x}(3x+2)$
Quotient	$\frac{\ln x}{x} \to \frac{1-\ln x}{x^2}$

The chain rule in layers

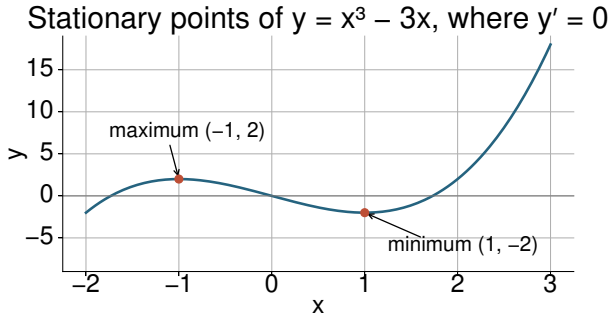
$u = 3x^2 + 1$ $\quad u^5$

outside: $\dfrac{d}{du}u^5 = 5u^4$ (keep the inside)

inside: $\dfrac{du}{dx} = 6x$ $\quad$ multiply: $30x(3x^2+1)^4$

Differentiate the outer, times the inner's **derivative**

Stationary points



- $f''(x) < 0$: maximum; > 0: minimum.
- Inflection: f'' changes sign.

Classifying

1. **Solve.** $3x^2 - 3 = 0$, so $x = \pm 1$
2. **Test** $x = -1$**.** $f'' = -6$: maximum $(-1, 2)$
3. **Test** $x = 1$**.** $f'' = 6$: minimum $(1, -2)$

Decision

Give both coordinates

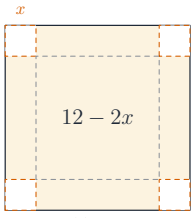
The largest box

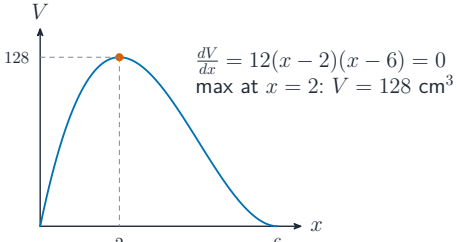
1. **Volume.** $V = 4x^3 - 48x^2 + 144x$
2. **Derivative.** $12(x-2)(x-6) = 0$, so $x = 2$
3. **Check.** $V'' = -48 < 0$; $V = 128$ cm^3

Decision

State the domain and justify the maximum

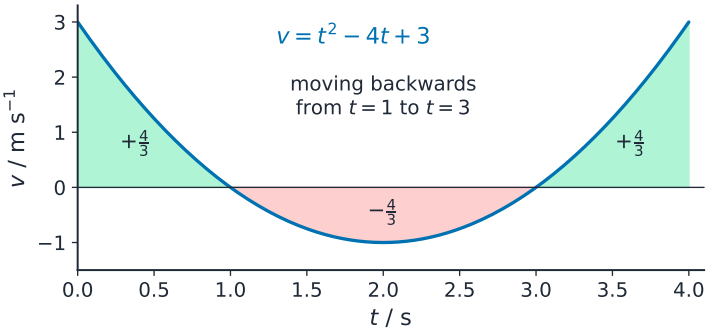
The open box





- $V = x(12 - 2x)^2$ for $0 < x < 6$.
- Maximum at $x = 2$: $V = 128$ cm^3.

Velocity and distance

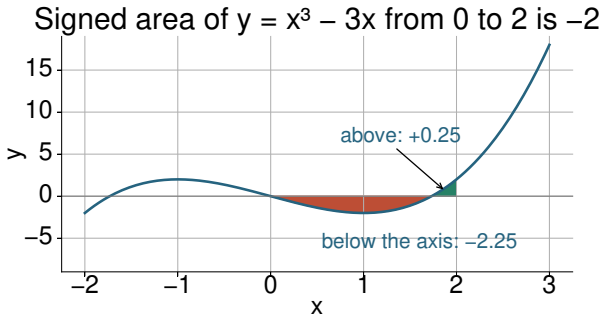


- $v = \frac{ds}{dt}$, $a = \frac{dv}{dt}$.
- **Displacement** $\frac{4}{3}$ m; distance 4 m.

Standard integrals

$f(x)$	$\int f(x)\,dx$
$x^n,\ n \neq -1$	$\frac{x^{n+1}}{n+1}$
$\frac{1}{x}$	$\ln \lvert x \rvert$
$\sin x$	$-\cos x$
$\cos x$	$\sin x$

Signed area



- Split where the curve crosses the axis.
- $\int_0^2 = -2$, but the area is $\frac{5}{2}$.

Area between curves

1. **Meet.** $x = 0$ and $x = 1$
2. **Integral.** $\int_0^1 (x - x^2)\,dx$
3. **Value.** $\frac{1}{2} - \frac{1}{3} = \frac{1}{6}$

Decision

$A = \int_a^b (f - g)\,dx$

Check: displacement or distance?

Your turn

Which gives the distance travelled from $t = 0$ to $t = 4$?

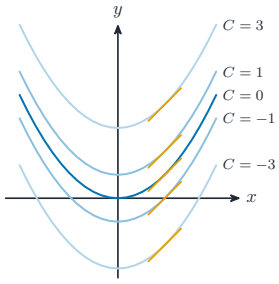
A. $\int_0^4 v\,dt$

B. $\int_0^4 |v|\,dt$

C. $v(4) - v(0)$

D. $a(4) \times 4$

Why + C



$\int 2x\,dx = x^2 + C$: every curve has gradient 2 at $x = 1$

- Every $x^2 + C$ has gradient $2x$.
- At $x = 1$ all are parallel: one condition fixes C.

First principles and limits

1. **Secant.** $\frac{(x+h)^3-x^3}{h}$
2. **Simplify.** $3x^2+3xh+h^2$
3. **Limit.** $3x^2$ as $h \to 0$

Decision

The **derivative** is a limit

Implicit and related rates

Differentiate each y term with the **chain rule**.

- $x^2 + y^2 = 25$: $\frac{dy}{dx} = -\frac{x}{y}$.
- Balloon: $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} = 200\pi$.

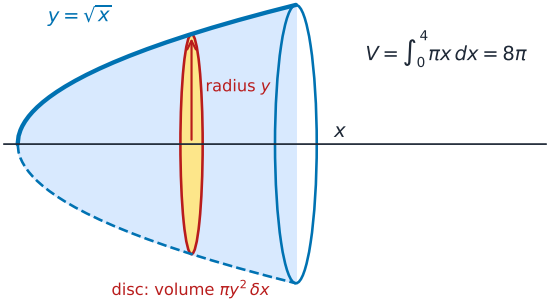
Parts and partial fractions

1. **Set up.** $u = x,\ v = e^x$
2. **Parts.** $[xe^x]_0^1 - \int_0^1 e^x\,dx$
3. **Value.** $e - (e-1) = 1$

Decision

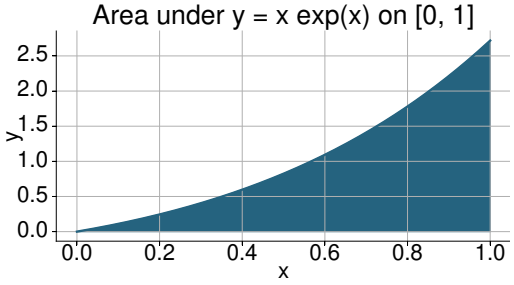
$\int uv'\,dx = uv - \int vu'\,dx$

Volumes of revolution



- About the x-axis: $\pi \int y^2 \, dx$; about the y-axis: $\pi \int x^2 \, dy$.

Integration by parts



The shaded area is exactly 1

Euler's method

1. **Step 1.** $2 + 0.5 \times 1 = 2.5$
2. **Step 2.** $2.5 + 0.5 \times 1.25 = 3.125$
3. **Compare.** Underestimate; smaller h helps

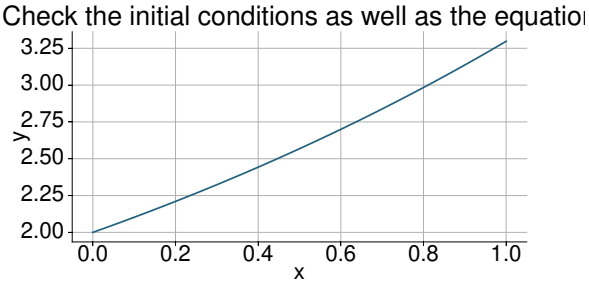
Decision

$y_{n+1} = y_n + hf(x_n, y_n)$

Exact methods and series

Type	Method
$\frac{dy}{dx} = f(x)g(y)$	Separate variables
$\frac{dy}{dx} = f(\frac{y}{x})$	Substitute $y = vx$
$y' + P(x)y = Q(x)$	**Integrating factor** $e^{\int P\,dx}$

Solving $\frac{dy}{dx} = 0.5y$



At $x = 1$, $y \approx 3.30$

Exam tips

Distance is $\int |v|\,dt$; always add $+C$.

- Rewrite roots and fractions as powers.
- Split area integrals at roots.
- Justify every maximum or minimum.

Check yourself

Try these

Find and classify the stationary points of $y = 2x^3 - 3x^2 - 12x + 4$.

A rectangle has perimeter 40 cm. Find the largest area.

Find the area between $y = x^2$ and $y = 2x$.