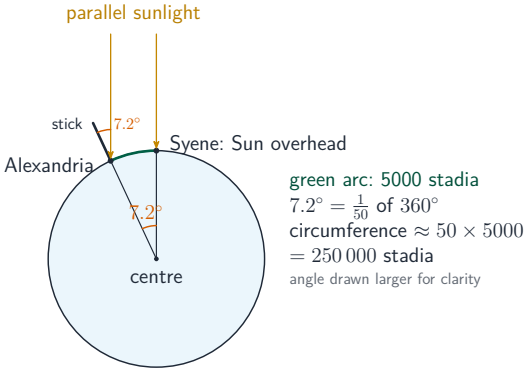
# Geometry and trigonometry

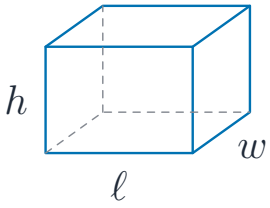
Topic 3 · SL and HL

# Measuring the Earth with a shadow

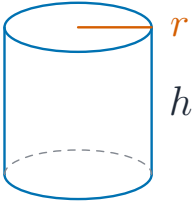



One angle and one distance gave the size of the Earth.

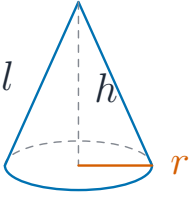
# Four solids, their dimensions



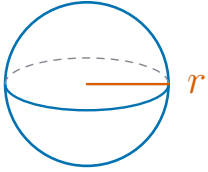

**cuboid**




**cylinder**




**cone**



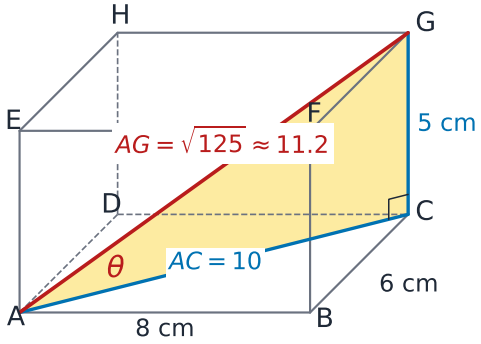

**sphere**

- Name $r$, $h$ and slant $l$ before using a formula.
- **Cone**: $V = \frac{1}{3}\pi r^2 h$, curved area $\pi rl$.

# Solids

| **Solid** | **Volume** |
|---|---|
| **Pyramid** | $\frac{1}{3} \times$ base $\times\, h$ |
| **Cone** | $\frac{1}{3}\pi r^2 h$; curved surface $\pi r l$ |
| **Sphere** | $\frac{4}{3}\pi r^3$; surface $4\pi r^2$ |
| Cone example | $96\pi \approx 302$ cm$^3$ |

# A line and a plane



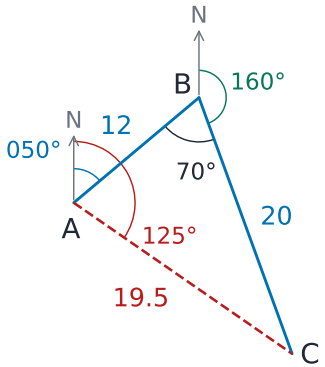

$\tan\theta = \frac{5}{10}$
$\theta \approx 26.6°$

- Find the right-angled triangle first.
- $AC = 10$, $CG = 5$, $\tan\theta = \frac{5}{10}$.

# Sine and cosine rules

| Rule | Formula |
| --- | --- |
| **Sine rule** | $\frac{a}{\sin A} = \frac{b}{\sin B}$ |
| **Cosine rule** | $c^2 = a^2 + b^2 - 2ab\cos C$ |
| Area | $\frac{1}{2}ab\sin C$ |

# Bearings




**1 angle at B**

back-bearing B→A = 050° + 180° = 230°

ABC = 230° − 160° = 70°

**2 cosine rule for AC**

$AC^2 = 12^2 + 20^2 - 2 \cdot 12 \cdot 20 \cdot \cos 70°$

$= 379.8$, so $AC = 19.5$ km

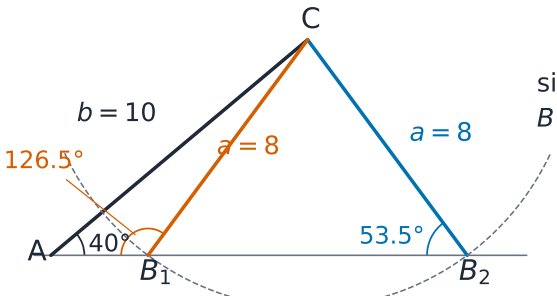
**3 sine rule, then the bearing**

$\sin A = 20 \sin 70° / 19.5 \rightarrow A = 74.6°$

bearing of C = 050° + 74.6° ≈ 125°

- 12 km on 050°, then 20 km on 160°: find AC and the **bearing** of C from A.

# The ambiguous case

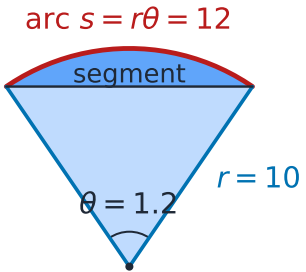



$\sin B = \frac{10 \sin 40°}{8}$

$B = 53.5°$ or $126.5°$

- $\sin B = \sin(180° - B)$.
- Check the obtuse angle leaves room.

# Arcs and sectors



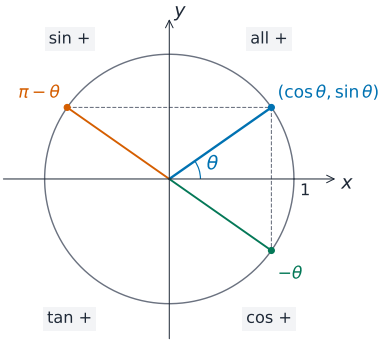

$$\text{sector} = \tfrac{1}{2}r^2\theta = 60$$

$$\text{triangle} = \tfrac{1}{2}r^2\sin\theta \approx 46.6$$

$$\text{segment} = \tfrac{1}{2}r^2(\theta - \sin\theta) \approx 13.4$$

- $\pi$ rad $= 180°$.
- $s = r\theta$, $A = \frac{1}{2}r^2\theta$, $\theta$ in radians.

# The unit circle




- $\sin(\pi - \theta) = \sin\theta$.
- $\cos(-\theta) = \cos\theta$.

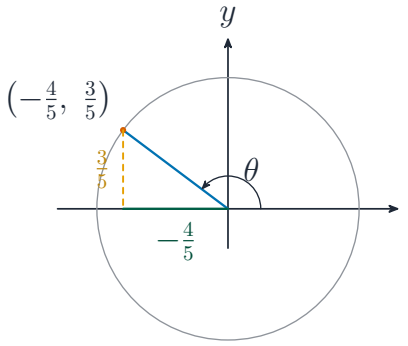
# Identities

1. **Cosine.** $\cos\theta = -\frac{4}{5}$
2. **Double angle.** $\sin 2\theta = -\frac{24}{25}$
3. **Cosine form.** $\cos 2\theta = 1 - 2\sin^2\theta = \frac{7}{25}$

## Decision

$\sin^2\theta + \cos^2\theta = 1$

# An obtuse angle on the unit circle



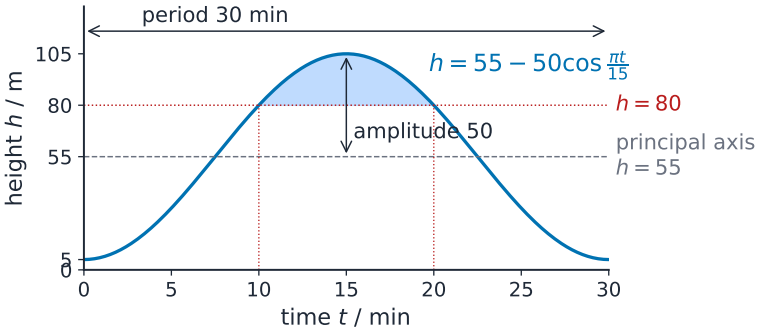

second quadrant: $\cos\theta < 0$

$\cos\theta = -\frac{4}{5},\ \tan\theta = -\frac{3}{4}$

$\sin 2\theta = -\frac{24}{25},\ \cos 2\theta = \frac{7}{25}$

- The point $(-\frac{4}{5}, \frac{3}{5})$: $\cos\theta = -\frac{4}{5}$, $\tan\theta = -\frac{3}{4}$.
- $\sin 2\theta = -\frac{24}{25}$, $\cos 2\theta = \frac{7}{25}$.

# Modelling a Ferris wheel




- **Amplitude** $|a|$, **period** $\frac{2\pi}{b}$, axis $y = d$.
- Above 80 m for $10 < t < 20$.

# Every solution

1. **Interval.** $0 \leq 2x \leq 4\pi$
2. **Solve.** $2x = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}$
3. **Divide.** $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

## Decision

Four answers, not two

# Quadratic in sine

1. **Identity.** $2\sin^2 x - 3\sin x + 1 = 0$
2. **Factorise.** $(2\sin x - 1)(\sin x - 1) = 0$
3. **Solve.** $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$

## Decision

One function, then factorise

# Check: how many solutions?

## Your turn

How many solutions has $\tan 3x = 1$ for $0 \leq x \leq \pi$?

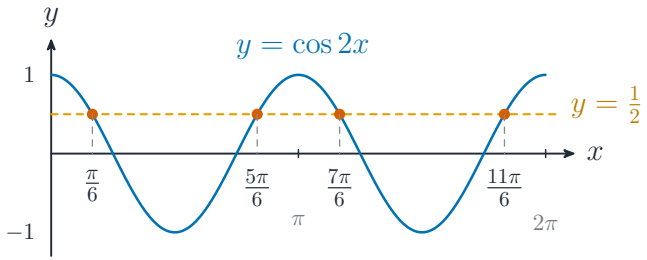
A. 3

B. 1

C. 2

D. 6

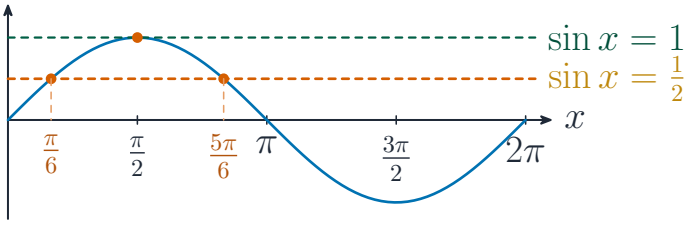
# Double the angle, double the solutions




two full turns of $u = 2x$ in $[0, 2\pi]$: four solutions, not two

- $\cos 2x = \frac{1}{2}$ has four solutions in $[0, 2\pi]$.
- $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$.

# Solving $\sin x = \frac{1}{2}$




$$(2\sin x - 1)(\sin x - 1) = 0$$

Read every solution in the interval

# Compound angles

1. **Split.** $\sin 75^\circ = \sin(45^\circ + 30^\circ)$
2. **Expand.** $\frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2}$
3. **Exact.** $\frac{\sqrt{6}+\sqrt{2}}{4}$

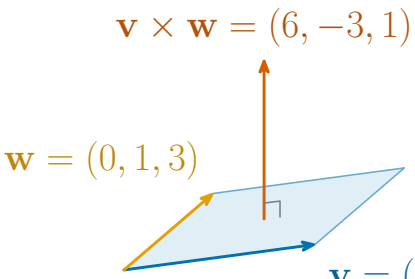
## Decision

$\sin(A + B) = \sin A \cos B + \cos A \sin B$

# Scalar and vector products

| **Product** | **Gives** |
|---|---|
| $\mathbf{v} \cdot \mathbf{w}$ | A number, $\lvert\mathbf{v}\rvert\lvert\mathbf{w}\rvert \cos\theta$ |
| $\mathbf{v} \times \mathbf{w}$ | A **vector** perpendicular to both |
| $\lvert\mathbf{v} \times \mathbf{w}\rvert$ | $\lvert\mathbf{v}\rvert\lvert\mathbf{w}\rvert \sin\theta$ |

# The vector product is a normal



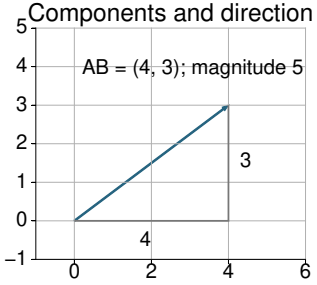

perpendicular to both $\mathbf{v}$ and $\mathbf{w}$

$|\mathbf{v} \times \mathbf{w}| = \sqrt{46} =$ parallelogram area

$\mathbf{w} \times \mathbf{v} = -\mathbf{v} \times \mathbf{w}$

- $(1, 2, 0) \times (0, 1, 3) = (6, -3, 1)$.
- Its length $\sqrt{46}$ is the parallelogram area.

# A vector and its magnitude




Pythagoras on the components

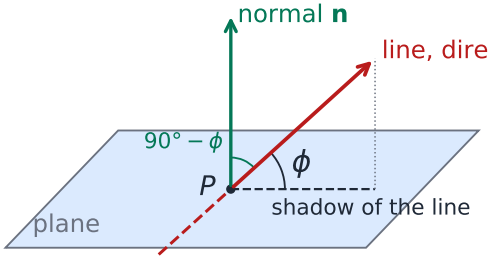
# Do the lines meet?

1. **Equate.** $1+\lambda=2$, $\lambda=3+\mu$
2. **Solve.** $\lambda=1$, $\mu=-2$
3. **Check third.** $2-1=3-2$, so they meet at $(2,1,1)$

**Decision**

Always test the unused equation

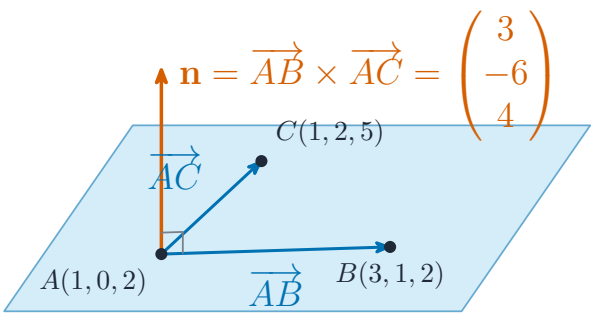
# Lines meeting planes




$$\sin\phi = \frac{|\mathbf{b}\cdot\mathbf{n}|}{|\mathbf{b}||\mathbf{n}|}$$

- Plane: $\mathbf{r}\cdot\mathbf{n} = \mathbf{a}\cdot\mathbf{n}$.
- $\mathbf{n} = \mathbf{b}\times\mathbf{c}$.

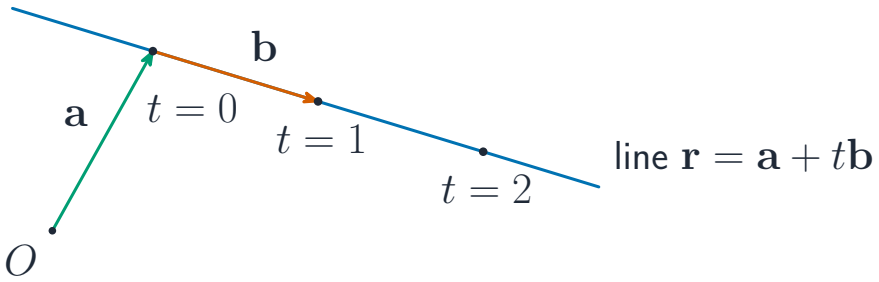
# A plane from three points




$\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$
$3x - 6y + 4z = 11$
check $B$: $9 - 6 + 8 = 11$

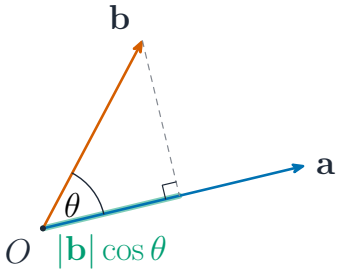
- $\vec{AB} \times \vec{AC} = (3, -6, 4)$.
- $3x - 6y + 4z = 11$; B checks: $9 - 6 + 8 = 11$.

# The vector equation of a line




$\mathbf{b}$ sets the direction

# The scalar product and the angle




Zero **scalar product**: perpendicular

# Exam tips

After the **sine rule**, test the obtuse angle too.

- Radians for arcs and calculus.
- Stretch the interval for $\sin kx$.
- Keep exact values on Paper 1.

# Check yourself

## Try these

In triangle $ABC$, $A = 30°$, $a = 6$, $b = 10$. Find both possible values of $B$.

A sector has radius 8 cm and area 40 cm$^2$. Find its angle and perimeter.

Solve $2\sin^2 x = 3\cos x$ for $0 \le x \le 2\pi$.