Functions

Topic 2 · SL and HL

A taxi fare as a function

> Each input gives exactly one output, so the fare is a function of distance.

- ¥14 for 3 km, then ¥2.50 per km.
- 10 km: $14 + 2.5 \times 7 = 31.5$ yuan.
- This unit: lines $\rightarrow$ functions $\rightarrow$ families $\rightarrow$ transformations.

Straight lines

Form	Equation
Gradient-intercept	$y = mx + c$
General	$ax + by + d = 0$
Point-gradient	$y - y_1 = m(x - x_1)$

A perpendicular line

1. **Gradient.** $-\frac{1}{2}$
2. **Line.** $y-5=-\frac{1}{2}(x-1)$
3. **General form.** $x+2y-11=0$

Decision

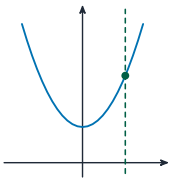
Negative reciprocal **gradient**

Domain and range

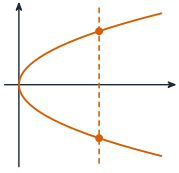
The **domain** is the allowed inputs; the **range** is the outputs it produces.

- Vertical line test: at most one meeting.
- A formula needs its domain: $\sqrt{x-2}$ needs $x \geq 2$.

The vertical line test



$y = x^2 + 1$: one meeting point, a function



$x = y^2$: two points, not a function

- $y = x^2 + 1$: each vertical line meets it once.
- $x = y^2$: some meet it twice, so not a function.

Composite functions

1. **f∘g.** $f(x^2) = 2x^2 + 1$, so 19 at 3
2. **g∘f.** $(2x+1)^2$, so 49 at 3
3. **Order.** $f \circ g \neq g \circ f$

Decision

Inner function first

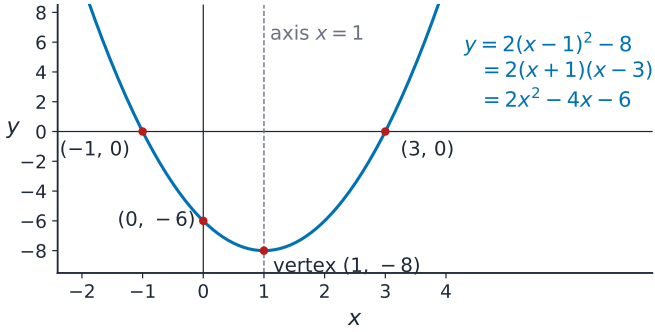
Inverse functions

1. **Function.** $f(x) = \sqrt{x-2},\ x \geq 2$
2. **Swap.** $x = \sqrt{y-2}$, so $y = x^2 + 2$
3. **Inverse.** $f^{-1}(x) = x^2 + 2,\ x \geq 0$

Decision

State the **domain** of the inverse

One parabola, three forms



- Axis of symmetry $x = -\frac{b}{2a}$.
- Complete the square for the **vertex**.

The discriminant

$\Delta = b^2 - 4ac$	**Roots**
$\Delta > 0$	Two distinct real roots
$\Delta = 0$	One repeated root
$\Delta < 0$	No real roots

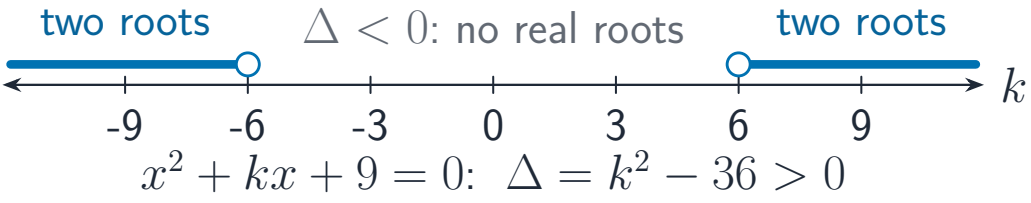
Two distinct roots

1. **Discriminant.** $k^2 - 36 > 0$
2. **Solve.** $k^2 > 36$
3. **Answer.** $k < -6$ or $k > 6$

Decision

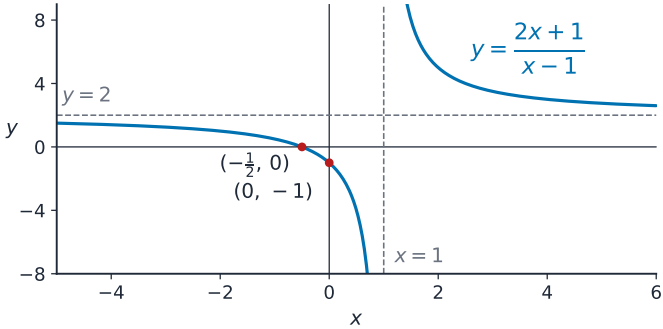
Sketch to solve the quadratic inequality

Where the discriminant is positive



- Two distinct roots for $k < -6$ or $k > 6$.
- No real roots for $-6 < k < 6$.

Rational functions



- Vertical **asymptote**: denominator zero.
- Horizontal asymptote: $y = \frac{a}{c}$.

Laws of logarithms

Law	**Rule**
Product	$\log_a(xy) = \log_a x + \log_a y$
Quotient	$\log_a \frac{x}{y} = \log_a x - \log_a y$
Power	$\log_a x^n = n \log_a x$
Change of base	$\log_a x = \frac{\ln x}{\ln a}$

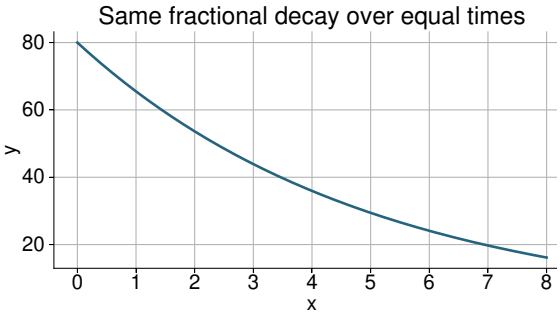
A half-life

1. **Divide.** $e^{-0.2t} = \frac{1}{2}$
2. **Take logs.** $-0.2t = -\ln 2$
3. **Solve.** $t = \frac{\ln 2}{0.2} \approx 3.47$ hours

Decision

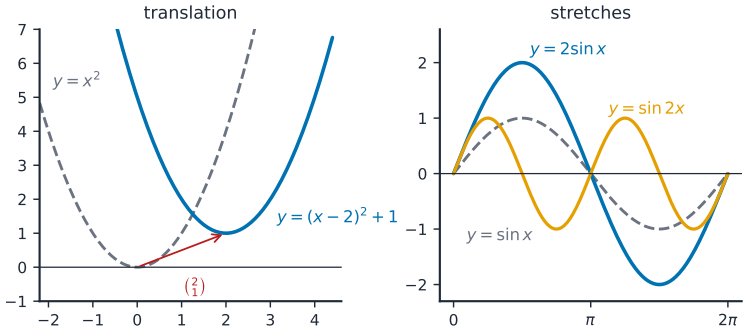
The same fraction is lost in each hour

Exponential decay



- $y = 80e^{-0.2x}$ starts at 80.
- At $x = 8$: $80e^{-1.6} \approx 16$.

Transformations



- Outside the function: acts on y as expected.
- Inside: acts on x the opposite way.

Follow a point

1. **Right 1.** $(5, 5)$
2. **Stretch $\times 3$.** $(5, 15)$
3. **Up 2.** $(5, 17)$

Decision

Order matters, so follow one point

Check: which transformation?

Your turn

Which transformation maps $y = f(x)$ onto $y = f(x - 3)$?

A. Translation 3 units left
B. Translation 3 units right
C. Translation 3 units up
D. Horizontal stretch, scale factor 3

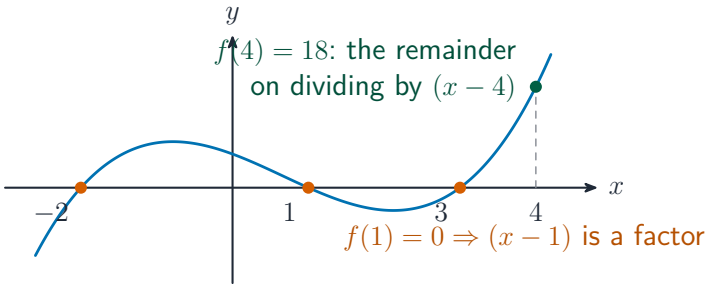
Factor and remainder theorems

1. **Test.** $P(1) = 1 - 2 - 5 + 6 = 0$
2. **Divide.** $(x-1)(x^2 - x - 6)$
3. **Factorise.** $(x-1)(x-3)(x+2)$

Decision

Sum of roots 2, product -6

Roots are factors; heights are remainders



- $f(x) = (x-1)(x+2)(x-3)$, so $f(1) = 0$.
- $f(4) = 18$ is the remainder on dividing by $(x-4)$.

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Odd and even functions

Even

$f(-x) = f(x)$

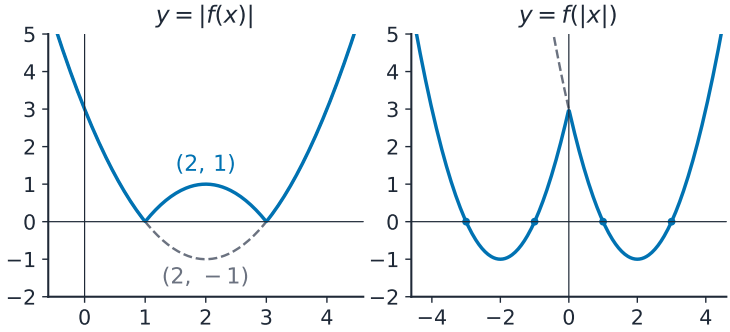
Symmetric in the y-axis

Odd

$f(-x) = -f(x)$

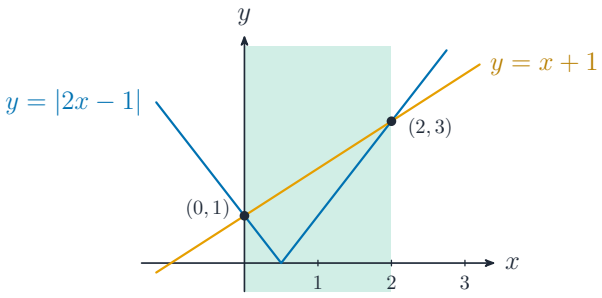
Half-turn about the origin

Modulus graphs



- $|f(x)|$: reflect the negative parts up.
- $f(|x|)$: reflect the right half across.

Solve a modulus inequality from the graph



V below the line: $0 < x < 2$

- $|2x - 1| = x + 1$ at $(0, 1)$ and $(2, 3)$.
- $|2x - 1| < x + 1$ for $0 < x < 2$.

Exam tips

> Composite $f(g(x))$: apply g first.

- Inverse **domain** $=$ original **range**.
- Reject log solutions with non-positive arguments.
- Describe transformations in full.

Check yourself

Try these

Write $y = x^2 + 6x + 5$ in vertex and factorised forms.

Solve $\log_2(x+3) + \log_2(x-3) = 4$.

Map $(1, 4)$ under $y = 2f(x+3) - 1$.