Number and algebra

Topic 1 · SL and HL

Rice on a chessboard

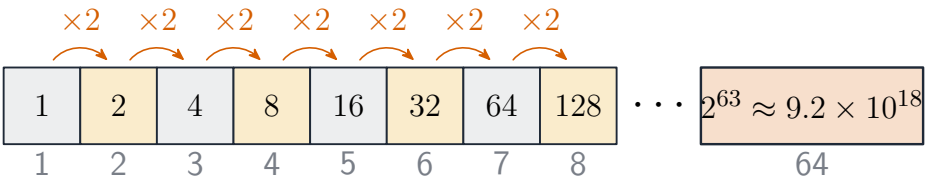
> Repeated multiplication grows faster than intuition expects.

- 1, 2, 4, 8, … grains for 64 squares.
- $2^{64} - 1 \approx 1.84 \times 10^{19}$ grains.
- This unit: exponents $\to$ logs $\to$ sequences $\to$ finance $\to$ binomial.

Laws of exponents

Law	Example
$a^m \times a^n = a^{m+n}$	$x^3 \times x^4 = x^7$
$(a^m)^n = a^{mn}$	$(2x^3)^4 = 16x^{12}$
$a^{-n} = \frac{1}{a^n}$	$2^{-3} = \frac{1}{8}$
$a^{\frac{m}{n}} = (\sqrt[n]{a})^m$	$8^{\frac{2}{3}} = 4$

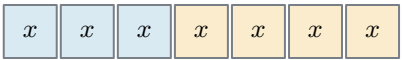
Doubling 64 times



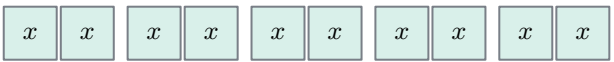
square n holds 2^{n-1} grains; the whole board holds $2^{64} - 1 \approx 1.84 \times 10^{19}$

- Square 64 holds $2^{63} \approx 9.2 \times 10^{18}$ grains.
- The board holds $2^{64} - 1 \approx 1.84 \times 10^{19}$.

Count the factors



$$x^3 \times x^4 = x^{3+4} = x^7$$



$$(x^2)^5 = x^{2\times 5} = x^{10}$$

count the factors: adding exponents is counting, not a rule to memorise

- $x^3 \times x^4$: three factors and four make x^7.
- $(x^2)^5$: five groups of two make x^{10}.

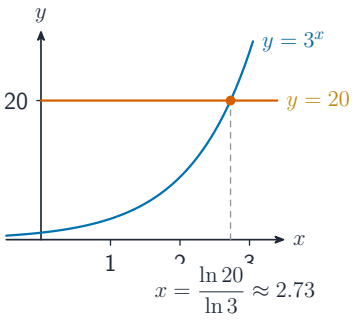
Logarithms

1. **Take ln.** $\ln 5 + x \ln 2 = (x+1) \ln 3$
2. **Collect.** $x(\ln 2 - \ln 3) = \ln 3 - \ln 5$
3. **Solve.** $x \approx 1.26$

Decision

Logarithms bring the **exponent** down

Solve with a log



- $3^x = 20$ gives $x = \frac{\ln 20}{\ln 3}$.
- The curve meets the line at $x \approx 2.73$.

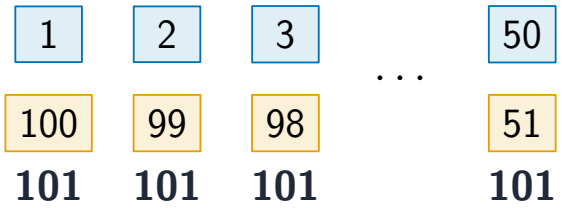
Arithmetic series

1. **Theatre.** 20 seats, $+2$ per row, 25 rows
2. **Last row.** $u_{25} = 20 + 24(2) = 68$
3. **Total.** $S_{25} = \frac{25}{2}(20 + 68) = 1100$

Decision

Pair first and last, like Gauss

Pair first and last



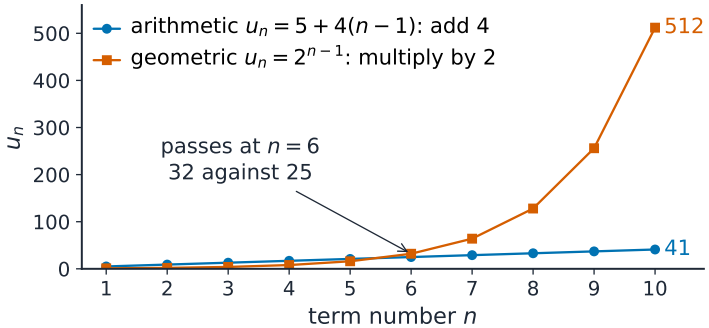
50 pairs, each 101

$$S_{100} = 50 \times 101 = 5050$$

$$S_n = \frac{n}{2}(u_1 + u_n)$$

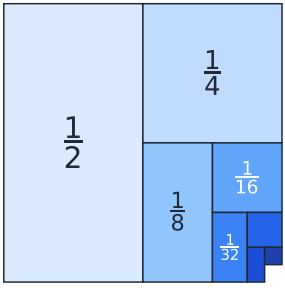
- $1 + 100$, $2 + 99$, ... each pair is 101.
- 50 pairs: $1 + 2 + \cdots + 100 = 5050$.

Add or multiply?



- $u_n = u_1 r^{n-1}$, $S_n = \frac{u_1(r^n-1)}{r-1}$.
- Geometric growth eventually wins.

Sum to infinity

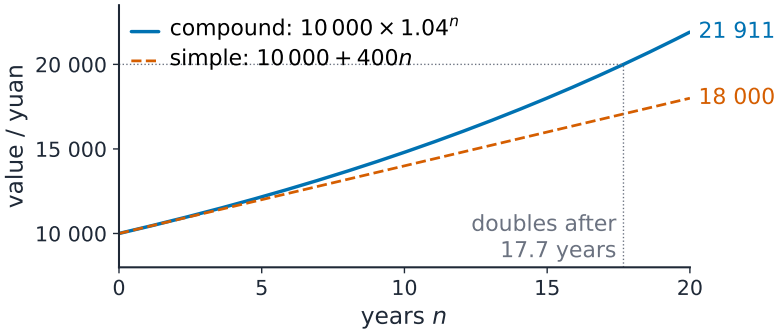


$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = 1$

$S_\infty = \frac{u_1}{1-r} = \frac{1/2}{1-1/2} = 1$

- $S_\infty = \frac{u_1}{1-r}$ only if $|r| < 1$.
- Bouncing ball: $2 + 2 \times \frac{1.2}{0.4} = 8$ m.

Compound interest



- $FV = PV(1 + \frac{r}{100k})^{kn}$.
- Doubles at 4 % in about 17.7 years.

Depreciation

1. **4 years.** $200\,000 \times 0.85^4 \approx$ ¥104 401
2. **Below ¥50 000.** $0.85^n < 0.25$
3. **Solve.** $n > 8.53$, so 9 years

Decision

Give a whole number of years

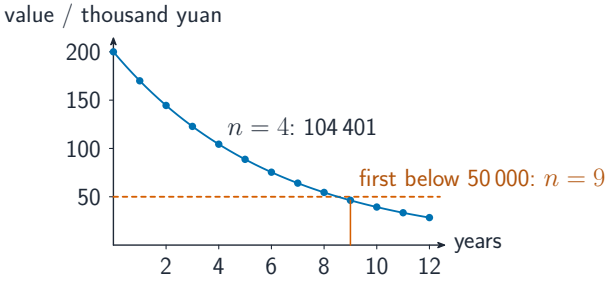
Check: which sequence?

Your turn

A car loses 15 % of its value each year. Its yearly values form which kind of sequence?

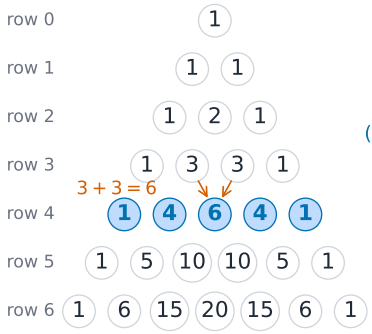
A. Geometric, $r = 0.85$
B. Arithmetic, $d = -15$
C. Geometric, $r = 0.15$
D. Arithmetic, $d = 0.85$

A car loses value each year



It first falls below 50 000 after 9 years

The binomial theorem



$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

- General term $\binom{n}{r} a^{n-r} b^r$.
- Bracket each term, with its sign.

Term independent of x

1. **General term.** $\binom{6}{r}2^r x^{12-3r}$
2. **Power zero.** $12 - 3r = 0$, so $r = 4$
3. **Term.** $15 \times 16 = 240$

Decision

Set the power of x to zero

Proof

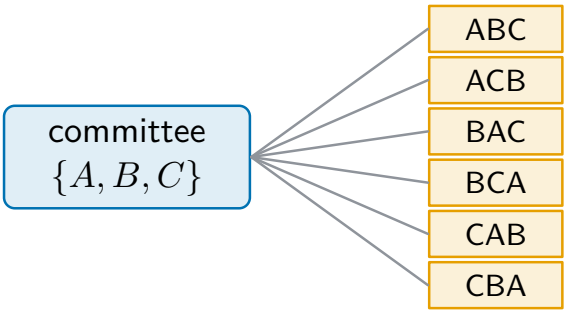
$n + (n+1) + (n+2) = 3(n+1)$, always a multiple of 3.

- Start from one side; end with "$=$ RHS".
- Examples are not a **proof**.

Counting

Choose 3 of 8	**Ways**
Order matters	${}^{8}P_{3} = 336$
Order does not matter	${}^{8}C_{3} = 56$
Arrange LEVEL	$\frac{5!}{2!\,2!} = 30$

Order multiplies by 3!

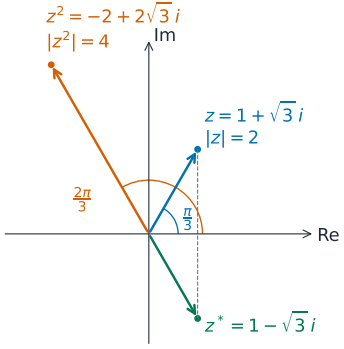


chair, secretary, treasurer:
6 orders per committee

${}^8C_3 = 56$ committees
${}^8P_3 = 56 \times 3! = 336$

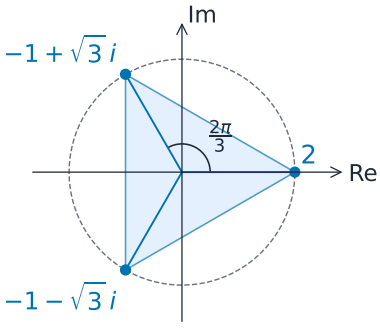
- ${}^8C_3 = 56$ committees of 3 from 8.
- Each gives $3! = 6$ role orders: ${}^8P_3 = 336$.

Complex numbers



- $z = re^{i\theta}$; multiply moduli, add arguments.
- Sketch first to get the quadrant.

Roots of a complex number



$z^3 = 8$

$z = 2\,e^{2k\pi i/3}$

$k = 0, 1, 2$

- $(re^{i\theta})^n = r^n e^{in\theta}$.
- Add $2k\pi$ before taking roots.

Proof by induction

1. **Base.** $n = 1$ gives 6
2. **Assume.** $7^k - 1 = 6m$
3. **Step.** $7^{k+1} - 1 = 7(7^k - 1) + 6 = 6(7m + 1)$

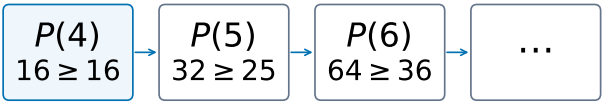
Decision

True for $n = 1$ and $k \Rightarrow k + 1$

Start where the claim starts

Claim: $2^n \geq n^2$ for integers $n \geq 4$

For every $k \geq 4$: $P(k) \Rightarrow P(k+1)$

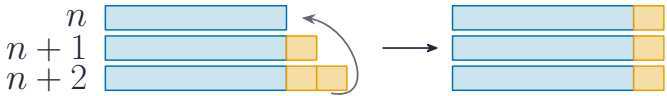


Base case Reached by the proved implication

Checked values alone do not justify the arrows.

- Base case: the first n the claim covers, here $n = 4$.
- One general step links every case to the next.

Three in a row



$3(n+1)$: a multiple of 3

- $n + (n+1) + (n+2) = 3(n+1)$.
- So the sum is always a multiple of 3.

Three equations

1. **Eliminate.** $2x+3y=8$ and $3x+y=5$
2. **Solve.** $x=1$, $y=2$
3. **Back-substitute.** $z=3$

Decision

Check in all three equations

Exam tips

First decide whether the pattern adds or multiplies.

- Same base, or take logarithms.
- State $|r| < 1$ for S_∞.
- Write every TVM value you enter.

Check yourself

Try these

In an arithmetic sequence, $u_3 = 11$ and $u_8 = 31$. Find S_{20}.

Find the sum to infinity of $24 + 18 + 13.5 + \ldots$

Find the coefficient of x^4 in $(2x + 3)^6$.