Course Companion

International Baccalaureate · IB Diploma · Mathematics: Analysis and Approaches · HL

Every topic handout in one volume

全部专题讲义合集

exercises.lol

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Number and algebra

International Baccalaureate · IB Diploma · Mathematics: Analysis and Approaches · HL

Scope and prerequisites

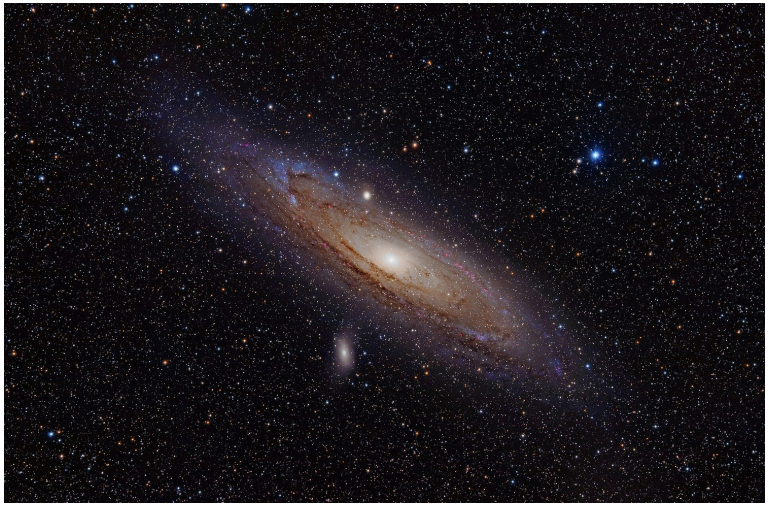
This handout teaches topic 1, Number and algebra, of Mathematics: Analysis and Approaches, the course first assessed in 2021: standard form, the laws of exponents, logarithms and exponential equations, arithmetic and geometric sequences and series, infinite geometric series, compound interest and depreciation, the binomial theorem and simple deductive proof, and, for HL, counting, the extended binomial theorem, partial fractions, complex numbers, proof by induction and contradiction, and systems of linear equations. It is written for both SL and HL students. HL students also apply the further results in the sections marked AHL.

Prerequisites: fractions, negative numbers, expanding brackets and factorising, and solving linear and quadratic equations.

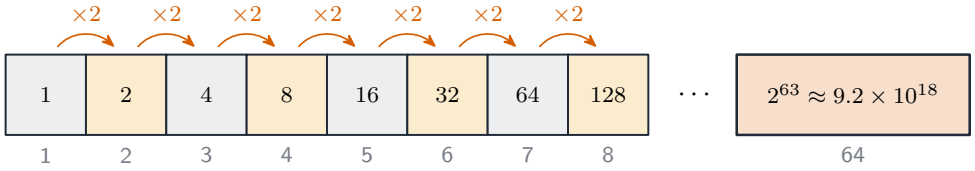
Exponents and standard form

Rice on a chessboard

An old legend tells of a clever servant who asked a king for a modest reward: one grain of rice on the first square of a chessboard, two on the second, four on the third, and so on, doubling for all 64 squares. The king agreed at once. The last square alone needs 2^{63} grains, and the whole board needs $2^{64}-1 \approx 1.84 \times 10^{19}$ grains. At about 0.025 g a grain, that is roughly 4.6×10^{11} tonnes of rice, hundreds of years of today's world harvest. Repeated multiplication grows faster than intuition expects. This topic gives us the tools to write such numbers (standard form), to undo repeated multiplication (logarithms) and to add up the squares (geometric series).



A galaxy: huge distances are written compactly in standard form.



Doubling 63 times: square n holds 2^{n-1} grains, a number best written in standard form.

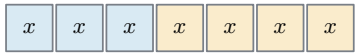
■ Standard form

A number in **standard form** 科学记数法 is written $a \times 10^k$, where $1 \le a < 10$ and k is an integer. For example, $3\,600\,000 = 3.6 \times 10^6$ and $0.000\,045\,2 = 4.52 \times 10^{-5}$.

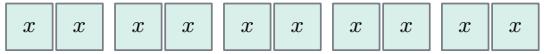
> **Worked example.** $(3.2 \times 10^5) \times (4 \times 10^{-8}) = 12.8 \times 10^{-3} = 1.28 \times 10^{-2}$. The first answer is not in standard form, because 12.8 is not less than 10.

■ Laws of exponents

For $a > 0$ and any rational m, n:



$x^3 \times x^4 = x^{3+4} = x^7$



$(x^2)^5 = x^{2\times5} = x^{10}$

count the factors: adding exponents is counting, not a rule to memorise

Exponent laws are counting factors

Law	Example
$a^m \times a^n = a^{m+n}$	$x^3 \times x^4 = x^7$
$a^m \div a^n = a^{m-n}$	$x^7 \div x^2 = x^5$
$(a^m)^n = a^{mn}$	$(x^2)^5 = x^{10}$
$a^0 = 1$, $a^{-n} = \dfrac{1}{a^n}$	$2^{-3} = \frac{1}{8}$
$a^{\frac{1}{n}} = \sqrt[n]{a}$, $a^{\frac{m}{n}} = (\sqrt[n]{a})^m$	$8^{\frac{2}{3}} = 2^2 = 4$

In a^n, a is the base and n is the **exponent** 指数.

> **Worked example.** Simplify $\dfrac{(2x^3)^4}{4x^5}$. The numerator is $16x^{12}$, so the result is $\dfrac{16x^{12}}{4x^5} = 4x^7$. Raise the 2 to the power 4 as well as the x^3.

> **Worked example.** Solve $4^{x+1} = 8^x$. Write both sides as powers of 2: $2^{2x+2} = 2^{3x}$, so $2x + 2 = 3x$ and $x = 2$.

Logarithms

■ What a logarithm is

A **logarithm** 对数 answers the question “what power?”:



Earthquake strength is measured on the logarithmic Richter scale.

$$a^x = b \iff x = \log_a b, \qquad a > 0,\ a \neq 1,\ b > 0$$

So $\log_2 32 = 5$ because $2^5 = 32$, and $\log_3 \frac{1}{9} = -2$ because $3^{-2} = \frac{1}{9}$. Two bases have their own names: $\log x$ means $\log_{10} x$, and the **natural logarithm** 自然对数 $\ln x$ means $\log_e x$, where $e \approx 2.718$. Logarithms are useful wherever numbers range over many powers of 10: the magnitude of an earthquake, $M = \log_{10} \frac{A}{A_0}$, rises by 2 when the amplitude A is 100 times larger.

Laws of logarithms

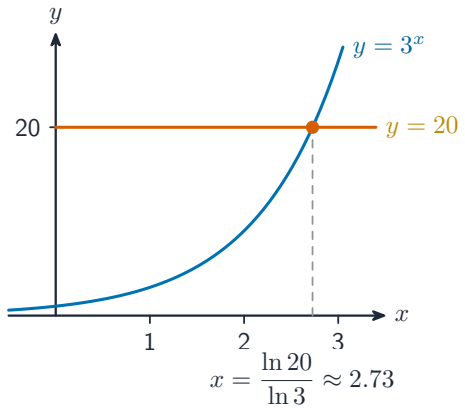
$$\log_a xy = \log_a x + \log_a y, \qquad \log_a \frac{x}{y} = \log_a x - \log_a y, \qquad \log_a x^m = m \log_a x$$

The **change of base** 换底公式 rule is $\log_a x = \dfrac{\log_b x}{\log_b a}$, so any logarithm can be found with ln on a calculator.

> **Worked example.** $\log_6 4 + \log_6 9 = \log_6 36 = 2$. And $2\log 3 + \log 4 - \log 6 = \log \dfrac{9 \times 4}{6} = \log 6$.

Solving exponential equations

When the bases cannot be matched, take logarithms of both sides.



Solving $3^x = 20$

> **Worked example.** Solve $3^x = 20$. $x \ln 3 = \ln 20$, so $x = \dfrac{\ln 20}{\ln 3} \approx 2.73$.

> **Worked example.** Solve $5 \times 2^x = 3^{x+1}$.

- Take ln: $\ln 5 + x \ln 2 = (x+1) \ln 3$.
- Collect the x terms: $x(\ln 2 - \ln 3) = \ln 3 - \ln 5$.
- $x = \dfrac{\ln 3 - \ln 5}{\ln 2 - \ln 3} \approx 1.26$.

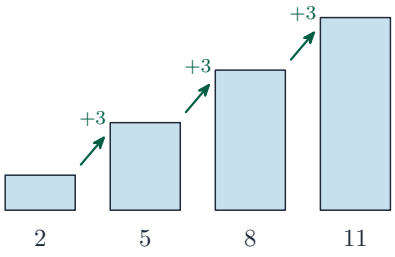
Common errors: $\log(x+y)$ is **not** $\log x + \log y$, and $\dfrac{\log x}{\log y}$ is **not** $\log x - \log y$. When solving equations with logarithms, check that every answer keeps the expressions inside the logarithms positive.

Arithmetic sequences and series

The general term

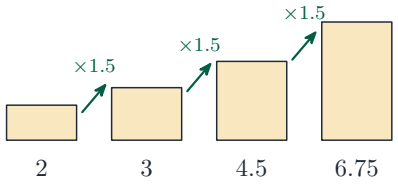
A **sequence** 数列 is an ordered list of numbers, its terms $u_1, u_2, u_3, \ldots$ An **arithmetic sequence** 等差数列 adds the same **common difference** 公差 d each time:

arithmetic: add d each step



$u_n = a + (n-1)d$: equal steps

geometric: multiply by r each step



$u_n = ar^{n-1}$: growing steps

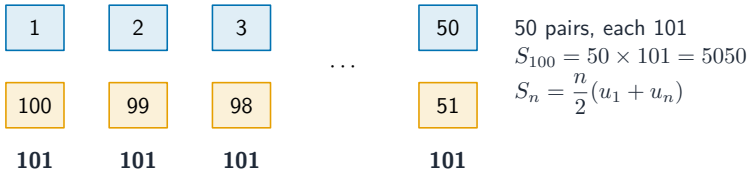
An arithmetic sequence climbs in equal steps; a geometric one multiplies by a ratio

$$u_n = u_1 + (n-1)d$$

Worked example. In an arithmetic sequence $u_4 = 17$ and $u_{10} = 41$. Then $6d = 41 - 17 = 24$, so $d = 4$ and $u_1 = 17 - 3(4) = 5$.

The sum of an arithmetic series

A **series** 级数 is the sum of the terms of a sequence. The story goes that the young Carl Friedrich Gauss added $1+2+\cdots+100$ in moments: pair the first and last terms, $1 + 100 = 101$, the second and second last, $2 + 99 = 101$, and so on. There are 50 pairs, so the sum is $50 \times 101 = 5050$. The same idea gives



Gauss's pairing

$$S_n = \frac{n}{2}\big(2u_1 + (n-1)d\big) = \frac{n}{2}(u_1 + u_n)$$

Worked example. A theatre has 20 seats in the first row and 2 more in each row behind, for 25 rows. The last row has $u_{25} = 20 + 24(2) = 68$ seats, and the theatre has $S_{25} = \frac{25}{2}(20 + 68) = 1100$ seats.

Sigma notation

Sigma notation 求和符号 writes a sum compactly: $\sum_{k=1}^{n} u_k = u_1 + u_2 + \cdots + u_n$. For example,

$$\sum_{k=1}^{10}(3k+2) = 5 + 8 + \cdots + 32 = \frac{10}{2}(5 + 32) = 185$$

Check the number of terms: $\sum_{k=3}^{10}$ has 8 terms, not 10.

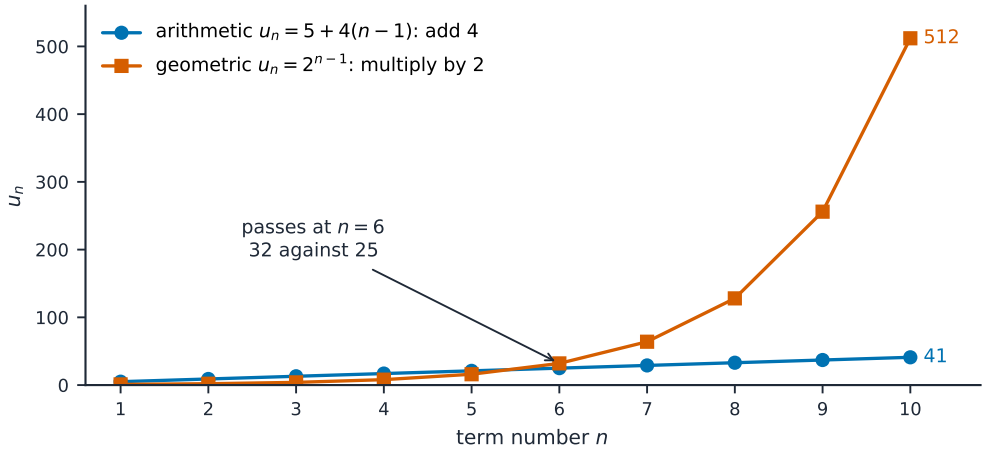
Geometric sequences and series

The general term and the sum

A **geometric sequence** 等比数列 multiplies by the same **common ratio** 公比 r each time:

$$u_n = u_1 r^{n-1}, \qquad S_n = \frac{u_1(r^n - 1)}{r - 1} = \frac{u_1(1 - r^n)}{1 - r}, \quad r \neq 1$$

On the chessboard, $u_1 = 1$ and $r = 2$, so $S_{64} = \dfrac{2^{64} - 1}{2 - 1} = 2^{64} - 1$.



Adding gives a straight line; multiplying eventually wins

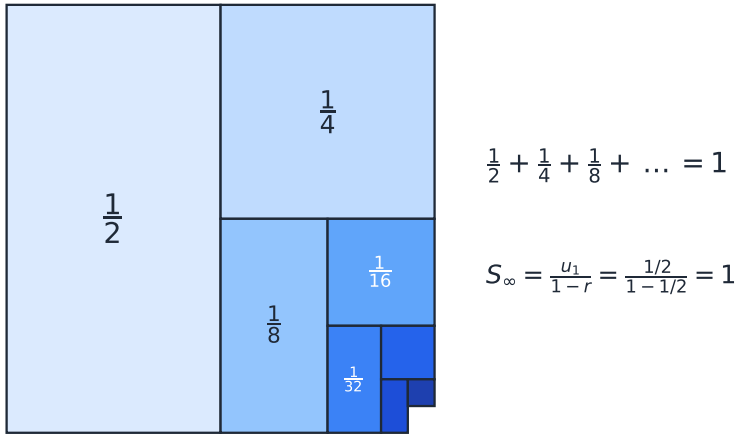
Worked example. For $3, 6, 12, \ldots$, $u_8 = 3 \times 2^7 = 384$ and $S_8 = 3(2^8 - 1) = 765$. For the smallest n with $S_n > 1000$: $3(2^n - 1) > 1000$ gives $2^n > 334.3$, so $n = 9$, since $2^8 = 256$ is too small and $2^9 = 512$ is enough. On a calculator paper you can also solve $n > \frac{\ln 334.3}{\ln 2} \approx 8.39$.

Infinite geometric series

When $|r| < 1$, the terms shrink so quickly that the sum approaches a limit. The series **converges** 收敛 and

$$S_\infty = \frac{u_1}{1 - r}, \qquad |r| < 1$$

If $|r| \geq 1$, there is no finite sum.



Infinitely many pieces, but a finite total

Worked example. A ball is dropped from 2 m, and each bounce reaches 60 % of the previous height. Find the total distance it travels.

- It falls 2 m, then rises and falls $1.2 + 1.2$, then $0.72 + 0.72$, and so on.
- The rebound heights form a geometric series with $u_1 = 1.2$ and $r = 0.6$, so their sum is $\dfrac{1.2}{1 - 0.6} = 3$ m.
- Total distance $= 2 + 2 \times 3 = 8$ m.

Worked example. $0.272727\cdots = 0.27 + 0.0027 + \cdots = \dfrac{0.27}{1 - 0.01} = \dfrac{27}{99} = \dfrac{3}{11}$.

When the ratio contains a variable, the condition $|r| < 1$ gives the values for which the series converges. For $1 + 2x + 4x^2 + \ldots$, $r = 2x$, so it converges for $|x| < \frac{1}{2}$.

Financial applications

Compound interest

With **compound interest** 复利, interest is added to the balance and then earns interest itself, so the balance grows geometrically. A present value PV invested at r % a year, compounded k times a year for n years, becomes



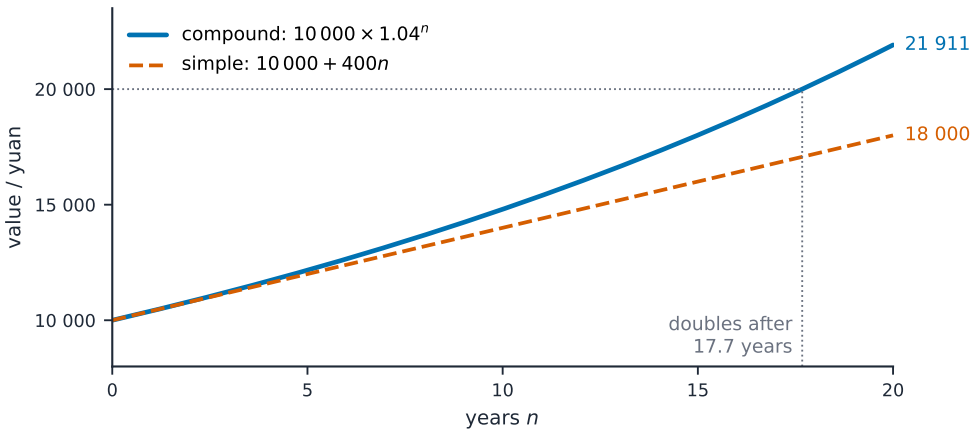
Shop discounts and interest rates are percentages: repeated percentage change is a geometric sequence

$$FV = PV\left(1 + \frac{r}{100k}\right)^{kn}$$

Worked example. ¥10 000 is invested at 4 % a year for 5 years.

- Compounded annually: $10\,000 \times 1.04^5 \approx$ ¥12 166.53.
- Compounded monthly: $10\,000\left(1 + \frac{4}{1200}\right)^{60} \approx$ ¥12 209.97.

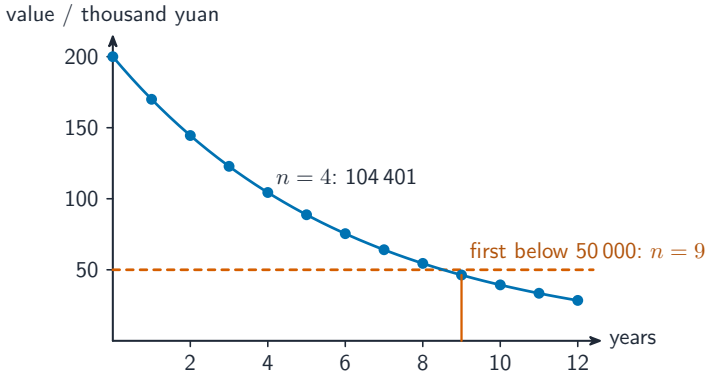
More frequent compounding gives slightly more. In exams you may use the calculator's finance (TVM) solver, but write down the values you enter: N, $I\%$, PV, PMT, FV, and the number of periods per year. Money paid in is entered as negative.



Compound interest is a geometric sequence; simple interest is arithmetic

Depreciation

Depreciation 折旧 is the loss of value over time. At r % a year, the value after n years is $V = V_0\left(1 - \frac{r}{100}\right)^n$.



15% depreciation a year

Worked example. A car bought for ¥200 000 loses 15 % of its value each year.

- After 4 years: $200\,000 \times 0.85^4 \approx$ ¥104 401.
- It is first worth less than ¥50 000 when $0.85^n < 0.25$, that is $n > \dfrac{\ln 0.25}{\ln 0.85} \approx 8.53$: after 9 years.

Worked example. How long does money take to double at 6 % a year, compounded annually? $1.06^n = 2$ gives $n = \dfrac{\ln 2}{\ln 1.06} \approx 11.9$, so 12 years.

Inflation works the same way: an amount of money buys less each year. A real interest rate is approximately the interest rate minus the inflation rate.

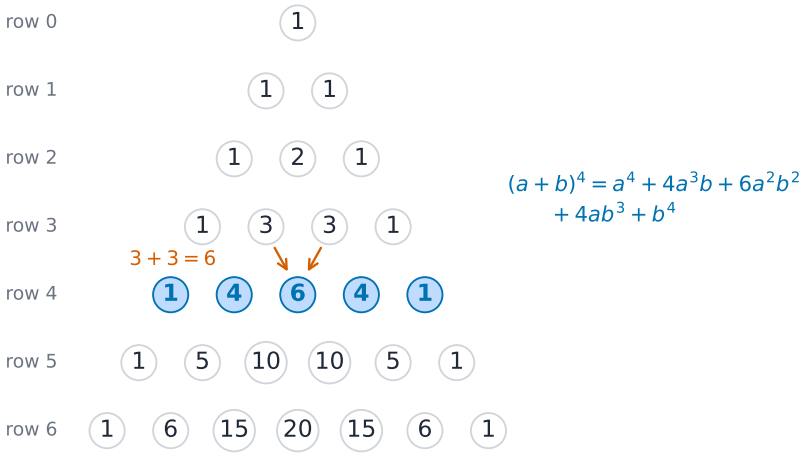
The binomial theorem

Pascal's triangle and binomial coefficients

Expanding $(a+b)^n$ by hand is slow. The **binomial theorem** 二项式定理 gives it directly:

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \cdots + \binom{n}{r}a^{n-r}b^r + \cdots + b^n$$

where the **binomial coefficient** 二项式系数 is $\binom{n}{r} = {}^nC_r = \dfrac{n!}{r!(n-r)!}$ and $n! = n(n-1)\cdots 2\times 1$. The coefficients also appear as the rows of **Pascal's triangle** 杨辉三角, where each number is the sum of the two above it.



Row n gives the coefficients of $(a+b)^n$

Worked example. Expand $(2x-3)^4$. Use row 4 with $a = 2x$ and $b = -3$:

$$(2x)^4 + 4(2x)^3(-3) + 6(2x)^2(-3)^2 + 4(2x)(-3)^3 + (-3)^4 = 16x^4 - 96x^3 + 216x^2 - 216x + 81$$

Put brackets round $2x$ and -3: the signs alternate because b is negative.

Finding one term

The general term is $\binom{n}{r}a^{n-r}b^r$.

Worked example. The coefficient of x^3 in $(1+2x)^7$ is $\binom{7}{3}(2)^3 = 35 \times 8 = 280$.

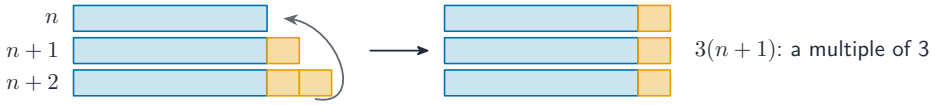
Worked example. Find the term independent of x in $\left(x^2 + \frac{2}{x}\right)^6$.

- General term: $\binom{6}{r}(x^2)^{6-r}\left(\frac{2}{x}\right)^r = \binom{6}{r}2^r x^{12-3r}$.
- Independent of x means $12 - 3r = 0$, so $r = 4$.
- Term: $\binom{6}{4}2^4 = 15 \times 16 = 240$.

Proof

Deductive proof

A **proof** 证明 shows that a statement is true in every case, not just in the examples you tried. A deductive proof starts from facts you know and uses correct algebra to reach the result.



Three consecutive integers make 3(n + 1)

Worked example. Prove that the sum of three consecutive integers is a multiple of 3. Let the integers be n, $n+1$, $n+2$. Their sum is $3n+3 = 3(n+1)$, which is a multiple of 3.

Worked example. Prove that $(n+3)^2 - (n-3)^2 = 12n$.

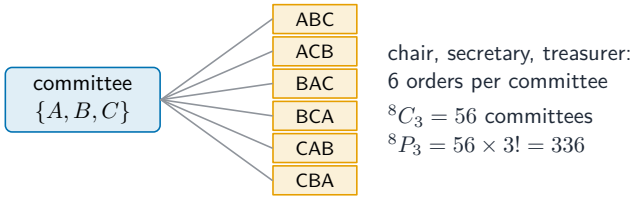
$$\text{LHS} = (n^2 + 6n + 9) - (n^2 - 6n + 9) = 12n = \text{RHS}$$

When proving an identity, start from one side and work to the other, ending with "= RHS". Do not start by writing the statement you want to prove and working on both sides at once. The symbol $\equiv$ means "is identically equal to": true for every value of the variable.

AHL: Counting and the extended binomial theorem

Permutations and combinations

- The number of ways to arrange n different objects in a line is $n!$.
- A **permutation** 排列 is an ordered selection: choosing r from n in order can be done in ${}^nP_r = \frac{n!}{(n-r)!}$ ways.
- A **combination** 组合 is an unordered selection: ${}^nC_r = \frac{n!}{r!(n-r)!}$ ways.



Why ${}^8P_3 = {}^8C_3 \times 3!$

Worked example. From 8 students: a committee of 3 can be chosen in ${}^8C_3 = 56$ ways, but a chair, secretary and treasurer in ${}^8P_3 = 336$ ways, because order now matters.

Worked example. Six people sit in a row, and two of them must sit together. Treat the pair as one block: 5! arrangements of the 5 "objects", times 2 for the order within the block, gives 240. The letters of LEVEL can be arranged in $\frac{5!}{2!\,2!} = 30$ ways, dividing by the orders of the repeated letters.

The extended binomial theorem

For any rational n, not only a positive integer,

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots, \qquad |x| < 1$$

The series is infinite when n is negative or a fraction, and it is valid only when $|x| < 1$.

Worked example. $(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \ldots$ With $x = 0.02$, $\sqrt{1.02} \approx 1 + 0.01 - 0.00005 = 1.00995$, correct to 5 decimal places.

Worked example. For $(2-x)^{-1}$, first take out the 2: $(2-x)^{-1} = \frac{1}{2}\left(1 - \frac{x}{2}\right)^{-1} = \frac{1}{2}\left(1 + \frac{x}{2} + \frac{x^2}{4} + \ldots\right)$, valid for $\left|\frac{x}{2}\right| < 1$, that is $|x| < 2$.

Partial fractions

A fraction with a factorised denominator can be split into **partial fractions** 部分分式. This makes such expressions easier to expand and to integrate (topic 5).

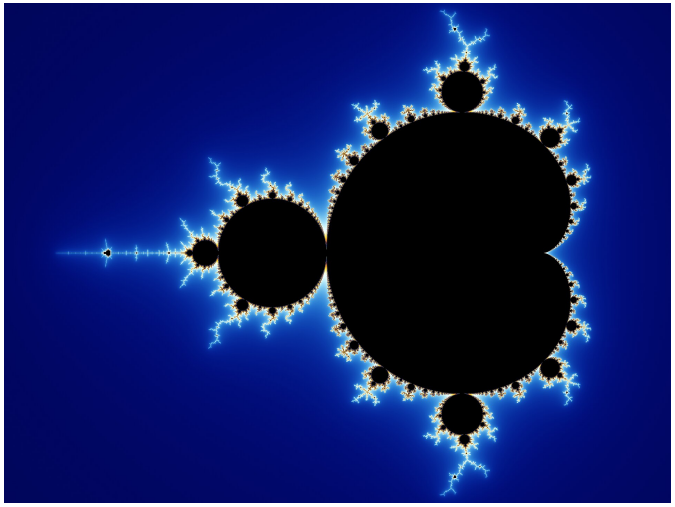
> **Worked example.** Write $\dfrac{5x+1}{(x-1)(x+2)}$ in partial fractions.

- Let $\dfrac{5x+1}{(x-1)(x+2)} = \dfrac{A}{x-1} + \dfrac{B}{x+2}$, so $5x+1 = A(x+2) + B(x-1)$.
- $x = 1$: $6 = 3A$, so $A = 2$. $x = -2$: $-9 = -3B$, so $B = 3$.
- Result: $\dfrac{2}{x-1} + \dfrac{3}{x+2}$.

AHL: Complex numbers

Cartesian form

The equation $x^2 = -1$ has no real solution, so mathematicians defined a new number i with $i^2 = -1$. A **complex number** 复数 is $z = a + bi$, with real part a and imaginary part b. Its **complex conjugate** 共轭复数 is $z^* = a - bi$.



The Mandelbrot set: the complex numbers c for which repeating $z \to z^2 + c$, starting from $z = 0$, stays bounded

- Add and multiply as with brackets, replacing i^2 by -1.
- Divide by multiplying the top and bottom by the conjugate of the denominator.

> **Worked example.** $(3+2i)(1-i) = 3 - 3i + 2i - 2i^2 = 5 - i$, and
>
> $$\frac{3+2i}{1-i} = \frac{(3+2i)(1+i)}{(1-i)(1+i)} = \frac{1+5i}{2}$$

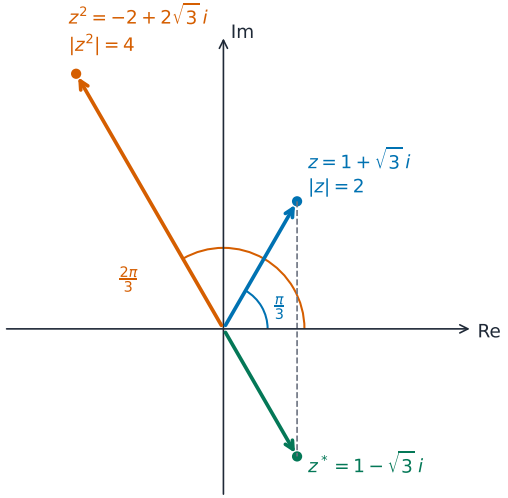
Modulus, argument and polar form

On an **Argand diagram** 阿尔冈图, $a + bi$ is the point (a, b). Its **modulus** 模 is its distance from the origin, $|z| = r = \sqrt{a^2 + b^2}$, and its **argument** 辐角 θ is the angle from the positive real axis. Then

$$z = r(\cos\theta + i\sin\theta) = r\,\mathrm{cis}\,\theta = re^{i\theta}$$

> **Worked example.** $z = 1 + \sqrt{3}\,i$ has $r = 2$ and $\theta = \frac{\pi}{3}$, so $z = 2e^{\frac{\pi}{3}i}$. Always sketch the point first: for $-1 - i$, $\arctan 1 = \frac{\pi}{4}$, but the point is in the third quadrant, so $\theta = -\frac{3\pi}{4}$.

To multiply, multiply the moduli and add the arguments; to divide, divide the moduli and subtract the arguments. So multiplying by $i = e^{\frac{\pi}{2}i}$ rotates a point 90° anticlockwise.



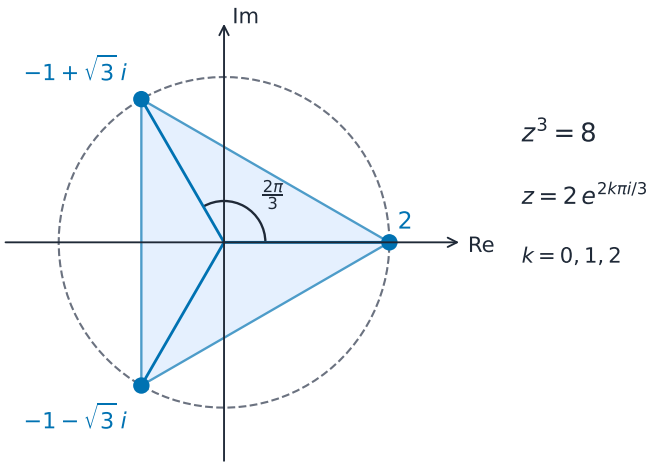
Squaring squares the modulus and doubles the argument

De Moivre's theorem, powers and roots

De Moivre's theorem 棣莫弗定理: $\big(r(\cos\theta + i\sin\theta)\big)^n = r^n(\cos n\theta + i\sin n\theta)$.

> **Worked example.** $(1 + \sqrt{3}\,i)^6 = 2^6\,\mathrm{cis}\,2\pi = 64$.

> **Worked example.** Solve $z^3 = 8$. Write $8 = 8e^{2k\pi i}$, so $z = 2e^{\frac{2k\pi}{3}i}$ for $k = 0, 1, 2$: $z = 2,\ -1 + \sqrt{3}\,i,\ -1 - \sqrt{3}\,i$.



The nth roots are equally spaced round a circle

De Moivre's theorem also proves identities. Expanding $(\cos\theta + i\sin\theta)^3$ and taking the real part gives $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta = 4\cos^3\theta - 3\cos\theta$.

Roots of polynomials

If a polynomial has real coefficients, its non-real roots come in conjugate pairs: if $2+i$ is a root, so is $2-i$.

> **Worked example.** $z^3 - 5z^2 + 9z - 5 = 0$ has the root $z = 1$. Dividing gives $(z-1)(z^2-4z+5) = 0$, and the quadratic formula gives $z = 2 \pm i$, a conjugate pair.

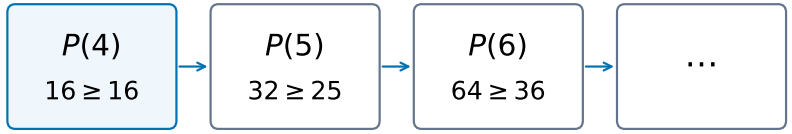
AHL: Proof by induction and contradiction

Proof by induction

Proof by induction 数学归纳法 proves a statement for every positive integer n in two steps, like knocking over a line of dominoes.

Claim: $2^n \geq n^2$ for integers $n \geq 4$

For every $k \geq 4$: $P(k) \Rightarrow P(k+1)$



Base case — Reached by the proved implication

Checked values alone do not justify the arrows.

1. Show it is true for $n = 1$.
2. Assume it is true for $n = k$, and use this to show it is true for $n = k+1$.

Then conclude: since it is true for $n = 1$, and true for $n = k+1$ whenever it is true for $n = k$, it is true for all positive integers n.

> **Worked example.** Prove that $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$.

- $n = 1$: LHS $= 1$, RHS $= \frac{1\times2\times3}{6} = 1$. True.
- Assume $\sum_{r=1}^{k} r^2 = \frac{k(k+1)(2k+1)}{6}$. Then

$$\sum_{r=1}^{k+1} r^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2 = \frac{(k+1)(2k^2+7k+6)}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$$

- This is the formula with $n = k+1$, so the statement is true for all positive integers n.

> **Worked example.** Prove that $7^n - 1$ is divisible by 6. For $n = 1$, $7 - 1 = 6$. If $7^k - 1 = 6m$, then $7^{k+1} - 1 = 7 \times 7^k - 1 = 7(7^k - 1) + 6 = 6(7m+1)$, a multiple of 6.

Contradiction and counterexample

In a **proof by contradiction** 反证法, assume the statement is false and show that this leads to something impossible.

> **Worked example.** Prove that $\sqrt{2}$ is irrational. Suppose $\sqrt{2} = \frac{p}{q}$ in lowest terms. Then $p^2 = 2q^2$, so p^2 is even and p is even: $p = 2s$. Then $4s^2 = 2q^2$, so $q^2 = 2s^2$ and q is even too. Both are even, so $\frac{p}{q}$ was not in lowest terms: a contradiction.

One **counterexample** 反例 is enough to disprove a statement. "$n^2 + n + 41$ is prime for every natural number n" works for $n = 0$ to 39, but $n = 40$ gives $1681 = 41^2$.

AHL: Systems of linear equations

Solving three equations

Up to three linear equations in three unknowns can be solved by elimination, or with the calculator's equation solver. Each equation in x, y, z is a plane, so the system has one solution, infinitely many (the planes share a line or coincide), or none.

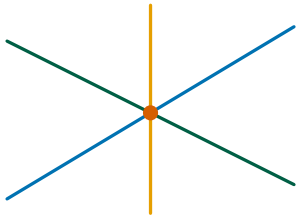
> **Worked example.** Solve $x + y + z = 6$, $2x - y + z = 3$, $x + 2y - z = 2$.

- Add the first and third: $2x + 3y = 8$. Add the second and third: $3x + y = 5$.
- From $y = 5 - 3x$: $2x + 15 - 9x = 8$, so $x = 1$, $y = 2$, and then $z = 3$.

■ No solution or infinitely many

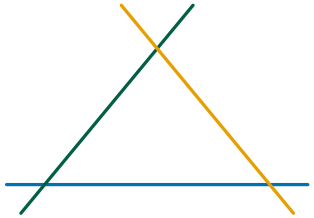
Worked example. For $x+y+z=2$, $x-y+2z=1$ and $2x+3z=k$:

$k=3$: **planes share a line**



the dot is the common line seen end-on: infinitely many solutions, $\mathbf{r}=\ldots+\lambda\,\mathbf{d}$

$k\neq 3$: **a triangular prism**



each pair meets, but no point lies on all three: no solution

Adding the first two equations gives $2x+3z=3$; the third says $2x+3z=k$

Seen end-on, three planes either share a line (infinitely many solutions) or form a prism (no solution)

- Adding the first two equations gives $2x+3z=3$.
- If $k\neq 3$, the third equation contradicts this: no solution.
- If $k=3$, the third equation adds nothing new: infinitely many solutions. Let $z=\lambda$; then $x=1.5-1.5\lambda$ and $y=0.5+0.5\lambda$.

The general solution with a parameter λ describes the line where the planes meet.

Common mistakes.

- Multiplying the indices instead of adding them: $2^3\times 2^4=2^7$, not 2^{12}.
- Writing $\log(a+b)=\log a+\log b$. The product law needs a product: $\log(ab)=\log a+\log b$.
- Using $u_n=u_1+nd$. The first term already counts once, so $u_n=u_1+(n-1)d$.
- Using $S_\infty=\dfrac{u_1}{1-r}$ when $|r|\geq 1$. Then the series has no sum to infinity.
- Entering an annual rate as if it were monthly: 6% a year compounded monthly is 0.5% a month for $12n$ months.
- Losing a sign in a binomial term: the x^3 term of $(2x-3)^5$ is $\binom{5}{2}(2x)^3(-3)^2=720x^3$.
- Giving a decimal for the smallest n: if $n>14.2$, the answer is $n=15$.
- **HL:** using $\binom{n}{r}$ when order matters. Arrangements need ${}^nP_r$.
- **HL:** expanding $(1+2x)^{-1}$ without its condition. The series is valid only for $|x|<\frac{1}{2}$.
- **HL:** taking $\arg(-1-\mathrm{i})$ from the calculator's $\arctan 1=\frac{\pi}{4}$. The point is in the third quadrant, so the argument is $-\frac{3\pi}{4}$.
- **HL:** assuming the $n=k+1$ statement inside its own induction proof.

Practice

1. Write 0.000 045 2 and 3 600 000 in standard form, and find $(6\times 10^5)\div(1.5\times 10^{-3})$ in standard form.
2. Simplify $\dfrac{(3x^2y)^3}{9x^4y}$.
3. Find the exact values of $16^{\frac{3}{4}}$ and $9^{-\frac{1}{2}}$, and solve $9^x=27^{x-1}$.
4. Find $\log_2 64$, $\log_5\frac{1}{125}$ and $\log 1000$. Write $2\log 3+\log 4-\log 6$ as a single logarithm.
5. Solve $2^x=15$ and $3^{x+2}=5^x$, giving answers to 3 significant figures.
6. Solve $\log_3(x+1)+\log_3(x-1)=1$.
7. In an arithmetic sequence, $u_3=11$ and $u_8=31$. Find u_1, d, u_{20} and S_{20}.
8. Evaluate $\displaystyle\sum_{k=1}^{30}(2k-1)$.
9. A salary starts at ¥60 000 and rises by ¥2400 each year. Find the total earned over 15 years.
10. In a geometric sequence, $u_2=12$ and $u_5=96$. Find r, u_1 and S_{10}.
11. Find the smallest n for which the sum of $5+10+20+\ldots$ to n terms exceeds 10 000.
12. Find the sum to infinity of $24+18+13.5+\ldots$, and write $0.181818\ldots$ as a fraction.
13. For which values of x does $1+3x+9x^2+\ldots$ converge? Find its sum when $x=0.2$.
14. ¥5000 is invested at 3.6 % a year, compounded quarterly. Find its value after 8 years, and the number of complete quarters until it first exceeds ¥8000.
15. Expand $(1-2x)^5$.
16. Find the coefficient of x^4 in $(2x+3)^6$, and the term independent of x in $\left(x-\frac{2}{x^2}\right)^6$.
17. Prove that $(n+1)^2-(n-1)^2=4n$, and that the sum of any two consecutive odd numbers is a multiple of 4.
18. **AHL.** Find the number of ways to (a) arrange the letters of ORANGE; (b) choose 4 people from 10; (c) arrange the letters of BANANA; (d) seat 6 people in a row if two of them must not sit together.
19. **AHL.** Expand $(1+2x)^{-2}$ up to the x^3 term and state the values of x for which it is valid.
20. **AHL.** Write $\dfrac{x+7}{(x-1)(x+3)}$ in partial fractions.
21. **AHL.** $z=2-3i$ and $w=1+2i$. Find zw, $\dfrac{z}{w}$ and $|z|$.
22. **AHL.** Write $z=-2+2i$ in the form $re^{i\theta}$, and find z^4.
23. **AHL.** Solve $z^3=-27$, giving the roots in Cartesian form.
24. **AHL.** $1+2i$ is a root of $z^3-3z^2+7z-5=0$. Find the other roots.
25. **AHL.** Use De Moivre's theorem to show that $\sin 3\theta=3\sin\theta-4\sin^3\theta$.

26. **AHL.** Prove by induction that $\displaystyle\sum_{r=1}^{n} r(r+1) = \frac{n(n+1)(n+2)}{3}$.
27. **AHL.** Prove by induction that $5^n - 1$ is divisible by 4 for every positive integer n.
28. **AHL.** Prove by contradiction that $\log_2 3$ is irrational, and give a counterexample to "$2^n + 1$ is prime for every positive integer n".
29. **AHL.** Solve $x + 2y - z = 3$, $2x - y + z = 4$, $3x + y + 2z = 9$.
30. **AHL.** The system $x + y - z = 1$, $2x + 3y + z = 4$, $3x + 4y + kz = 5$ does not have a unique solution. Find k and the general solution.

Answers.

1. 4.52×10^{-5}; 3.6×10^{6}; 4×10^{8}.
2. $\dfrac{27x^6y^3}{9x^4y} = 3x^2y^2$.
3. $16^{\frac{3}{4}} = 2^3 = 8$; $9^{-\frac{1}{2}} = \frac{1}{3}$. $3^{2x} = 3^{3x-3}$, so $x = 3$.
4. 6, -3, 3. $\log \dfrac{9 \times 4}{6} = \log 6$.
5. $x = \dfrac{\ln 15}{\ln 2} \approx 3.91$. $(x+2)\ln 3 = x \ln 5$, so $x = \dfrac{2\ln 3}{\ln 5 - \ln 3} \approx 4.30$.
6. $\log_3(x^2 - 1) = 1$, so $x^2 - 1 = 3$ and $x = \pm 2$. Reject $x = -2$, since $\log_3(-1)$ is undefined: $x = 2$.
7. $5d = 20$, so $d = 4$ and $u_1 = 3$. $u_{20} = 3 + 19(4) = 79$; $S_{20} = 10(3 + 79) = 820$.
8. The first 30 odd numbers: $\frac{30}{2}(1 + 59) = 900$.
9. $S_{15} = \frac{15}{2}\big(2(60\,000) + 14(2400)\big) = 7.5 \times 153\,600 = ¥1\ 152\ 000$.
10. $r^3 = 8$, so $r = 2$ and $u_1 = 6$. $S_{10} = 6(2^{10} - 1) = 6138$.
11. $5(2^n - 1) > 10\,000$ gives $2^n > 2001$, so $n = 11$ ($2^{10} = 1024$, $2^{11} = 2048$).
12. $r = 0.75$, so $S_\infty = \dfrac{24}{0.25} = 96$. $0.181818\cdots = \dfrac{0.18}{0.99} = \dfrac{2}{11}$.
13. $r = 3x$, so $|x| < \frac{1}{3}$. When $x = 0.2$, $S_\infty = \dfrac{1}{1 - 0.6} = 2.5$.
14. $5000(1.009)^{32} \approx ¥6660.20$. $1.009^q > 1.6$ gives $q > \dfrac{\ln 1.6}{\ln 1.009} \approx 52.5$: 53 quarters.
15. $1 - 10x + 40x^2 - 80x^3 + 80x^4 - 32x^5$.
16. $\binom{6}{4}2^4 3^2 = 15 \times 16 \times 9 = 2160$. General term $\binom{6}{r}(-2)^r x^{6-3r}$; $r = 2$ gives $15 \times 4 = 60$.
17. $(n^2 + 2n + 1) - (n^2 - 2n + 1) = 4n$. $(2n - 1) + (2n + 1) = 4n$, a multiple of 4.
18. **AHL.** (a) $6! = 720$; (b) ${}^{10}C_4 = 210$; (c) $\dfrac{6!}{3!\,2!} = 60$; (d) $720 - 2 \times 5! = 480$.
19. **AHL.** $1 - 4x + 12x^2 - 32x^3$, valid for $|2x| < 1$, that is $|x| < \frac{1}{2}$.
20. **AHL.** $x + 7 = A(x+3) + B(x-1)$: $x = 1$ gives $A = 2$; $x = -3$ gives $B = -1$. So $\dfrac{2}{x-1} - \dfrac{1}{x+3}$.
21. **AHL.** $zw = 2 + 4i - 3i + 6 = 8 + i$. $\dfrac{z}{w} = \dfrac{(2-3i)(1-2i)}{5} = \dfrac{-4-7i}{5}$. $|z| = \sqrt{13}$.
22. **AHL.** $r = 2\sqrt{2}$, $\theta = \frac{3\pi}{4}$: $z = 2\sqrt{2}\,e^{\frac{3\pi}{4}i}$. $z^4 = 64e^{3\pi i} = -64$.
23. **AHL.** $-27 = 27e^{(\pi + 2k\pi)i}$, so $z = 3e^{\frac{(2k+1)\pi}{3}i}$: $z = \frac{3}{2} + \frac{3\sqrt{3}}{2}i$, -3, $\frac{3}{2} - \frac{3\sqrt{3}}{2}i$.
24. **AHL.** $1 - 2i$ is also a root, giving the factor $z^2 - 2z + 5$. $z^3 - 3z^2 + 7z - 5 = (z-1)(z^2 - 2z + 5)$, so the roots are $1 \pm 2i$ and 1.
25. **AHL.** The imaginary part of $(\cos\theta + i\sin\theta)^3$ is $3\cos^2\theta\sin\theta - \sin^3\theta = 3(1 - \sin^2\theta)\sin\theta - \sin^3\theta = 3\sin\theta - 4\sin^3\theta$, and by De Moivre it equals $\sin 3\theta$.
26. **AHL.** $n = 1$: $2 = \frac{1\times2\times3}{3}$. Assuming it for k: $\frac{k(k+1)(k+2)}{3} + (k+1)(k+2) = \frac{(k+1)(k+2)(k+3)}{3}$, the formula for $k+1$; so it holds for all positive integers n.
27. **AHL.** $n = 1$: $5 - 1 = 4$. If $5^k - 1 = 4m$, then $5^{k+1} - 1 = 5(5^k - 1) + 4 = 4(5m + 1)$; so it holds for all positive integers n.
28. **AHL.** If $\log_2 3 = \frac{p}{q}$ with p, q positive integers, then $2^p = 3^q$; the left side is even and the right side is odd, a contradiction. Counterexample: $n = 3$ gives $9 = 3 \times 3$.
29. **AHL.** $x = 2$, $y = 1$, $z = 1$.
30. **AHL.** The third equation must be the sum of the first two, so $k = 0$. Let $z = \lambda$: $x = -1 + 4\lambda$, $y = 2 - 3\lambda$, $z = \lambda$.

Exam tips

- Give final answers in standard form when asked, with $1 \le a < 10$.
- Write numbers as powers of the same base before equating exponents; otherwise take logarithms.
- In sequence questions, first decide: does it add (arithmetic) or multiply (geometric)?
- State $|r| < 1$ before using the sum to infinity.
- For "smallest n" questions, give an integer and check the values either side.
- For finance questions, write down every value you enter in the TVM solver, and round money to 2 decimal places only at the end.
- In binomial expansions, bracket each term, including its sign: $(2x)^4$, $(-3)^2$.
- **HL:** in an induction proof, show the base case, state the assumption, prove the $k+1$ case and write the conclusion.
- **HL:** sketch a complex number before finding its argument, so you choose the correct quadrant.

Functions

International Baccalaureate · IB Diploma · Mathematics: Analysis and Approaches · HL

Scope and prerequisites

This handout teaches topic 2, Functions, of Mathematics: Analysis and Approaches, the course first assessed in 2021: straight lines, the language of functions, composite and inverse functions, quadratic, rational, exponential and logarithmic functions, transformations of graphs, and, for HL, polynomials, odd and even functions and further graphs. It is written for both SL and HL students. HL students also apply the further results in the sections marked AHL.

Prerequisites: algebraic manipulation, solving linear and simultaneous equations, indices and surds (topic 1).

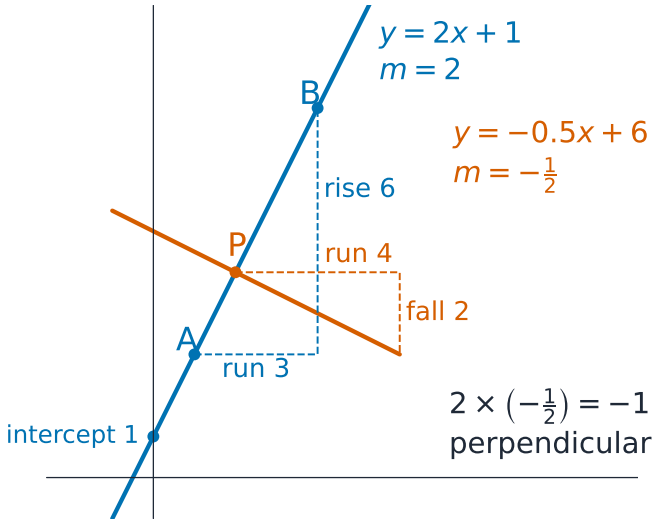
Lines and functions

A taxi fare as a function

A city taxi charges a fixed ¥14 for the first 3 km and ¥2.50 for each extra kilometre. The fare F depends on the distance d: for every distance there is exactly one fare. That is what makes it a **function** 函数. For $d \geq 3$ the rule is a straight line, $F = 14 + 2.5(d-3)$, so a 10 km trip costs $14 + 2.5 \times 7 = 31.5$ yuan. The graph is flat up to 3 km and then rises with gradient 2.5. Much of this topic is about reading a rule like this, drawing its graph and turning questions about the situation into questions about the graph.

Straight lines

The **gradient** 斜率 of the line through (x_1, y_1) and (x_2, y_2) is



Perpendicular gradients multiply to -1

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

A line can be written in three forms:

- gradient-intercept form $y = mx + c$, where c is the y-intercept;
- general form $ax + by + d = 0$;
- point-gradient form $y - y_1 = m(x - x_1)$.

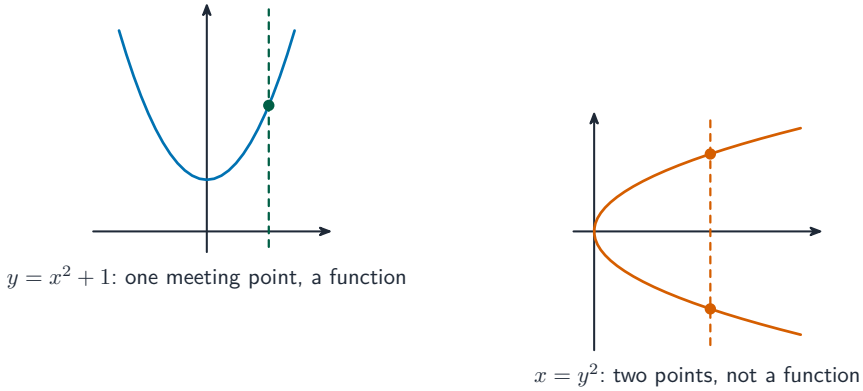
Parallel lines have equal gradients. Perpendicular lines have gradients whose product is -1, so $m_2 = -\dfrac{1}{m_1}$.

> **Worked example.** Find the line through $A(1, 5)$ perpendicular to $y = 2x - 3$.

- The given gradient is 2, so the perpendicular gradient is $-\frac{1}{2}$.
- $y - 5 = -\frac{1}{2}(x - 1)$, so $y = -\frac{1}{2}x + \frac{11}{2}$, or $x + 2y - 11 = 0$.

The language of functions

A function maps each input x in its **domain** 定义域 to exactly one output $f(x)$. The set of outputs is the **range** 值域. We write $f(x) = x^2 + 1$ or $f : x \mapsto x^2 + 1$.



The vertical line test

- A graph shows a function if every vertical line meets it at most once.
- A formula alone does not define a function: the domain must be stated or understood. $f(x) = \sqrt{x-2}$ needs $x \geq 2$; $g(x) = \dfrac{1}{x-3}$ needs $x \neq 3$.
- The range follows from the domain and the graph: for $f(x) = x^2 + 1$ with $x \in \mathbb{R}$, the range is $f(x) \geq 1$.

On calculator papers, use the graphing calculator to sketch a graph, find intercepts, maximum and minimum points and intersections. Draw a sketch with a labelled scale, and state coordinates to three significant figures unless they are exact.

Composite and inverse functions

Composite functions

A **composite function** 复合函数 applies one function to the output of another:



Nesting represents the order: the inner function acts before the outer function.

$$(f \circ g)(x) = f(g(x))$$

Apply g first, then f. In general $f \circ g \neq g \circ f$.

> **Worked example.** $f(x) = 2x + 1$ and $g(x) = x^2$.

- $(f \circ g)(x) = f(x^2) = 2x^2 + 1$, so $(f \circ g)(3) = 19$.
- $(g \circ f)(x) = g(2x + 1) = (2x + 1)^2$, so $(g \circ f)(3) = 49$.

The output of g must lie in the domain of f: with $f(x) = \sqrt{x}$ and $g(x) = x - 5$, $(f \circ g)(x) = \sqrt{x-5}$ needs $x \geq 5$.

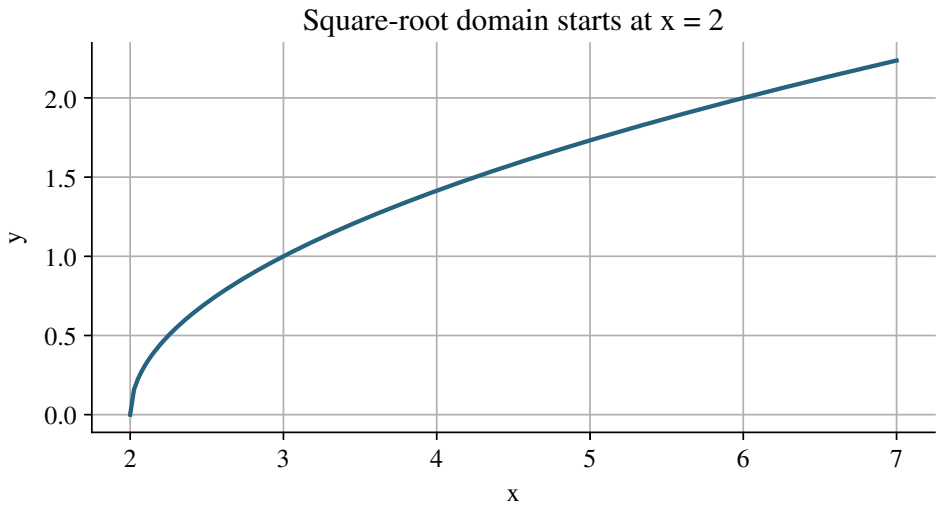
Inverse functions

The **inverse function** 反函数 f^{-1} undoes f: if $f(a) = b$ then $f^{-1}(b) = a$, and

$$(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$$

An inverse exists only if f is **one-to-one**: each output comes from exactly one input, so every horizontal line meets the graph at most once. A quadratic on all real numbers is many-to-one ($f(1) = f(-1)$ for x^2), so its domain must be restricted first. The graph of $y = f^{-1}(x)$ is the reflection of $y = f(x)$ in the line $y = x$, and the domain of f^{-1} is the range of f.

To find an inverse: write $y = f(x)$, swap x and y, and make y the subject.



$y = \sqrt{(x-2)}$ exists only for $x \geq 2$, so its domain starts at 2 and its range starts at 0

Worked example. $f(x) = \sqrt{x-2}$, $x \geq 2$.

- The range is $f(x) \geq 0$.
- Swap: $x = \sqrt{y-2}$, so $x^2 = y - 2$ and $y = x^2 + 2$.
- $f^{-1}(x) = x^2 + 2$ with domain $x \geq 0$ (the range of f). Without the restriction $x \geq 0$, $x^2 + 2$ would not be the inverse.

Quadratic functions

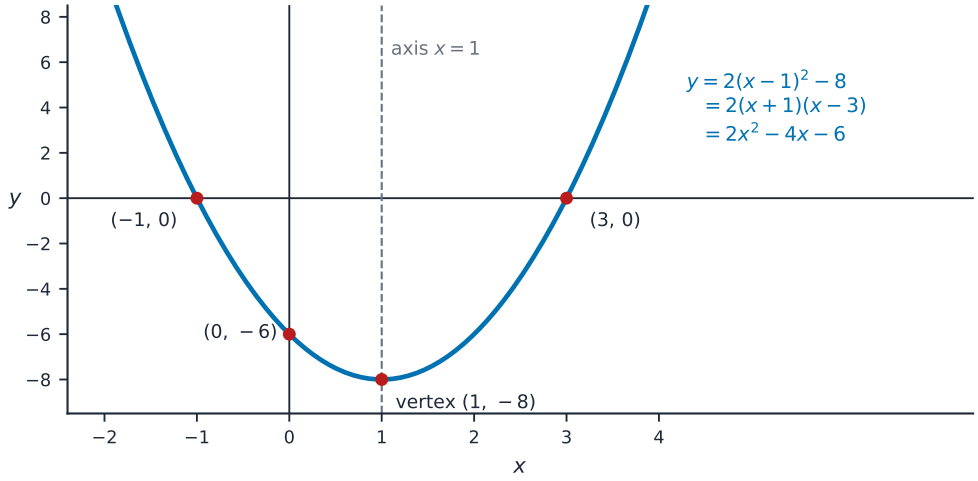
Three forms

A **quadratic function** 二次函数 $f(x) = ax^2 + bx + c$ with $a \neq 0$ has a parabola as its graph, opening upwards if $a > 0$ and downwards if $a < 0$. Each form shows one feature directly:



A suspension bridge cable hangs in a parabola – the graph of a quadratic.

Form	Shows
$y = ax^2 + bx + c$	y-intercept c; axis of symmetry $x = -\dfrac{b}{2a}$
$y = a(x-p)(x-q)$	x-intercepts p and q
$y = a(x-h)^2 + k$	**vertex** 顶点 (h, k); axis $x = h$



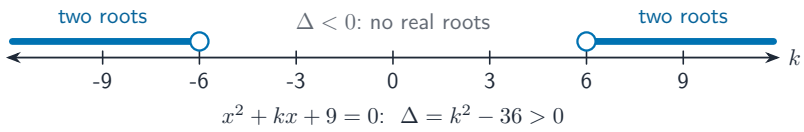
One parabola, three forms

Worked example. Write $y = 2x^2 - 4x - 6$ in vertex form and factorised form.

- Complete the square: $2(x^2-2x)-6 = 2[(x-1)^2-1]-6 = 2(x-1)^2-8$. Vertex $(1,-8)$.
- Factorise: $2(x^2-2x-3) = 2(x+1)(x-3)$. Roots $x=-1$ and $x=3$.
- The range is $y \geq -8$.

The discriminant

The solutions of $ax^2+bx+c=0$ are



Values of k giving two distinct roots

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

The **discriminant** 判别式 $\Delta = b^2-4ac$ tells you how many real roots there are:

- $\Delta > 0$: two distinct real roots (the graph crosses the x-axis twice);
- $\Delta = 0$: one repeated real root (the graph touches the axis);
- $\Delta < 0$: no real roots.

Worked example. For which k does $x^2+kx+9=0$ have two distinct real roots?

- $\Delta = k^2-36 > 0$, so $k^2 > 36$.
- $k < -6$ or $k > 6$.

The same idea finds when a line meets a curve: substitute the line into the curve and use the discriminant of the resulting quadratic. A line is a **tangent** when $\Delta = 0$.

Quadratic inequalities

To solve $x^2-x-6<0$: factorise $(x-3)(x+2)<0$, sketch the parabola with roots -2 and 3, and read off where it is below the axis: $-2<x<3$. Always sketch: dividing both sides by an expression that might be negative reverses the inequality and causes errors.

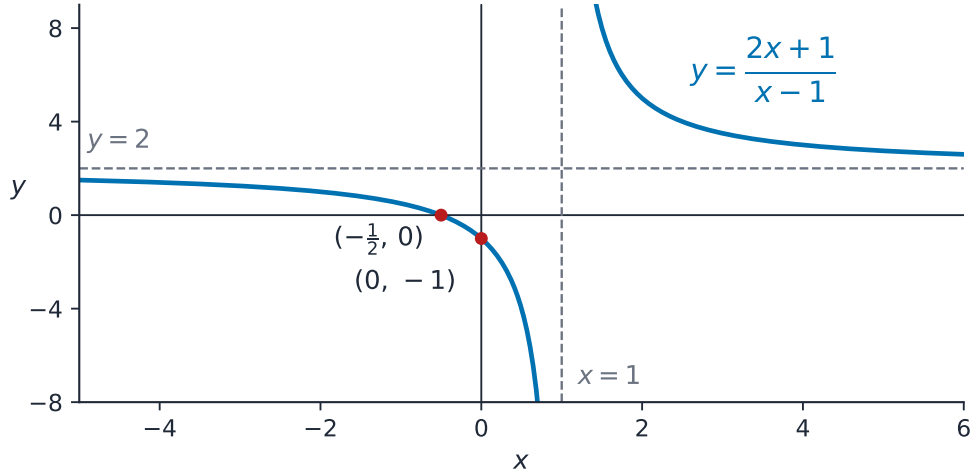
Rational, exponential and logarithmic functions

Reciprocal and rational functions

The **reciprocal function** $f(x)=\dfrac{1}{x}$, $x \neq 0$, has the axes as asymptotes. It is **self-inverse**: $f^{-1}(x)=f(x)$, because its graph is symmetric in $y=x$.

For $c \neq 0$ and $ad-bc \neq 0$, a rational function $f(x)=\dfrac{ax+b}{cx+d}$ has:

- a vertical **asymptote** 渐近线 where the denominator is zero, $x=-\dfrac{d}{c}$;
- a horizontal asymptote $y=\dfrac{a}{c}$, the value approached as $x \to \pm\infty$;
- an x-intercept where the numerator is zero, and, when $d \neq 0$, a y-intercept at $\dfrac{b}{d}$.



Asymptotes first, then intercepts, then the two branches

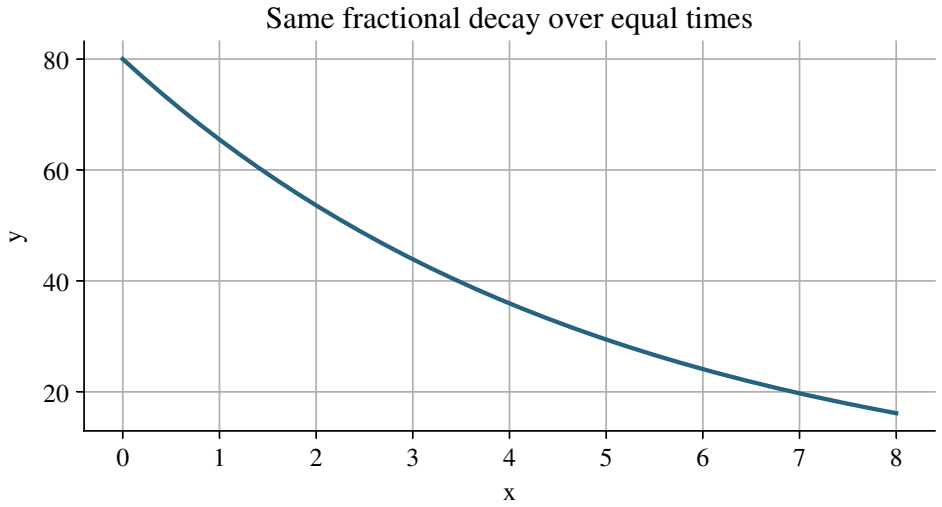
Worked example. $f(x)=\dfrac{2x+1}{x-1}$. Vertical asymptote $x=1$; horizontal asymptote $y=2$; x-intercept $-\frac{1}{2}$; y-intercept -1. The domain is $x \neq 1$ and the range is $y \neq 2$.

Exponential functions

An **exponential function** 指数函数 $f(x)=a^x$ with $a>0$, $a \neq 1$, has domain all real numbers, range $f(x)>0$ and the horizontal asymptote $y=0$. The most important base is $e \approx 2.718$. In a model $y=Ae^{kt}$, A is the initial value; $k>0$ gives growth and $k<0$ gives decay. The quantity changes by the same percentage in each equal time step, so it never becomes negative.



Populations can grow exponentially when resources are plentiful.



y = 80e^(−0.2x) loses the same fraction, about 18%, in every unit of x

Logarithms

The **logarithm** 对数 is the inverse of the exponential:

$$a^x = b \iff x = \log_a b \qquad (a > 0,\ a \neq 1,\ b > 0)$$

$\ln x$ means $\log_e x$, so $y = \ln x$ is the reflection of $y = e^x$ in $y = x$. The laws of logarithms:

$$\log_a(xy) = \log_a x + \log_a y \qquad \log_a \frac{x}{y} = \log_a x - \log_a y \qquad \log_a x^n = n \log_a x$$

and the change of base $\log_a x = \dfrac{\ln x}{\ln a}$. Logarithms need positive arguments, and $\log(x + y)$ is not $\log x + \log y$.

> **Worked example.** A medicine's concentration is $C = 80e^{-0.2t}$ mg/L after t hours. When does it fall to 40 mg/L?

- $80e^{-0.2t} = 40$, so $e^{-0.2t} = \frac{1}{2}$.
- $-0.2t = \ln \frac{1}{2} = -\ln 2$, so $t = \dfrac{\ln 2}{0.2} \approx 3.47$ hours. This is the half-life.

> **Worked example.** Solve $3^{x+1} = 5^x$.

- Take logs: $(x + 1)\ln 3 = x \ln 5$.
- $x(\ln 5 - \ln 3) = \ln 3$, so $x = \dfrac{\ln 3}{\ln 5 - \ln 3} \approx 2.15$.

Transformations of graphs

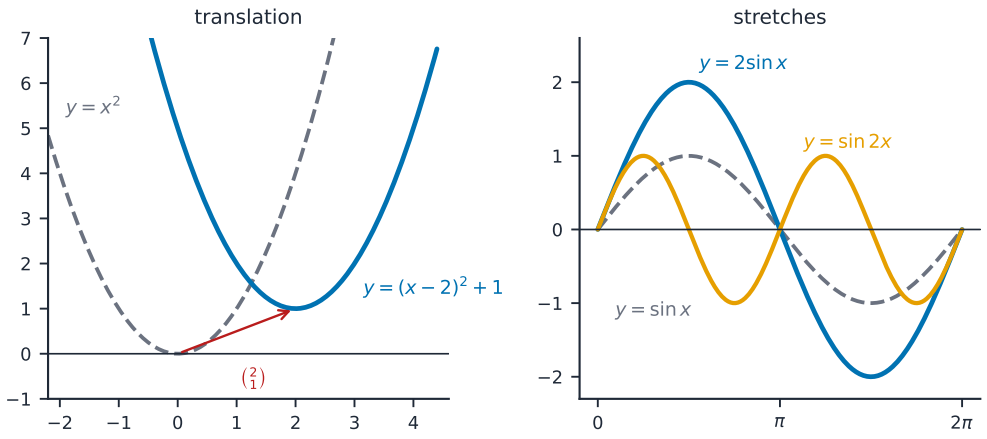
Translations, stretches and reflections

Starting from $y = f(x)$:

New graph	Transformation
$y = f(x) + b$	translation by b units up
$y = f(x - a)$	translation by a units **right**
$y = pf(x)$	vertical stretch, scale factor p
$y = f(qx)$	horizontal stretch, scale factor $\dfrac{1}{q}$
$y = -f(x)$	reflection in the x-axis
$y = f(-x)$	reflection in the y-axis

Changes **outside** the function act on y in the way you expect. Changes **inside** the function act on x in the opposite way: $f(x - 2)$ moves the graph 2 to the right, and $f(2x)$ squashes it horizontally by a factor of $\frac{1}{2}$.

exercises.lol exercises.com.cn



Outside changes act on y; inside changes act on x in the opposite way

Combining transformations

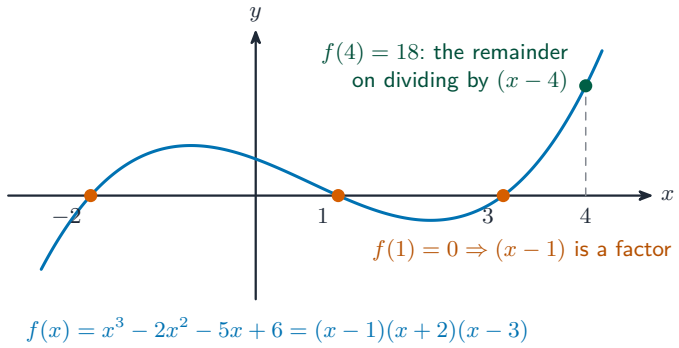
When several transformations are combined, the order matters. A safe method is to follow a point. To get $y = 3f(x-1)+2$ from $y = f(x)$: translate 1 right, stretch vertically by factor 3, then translate 2 up. The point $(4,5)$ on $y = f(x)$ goes to $(5,5)$, then $(5,15)$, then $(5,17)$.

> **Worked example.** Describe how $y = (x-2)^2+1$ is obtained from $y = x^2$, and find its vertex. A translation by the vector $\binom{2}{1}$; the vertex moves from $(0,0)$ to $(2,1)$.

AHL: Polynomials

Factor and remainder theorems

A **polynomial** 多项式 of degree n is $P(x) = a_n x^n + \cdots + a_1 x + a_0$ with $a_n \neq 0$. A polynomial of degree n has at most n real roots, and its graph has at most $n-1$ turning points.



Roots are factors; the height at $x = b$ is the remainder on dividing by $(x-b)$

- **Remainder theorem** 余数定理: when $P(x)$ is divided by $(x-a)$, the remainder is $P(a)$.
- **Factor theorem** 因式定理: $(x-a)$ is a factor of $P(x)$ if and only if $P(a) = 0$.

> **Worked example.** $P(x) = x^3 - 2x^2 - 5x + 6$.

- $P(1) = 1 - 2 - 5 + 6 = 0$, so $(x-1)$ is a factor.
- Divide: $P(x) = (x-1)(x^2 - x - 6) = (x-1)(x-3)(x+2)$.
- Roots 1, 3 and -2. Since the leading coefficient is positive, the graph starts low on the left and ends high on the right.

> **Worked example.** When $x^3 + kx^2 - 4$ is divided by $(x+2)$ the remainder is 4. Find k: $(-2)^3 + 4k - 4 = 4$, so $4k = 16$ and $k = 4$.

Sum and product of roots

For $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$, including complex roots:

$$\text{sum of roots} = -\frac{a_{n-1}}{a_n} \qquad \text{product of roots} = \frac{(-1)^n a_0}{a_n}$$

For the cubic above, the sum is $1+3-2 = 2 = -\frac{-2}{1}$ and the product is $1\times 3\times(-2) = -6 = \frac{(-1)^3\times 6}{1}$.

AHL: Further functions and graphs

Odd and even functions

- An **even function** 偶函数 satisfies $f(-x) = f(x)$; its graph is symmetric in the y-axis (x^2, $\cos x$).
- An **odd function** 奇函数 satisfies $f(-x) = -f(x)$; its graph has rotational symmetry of order 2 about the origin (x^3, $\sin x$).

Most functions are neither. A function is self-inverse if $f(f(x)) = x$, such as $\dfrac{1}{x}$ and $\dfrac{x+3}{2x-1}$ (check: the numerator coefficient of x equals minus the constant in the denominator).

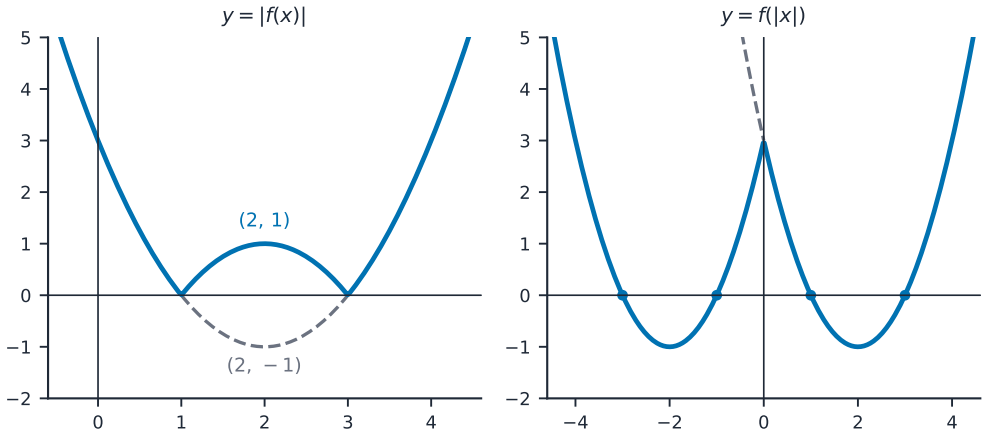
Rational functions with quadratic parts

For $f(x) = \dfrac{ax+b}{cx^2+dx+e}$ the horizontal asymptote is $y = 0$, and there are vertical asymptotes at real roots of the denominator that remain after cancelling common factors. A cancelled factor can instead leave a hole. For $f(x) = \dfrac{ax^2+bx+c}{dx+e}$, divide to find an **oblique asymptote**:

$$\frac{x^2+3x+1}{x+1} = x+2-\frac{1}{x+1}$$

so the oblique asymptote is $y = x+2$, and the vertical asymptote is $x = -1$.

Graphs of related functions

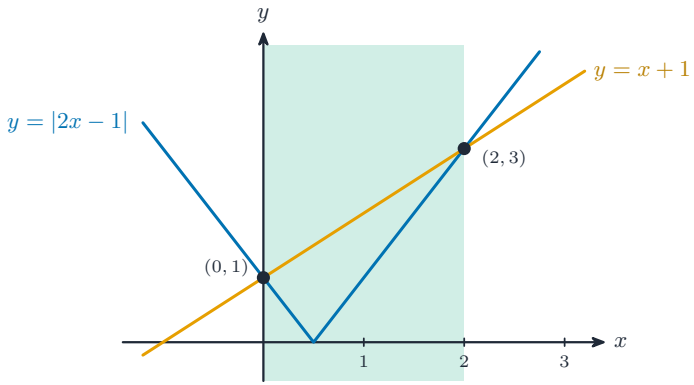


Modulus graphs

- $y = |f(x)|$: reflect the parts of $y = f(x)$ below the x-axis in the x-axis.
- $y = f(|x|)$: keep the part for $x \geq 0$ and reflect it in the y-axis.
- $y = \dfrac{1}{f(x)}$: roots of f become vertical asymptotes; a local maximum at height $k \neq 0$ becomes a local minimum at height $\dfrac{1}{k}$; where $f(x) = \pm 1$ the graphs meet; the sign is unchanged.
- $y = [f(x)]^2$: never negative; roots of f become points where the graph touches the axis.

Solving inequalities

To solve $f(x) > g(x)$, find where the graphs intersect, sketch both, and read off where $y = f(x)$ is above $y = g(x)$.



V below the line: $0 < x < 2$

Find the intersections, then read off where the V lies below the line: $0 < x < 2$

Worked example. Solve $|2x-1| < x+1$.

- Intersections: $2x-1 = x+1$ gives $x = 2$; $-(2x-1) = x+1$ gives $x = 0$.
- A sketch shows the V-shaped graph below the line between the intersections, so $0 < x < 2$.

Common mistakes.

- Reading $f^{-1}(x)$ as $\dfrac{1}{f(x)}$. It is the inverse function, which undoes f.
- Composing in the wrong order: $(f \circ g)(x) = f(g(x))$ applies g first.
- Moving $y = f(x-3)$ three units left. It moves the graph three units **right**; $y = f(2x)$ halves horizontal distances.
- Solving $x^2 < 9$ as $x < \pm 3$. The solution is $-3 < x < 3$.
- Using $\Delta > 0$ for real roots. Real roots need $\Delta \geq 0$; two **distinct** real roots need $\Delta > 0$.
- Forgetting a domain: $\ln(x-2)$ exists only for $x > 2$, so check every root against it.
- Swapping the asymptotes of $y = \dfrac{ax+b}{cx+d}$: the vertical one is $x = -\dfrac{d}{c}$, the horizontal one $y = \dfrac{a}{c}$.
- **HL:** dividing $P(x)$ by $2x-1$ and evaluating $P(1)$ or $P(-\frac{1}{2})$. The remainder is $P(\frac{1}{2})$.

- **HL:** sign errors in the roots of $a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0$: the sum is $-\dfrac{a_{n-1}}{a_n}$ and the product $(-1)^n\dfrac{a_0}{a_n}$.
- **HL:** multiplying $\dfrac{1}{x-1} > 2$ by $x-1$. Its sign is unknown, so multiply by $(x-1)^2$ or use a sign diagram.
- **HL:** confusing $y = |f(x)|$ (reflect the parts below the x-axis) with $y = f(|x|)$ (reflect the right half in the y-axis).

Practice

1. Find the equation of the line through $(2,-1)$ parallel to $3x+y=7$, in the form $ax+by+d=0$.
2. Find the line through $(4,3)$ perpendicular to the line through $(0,1)$ and $(2,5)$.
3. State the largest possible domain and the corresponding range of $f(x)=\sqrt{6-2x}$.
4. $f(x)=3x-2$ and $g(x)=x^2+1$. Find $(f\circ g)(x)$, $(g\circ f)(x)$ and $(g\circ f)(-1)$.
5. Find $f^{-1}(x)$ for $f(x)=\dfrac{x+4}{3}$ and verify that $(f\circ f^{-1})(x)=x$.
6. $h(x)=(x-1)^2$, $x\geq 1$. Find $h^{-1}(x)$ and state its domain. Explain why the restriction $x\geq 1$ is needed.
7. Write $y=x^2+6x+5$ in vertex form and factorised form, and state the vertex, roots and range.
8. Find the values of k for which $2x^2-kx+8=0$ has equal roots.
9. Find the values of c for which the line $y=2x+c$ is a tangent to $y=x^2+3$.
10. Solve $2x^2-5x-3\geq 0$.
11. For $f(x)=\dfrac{3x-6}{x+2}$, state the asymptotes and intercepts, and the range.
12. Solve $\log_2(x+3)+\log_2(x-3)=4$.
13. Solve $2^{3x}=7$, giving x exactly and to 3 significant figures.
14. A population is $P=500e^{0.04t}$ after t years. Find when it doubles.
15. Describe fully the transformations that map $y=f(x)$ onto $y=2f(x+3)-1$, and find the image of the point $(1,4)$.
16. The graph of $y=\sin x$ is stretched horizontally by scale factor $\frac{1}{3}$ and then translated 2 units up. Write the new equation.
17. Explain why $f(x)=x^2$, $x\in\mathbb{R}$, has no inverse, and suggest a domain on which it has one.
18. **AHL.** Show that $(x+1)$ is a factor of $P(x)=x^3-7x-6$, and factorise $P(x)$ fully.
19. **AHL.** When $2x^3+ax^2-x+3$ is divided by $(x-1)$ the remainder is 6. Find a.
20. **AHL.** The cubic $x^3-6x^2+kx-10=0$ has roots α, β, γ. Find $\alpha+\beta+\gamma$ and $\alpha\beta\gamma$.
21. **AHL.** Decide whether each function is odd, even or neither: x^4-x^2, x^3+x, x^2+x.
22. **AHL.** Find the asymptotes of $f(x)=\dfrac{x^2+1}{x-1}$.
23. **AHL.** Sketch $y=|x^2-4|$ and solve $|x^2-4|=3$.

Answers.

1. Gradient -3: $y+1=-3(x-2)$, so $3x+y-5=0$.
2. The given line has gradient $\frac{5-1}{2-0}=2$, so the perpendicular gradient is $-\frac{1}{2}$. $y-3=-\frac{1}{2}(x-4)$, so $y=-\frac{1}{2}x+5$.
3. Need $6-2x\geq 0$, so the domain is $x\leq 3$; the range is $f(x)\geq 0$.
4. $(f\circ g)(x)=3(x^2+1)-2=3x^2+1$. $(g\circ f)(x)=(3x-2)^2+1$. $(g\circ f)(-1)=(-5)^2+1=26$.
5. $x=\frac{y+4}{3}$ gives $y=3x-4$, so $f^{-1}(x)=3x-4$. Check: $f(3x-4)=\frac{3x-4+4}{3}=x$.
6. $x=(y-1)^2$ with $y\geq 1$ gives $y=1+\sqrt{x}$, so $h^{-1}(x)=1+\sqrt{x}$, domain $x\geq 0$. Without $x\geq 1$, h is many-to-one ($h(0)=h(2)=1$), so no inverse exists.
7. $y=(x+3)^2-4=(x+1)(x+5)$. Vertex $(-3,-4)$, roots -1 and -5, range $y\geq -4$.
8. $\Delta=k^2-64=0$, so $k=\pm 8$.
9. $x^2+3=2x+c$ gives $x^2-2x+3-c=0$. Tangent when $\Delta=4-4(3-c)=0$, so $c=2$.
10. $(2x+1)(x-3)\geq 0$; the parabola is above the axis outside its roots: $x\leq -\frac{1}{2}$ or $x\geq 3$.
11. Vertical asymptote $x=-2$; horizontal asymptote $y=3$; x-intercept 2; y-intercept -3. Range $y\neq 3$.
12. $\log_2[(x+3)(x-3)]=4$, so $x^2-9=16$ and $x=\pm 5$. Both logs need positive arguments, so $x=5$ only.
13. $3x\ln 2=\ln 7$, so $x=\dfrac{\ln 7}{3\ln 2}\approx 0.936$.
14. $e^{0.04t}=2$, so $t=\dfrac{\ln 2}{0.04}\approx 17.3$ years.
15. Translation 3 units left, vertical stretch scale factor 2, translation 1 unit down. $(1,4)\to(-2,4)\to(-2,8)\to(-2,7)$.
16. $y=\sin 3x+2$.
17. x^2 is many-to-one ($f(2)=f(-2)=4$), so it fails the horizontal line test. On $x\geq 0$ it is one-to-one, with inverse $\sqrt{x}$.
18. **AHL.** $P(-1)=-1+7-6=0$, so $(x+1)$ is a factor. $P(x)=(x+1)(x^2-x-6)=(x+1)(x-3)(x+2)$.
19. **AHL.** $2+a-1+3=6$, so $a=2$.
20. **AHL.** Sum $=-\frac{-6}{1}=6$. Product $=\frac{(-1)^3\times(-10)}{1}=10$.
21. **AHL.** x^4-x^2 is even; x^3+x is odd; x^2+x is neither, since $(-x)^2+(-x)=x^2-x$ is neither $f(x)$ nor $-f(x)$.
22. **AHL.** Vertical asymptote $x=1$. Dividing, $\frac{x^2+1}{x-1}=x+1+\frac{2}{x-1}$, so the oblique asymptote is $y=x+1$.

23. **AHL.** The sketch is $y = x^2 - 4$ with the part between -2 and 2 reflected above the axis (peak at $(0, 4)$). $x^2 - 4 = 3$ gives $x = \pm\sqrt{7}$; $x^2 - 4 = -3$ gives $x = \pm 1$. Four solutions: ± 1, $\pm\sqrt{7}$.

Exam tips

- Always state the domain when you give an inverse function: it is the range of the original function.
- Apply the inner function first in a composite: $f(g(x))$ means g then f.
- Use the discriminant for "how many roots" and "tangent" questions, and sketch for inequalities.
- For rational functions, give asymptotes as equations, $x = \ldots$ and $y = \ldots$, not just numbers.
- Check log equations for solutions that make an argument zero or negative, and reject them.
- For transformations, follow one point and say "translation by the vector", "vertical stretch, scale factor" and "reflection in the x-axis" in full.
- On Paper 1, keep answers exact (ln, surds, fractions); on calculator papers, give three significant figures.

Geometry and trigonometry

International Baccalaureate · IB Diploma · Mathematics: Analysis and Approaches · HL

Scope and prerequisites

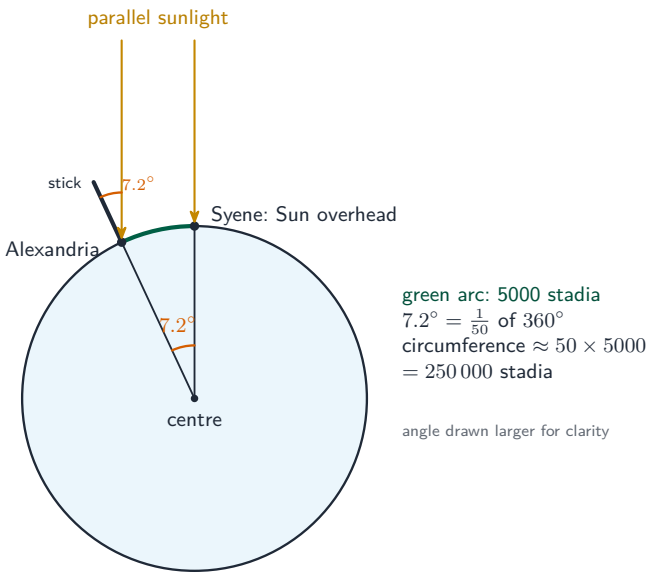
This handout teaches topic 3, Geometry and trigonometry, of Mathematics: Analysis and Approaches, the course first assessed in 2021: distance and volume in three dimensions, right-angled and non-right-angled triangles, bearings, radians, arcs and sectors, the unit circle, identities, trigonometric graphs and equations, and, for HL, reciprocal and inverse trigonometric functions, compound angle identities, vectors, the scalar and vector products, and lines and planes in space. It is written for both SL and HL students. HL students also apply the further results in the sections marked AHL.

Prerequisites: Pythagoras' theorem, similar triangles, solving linear and quadratic equations, and the function transformations of topic 2.

Geometry in three dimensions

■ Measuring the Earth with a shadow

Around 240 BC the Greek scholar Eratosthenes heard that at noon on midsummer's day the Sun shone straight down a well at Syene, in the south of Egypt, so it was directly overhead. At the same time in Alexandria, about 5000 stadia further north, a vertical stick cast a short shadow: the Sun was 7.2° from overhead. Because the Sun's rays are parallel, the angle at the centre of the Earth between the two cities is also 7.2°, which is $\frac{1}{50}$ of a full turn. So the distance round the Earth is $50 \times 5000 = 250\,000$ stadia. If a stadion was about 160 m, as some historians think, that is about 40 000 km, very close to the modern value. Eratosthenes never left Egypt: he measured one angle and one distance, and geometry did the rest. This topic is about exactly that, finding lengths, angles and areas that cannot be measured directly. We finish his calculation in radians later.



Parallel rays make the shadow angle at Alexandria equal to the angle at the Earth's centre

Distance and midpoint

In three dimensions a point has coordinates (x, y, z). For points (x_1, y_1, z_1) and (x_2, y_2, z_2):

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}, \qquad M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

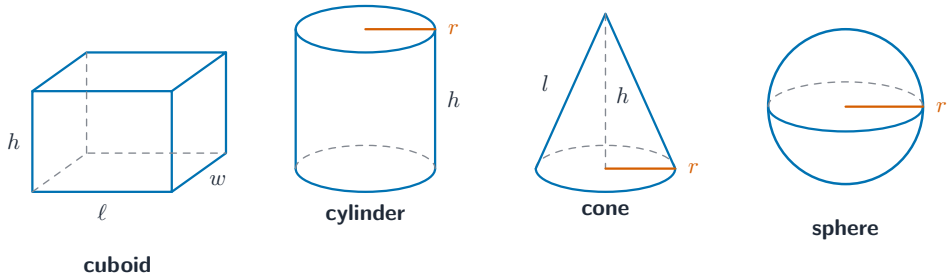
Worked example. For $P(1, 2, 3)$ and $Q(4, 6, 15)$, the differences are 3, 4 and 12, so $PQ = \sqrt{9 + 16 + 144} = \sqrt{169} = 13$. The **midpoint** 中点 is $(2.5, 4, 9)$.

Volume and surface area

The formula booklet gives these. Here r is a radius, h a perpendicular height and l the **slant height** 斜高 of a cone.



The pyramids of Giza are square-based pyramids – a 3-D solid.



Common solids and the dimensions (r, h, ℓ, slant l) used in their volume and surface-area formulas.

Solid	Volume	Surface area
Prism, cylinder	area of cross-section × length; cylinder $\pi r^2 h$	cylinder curved surface $2\pi rh$
Pyramid 棱锥	$\frac{1}{3}\times$ base area $\times h$	add the areas of the faces
Cone 圆锥	$\frac{1}{3}\pi r^2 h$	curved surface πrl
Sphere 球	$\frac{4}{3}\pi r^3$	$4\pi r^2$

A **hemisphere** 半球 is half a sphere: volume $\frac{2}{3}\pi r^3$, curved surface $2\pi r^2$, plus a flat circle πr^2 if the question asks for the total.

Worked example. A cone has radius 6 cm and height 8 cm.

- Slant height: $l = \sqrt{6^2 + 8^2} = 10$ cm.
- Volume: $\frac{1}{3}\pi(6^2)(8) = 96\pi \approx 302$ cm^3.
- Total surface area: $\pi(6)(10) + \pi(6^2) = 60\pi + 36\pi = 96\pi \approx 302$ cm^2.

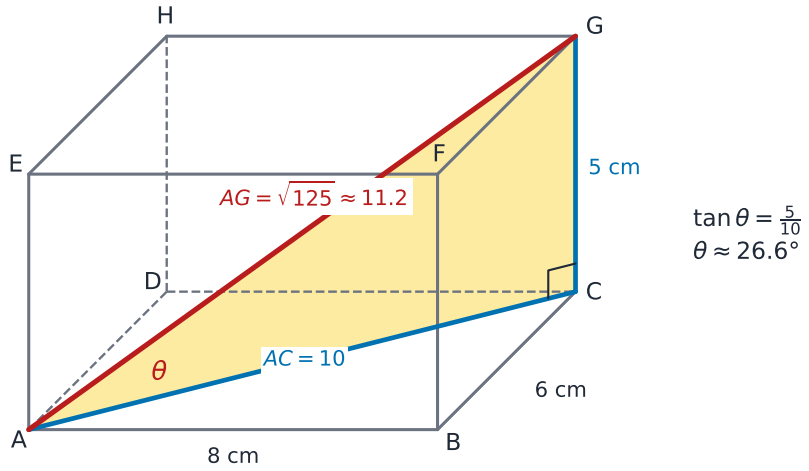
The two numbers are equal only by chance: one is in cm^3 and the other in cm^2. Always write the units, and read whether the question wants the curved surface only or the total.

The angle between a line and a plane

To find the angle between a line and a plane, drop a perpendicular from a point on the line to the plane. The angle is between the line and its "shadow" in the plane, and it sits in a right-angled triangle.

> **Worked example.** A cuboid is 8 cm by 6 cm by 5 cm. Find the angle between the **space diagonal** 体对角线 AG and the base.

- Base diagonal: $AC = \sqrt{8^2 + 6^2} = 10$ cm.
- $CG = 5$ cm is vertical, so triangle ACG has a right angle at C.
- $\tan\theta = \frac{5}{10}$, so $\theta \approx 26.6°$, and $AG = \sqrt{125} \approx 11.2$ cm.



$\tan\theta = \frac{5}{10}$
$\theta \approx 26.6°$

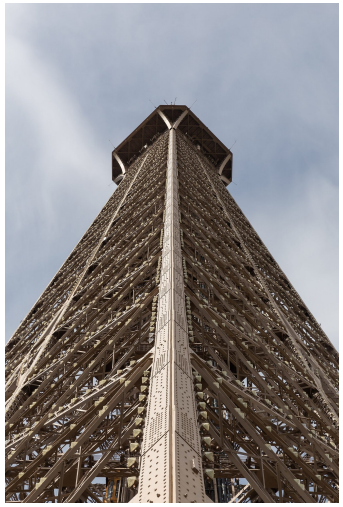
Find the right-angled triangle that contains the angle, then use it

Always draw the right-angled triangle on its own, flat on the page, before you calculate.

Triangle trigonometry

Right-angled triangles

Label the sides from the angle θ you are using: the **hypotenuse** 斜边 is opposite the right angle, the opposite side faces θ, and the adjacent side is next to it.



Looking up at a tower involves an angle of elevation.

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}, \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}, \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

- An **angle of elevation** 仰角 is measured up from the horizontal; an **angle of depression** 俯角 is measured down from the horizontal.
- A **bearing** 方位角 is an angle measured clockwise from north, written with three figures: 050°, 160°, 305°.

> **Worked example.** From a point 40 m from the foot of a tower, the angle of elevation of the top is 32°. The height is $40\tan 32° \approx 25.0$ m.

The sine rule, the cosine rule and area

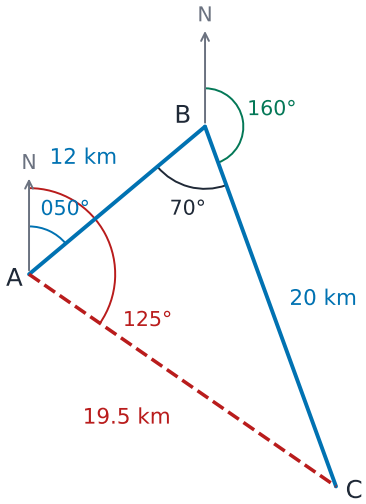
For any triangle with sides a, b, c opposite angles A, B, C:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}, \qquad c^2 = a^2 + b^2 - 2ab\cos C, \qquad \text{Area} = \frac{1}{2}ab\sin C$$

- Use the **sine rule** 正弦定理 when you know a side and the angle opposite it.
- Use the **cosine rule** 余弦定理 when you know two sides and the angle between them, or all three sides. To find an angle, rearrange: $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$.

Worked example (bearings). A ship sails 12 km from A on a bearing of 050° to B, then 20 km on a bearing of 160° to C. Find the distance and bearing of C from A, and the area of triangle ABC.

- At B, the bearing back to A is $050° + 180° = 230°$, so angle $ABC = 230° - 160° = 70°$.
- Cosine rule: $AC^2 = 12^2 + 20^2 - 2(12)(20)\cos 70° \approx 379.8$, so $AC \approx 19.5$ km.
- Angle at A: $\cos A = \dfrac{12^2 + 19.49^2 - 20^2}{2(12)(19.49)} \approx 0.265$, so $A \approx 74.6°$.
- Bearing of C from A: $050° + 74.6° \approx 125°$.
- Area: $\frac{1}{2}(12)(20)\sin 70° \approx 113$ km^2.



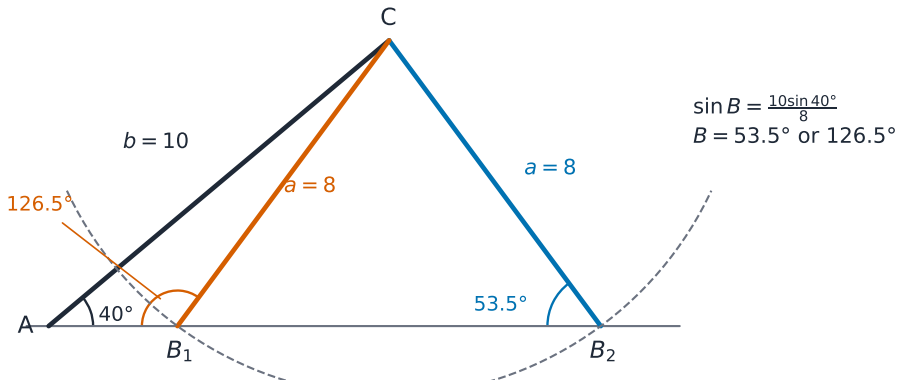
Draw a north line at every point where a bearing is measured

The ambiguous case

When you use the sine rule to find an angle, remember that $\sin B = \sin(180° - B)$. A value of $\sin B$ can come from an acute angle **or** an obtuse one, and sometimes both make a real triangle.

Worked example. In triangle ABC, $A = 40°$, $b = 10$ and $a = 8$. Find angle B.

- $\sin B = \dfrac{10\sin 40°}{8} \approx 0.803$, so $B \approx 53.5°$ or $B \approx 180° - 53.5° = 126.5°$.
- Check: $126.5° + 40° < 180°$, so both triangles exist.
- $C = 86.5°$, $c \approx 12.4$, or $C = 13.5°$, $c \approx 2.90$.



Side $a = 8$ can swing to two positions, so two triangles fit the data

The obtuse answer is possible only when it leaves room for the third angle. If the side opposite the given angle is the longest given side, there is only one triangle.

Radians, arcs and sectors

What a radian is

A **radian** 弧度 is the angle at the centre of a circle made by an arc as long as the radius. A full turn is an arc of $2\pi r$, so

$$2\pi \text{ rad} = 360°, \qquad \pi \text{ rad} = 180°$$

To change degrees to radians multiply by $\frac{\pi}{180}$; to change back multiply by $\frac{180}{\pi}$. For example, $60° = \frac{\pi}{3}$ and $135° = \frac{3\pi}{4}$. In calculus (topic 5) angles must be in radians, so from now on assume radians unless you see the degree sign.

Arc length and sector area

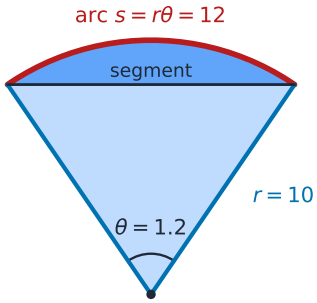
With θ in radians:

$$\text{arc length } s = r\theta, \qquad \text{sector area } A = \frac{1}{2}r^2\theta$$

A **sector** 扇形 is the "slice" between two radii. The **segment** 弓形 is the part between the chord and the arc: its area is the sector minus the triangle, $\frac{1}{2}r^2(\theta - \sin\theta)$.

Worked example. A circle has radius 10 cm, and a sector has angle 1.2 rad.

- Arc length $= 10 \times 1.2 = 12$ cm, so the sector's perimeter is $12 + 10 + 10 = 32$ cm.
- Sector area $= \frac{1}{2}(10^2)(1.2) = 60$ cm^2.
- Segment area $= \frac{1}{2}(10^2)(1.2 - \sin 1.2) \approx 13.4$ cm^2.



sector $= \frac{1}{2}r^2\theta = 60$

triangle $= \frac{1}{2}r^2\sin\theta \approx 46.6$

segment $= \frac{1}{2}r^2(\theta - \sin\theta) \approx 13.4$

Segment = sector − triangle

Back to Eratosthenes. His angle was $7.2° = \frac{\pi}{25}$ rad and his arc was 5000 stadia. From $s = r\theta$, the Earth's radius is $r = \dfrac{5000}{\pi/25} \approx 39\,800$ stadia, and the circumference is $2\pi r = 250\,000$ stadia, as before.

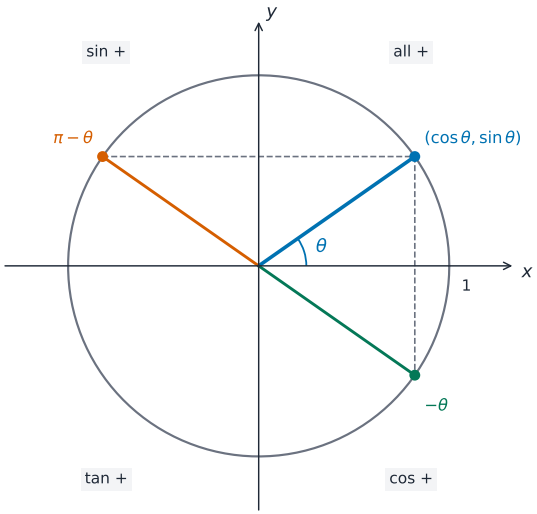
The unit circle and identities

Sine and cosine for any angle

Draw a circle of radius 1 with its centre at the origin: the **unit circle** 单位圆. Turn anticlockwise through angle θ from the positive x-axis. The point you reach is $(\cos\theta, \sin\theta)$, and

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

This defines sine and cosine for every angle, not just acute ones. The signs follow from the coordinates: all three ratios are positive in the first quadrant, only sine in the second, only tangent in the third and only cosine in the fourth. Clockwise turns give negative angles.



Reflections in the unit circle give the symmetry rules

The reflections in the figure give rules you will use constantly:

- $\sin(\pi - \theta) = \sin\theta$ and $\cos(\pi - \theta) = -\cos\theta$.
- $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$.

Exact values

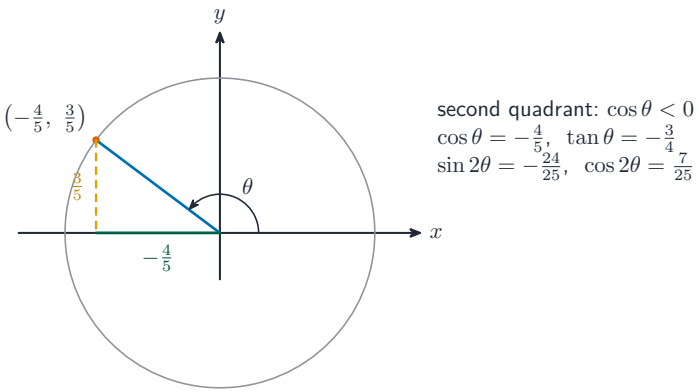
Learn these: they come from the half-square and the half-equilateral triangle, and Paper 1 has no calculator.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined

For other angles, find the reference angle in the first quadrant and attach the sign. For example, $\cos\frac{4\pi}{3}$ is in the third quadrant with reference angle $\frac{\pi}{3}$, so $\cos\frac{4\pi}{3} = -\frac{1}{2}$.

Identities

An **identity** 恒等式 is true for every value of the variable. Since the point $(\cos\theta, \sin\theta)$ is on a circle of radius 1, Pythagoras gives



An obtuse angle with $\sin\theta = 3/5$

$$\sin^2\theta + \cos^2\theta = 1$$

The **double angle identities** 二倍角公式 are

$$\sin 2\theta = 2\sin\theta\cos\theta, \qquad \cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$

Worked example. θ is obtuse and $\sin\theta = \frac{3}{5}$. Find $\cos\theta$, $\tan\theta$, $\sin 2\theta$ and $\cos 2\theta$.

- $\cos^2\theta = 1 - \frac{9}{25} = \frac{16}{25}$. θ is in the second quadrant, so $\cos\theta = -\frac{4}{5}$.
- $\tan\theta = \dfrac{3/5}{-4/5} = -\frac{3}{4}$.
- $\sin 2\theta = 2\left(\frac{3}{5}\right)\left(-\frac{4}{5}\right) = -\frac{24}{25}$; $\cos 2\theta = 1 - 2\left(\frac{9}{25}\right) = \frac{7}{25}$.

The square root gives two signs: the quadrant decides which one.

Trigonometric graphs

The three basic graphs

- $y = \sin x$ and $y = \cos x$ repeat every 2π: their **period** 周期 is 2π. They stay between -1 and 1, so their **amplitude** 振幅 is 1. The cosine graph is the sine graph moved $\frac{\pi}{2}$ to the left.
- $y = \tan x$ has period π and vertical asymptotes at $x = \frac{\pi}{2} + k\pi$, where $\cos x = 0$.



A point on a turning wheel can be modelled with a sine or cosine function.

Transformed graphs

Using the transformations of topic 2, the graph of

$$y = a\sin\big(b(x - c)\big) + d$$

has amplitude $|a|$, period $\dfrac{2\pi}{|b|}$ (for $a \neq 0$, $b \neq 0$), a horizontal shift of c and a **principal axis** 主轴 $y = d$. Its maximum is $d + |a|$ and its minimum is $d - |a|$. The same holds for cosine.

Worked example. For $y = 3\sin\left(2(x - \frac{\pi}{4})\right) + 1$: amplitude 3, period $\frac{2\pi}{2} = \pi$, shift $\frac{\pi}{4}$ to the right, principal axis $y = 1$, maximum 4 and minimum -2.

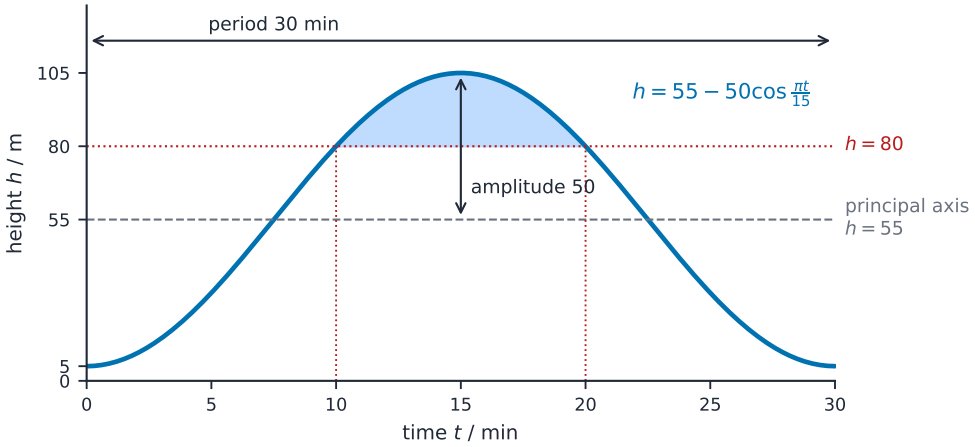
Modelling with a trigonometric function

Anything that repeats, such as tides, daylight hours or a turning wheel, can be modelled this way.

Worked example. A Ferris wheel has diameter 100 m, and its centre is 55 m above the ground. It turns once every 30 minutes. A passenger boards at the bottom at $t = 0$.

- The centre gives $d = 55$ and the radius gives the amplitude 50. The period is 30, so $b = \frac{2\pi}{30} = \frac{\pi}{15}$.
- Starting at the bottom means starting at a minimum, so use $-\cos$: $h = 55 - 50\cos\dfrac{\pi t}{15}$.
- Check: $h(0) = 5$ m (the lowest point) and $h(15) = 105$ m (the top).

- When is the passenger above 80 m? $55 - 50\cos\frac{\pi t}{15} > 80$ gives $\cos\frac{\pi t}{15} < -\frac{1}{2}$, so $\frac{2\pi}{3} < \frac{\pi t}{15} < \frac{4\pi}{3}$ and $10 < t < 20$: 10 minutes of each turn.



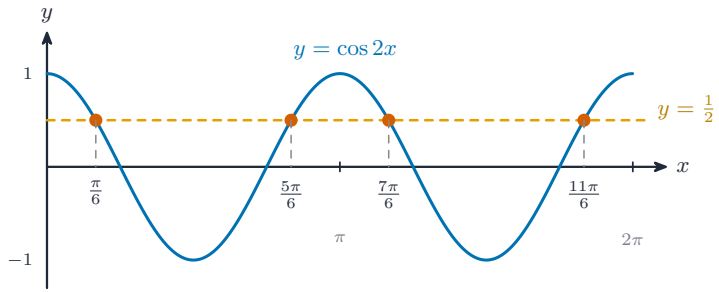
The centre gives the principal axis and the radius gives the amplitude

The same wheel can be written as $h = 50\sin\left(\frac{\pi}{15}(t-7.5)\right)+55$. Different correct forms exist, so check yours with one or two values of t.

Trigonometric equations

A method that finds every solution

A calculator's inverse sine, cosine or tangent gives only one **principal value** 主值. A trigonometric equation usually has more solutions in the interval.



two full turns of $u = 2x$ in $[0, 2\pi]$: four solutions, not two

$u = 2x$ runs through two full turns, so cos 2x = ½ has four solutions in [0, 2π]

1. Rearrange to get one trigonometric function on its own.
2. Find the reference angle.
3. Use the quadrants (or the graph) to find every angle in the interval.
4. For $\sin kx$ or $\cos kx$, first stretch the interval by k, solve for kx, then divide by k.

> **Worked example.** Solve $2\sin x = 1$ for $0 \le x \le 2\pi$. $\sin x = \frac{1}{2}$; sine is positive in quadrants 1 and 2, so $x = \frac{\pi}{6}$ or $x = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$.

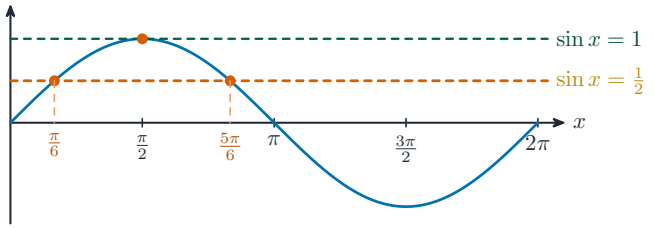
> **Worked example.** Solve $\cos 2x = \frac{1}{2}$ for $0 \le x \le 2\pi$.

- Let $u = 2x$, so $0 \le u \le 4\pi$: two full turns.
- $u = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}$.
- $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$.

Without the larger interval you would find only two of the four answers.

Equations that become quadratics

Use an identity to get one function, then factorise.



$(2\sin x - 1)(\sin x - 1) = 0$

Three solutions from two factors

> **Worked example.** Solve $2\cos^2 x + 3\sin x - 3 = 0$ for $0 \le x \le 2\pi$.

- Replace $\cos^2 x$ with $1-\sin^2 x$: $2-2\sin^2 x+3\sin x-3 = 0$, so $2\sin^2 x-3\sin x+1 = 0$.
- Factorise: $(2\sin x - 1)(\sin x - 1) = 0$, so $\sin x = \frac{1}{2}$ or $\sin x = 1$.
- $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$.

> **Worked example.** Solve $\sin 2x = \sin x$ for $0 \le x \le 2\pi$.

- $2\sin x\cos x - \sin x = 0$, so $\sin x(2\cos x - 1) = 0$.
- $\sin x = 0$ gives $x = 0, \pi, 2\pi$; $\cos x = \frac{1}{2}$ gives $x = \frac{\pi}{3}, \frac{5\pi}{3}$.

Never divide both sides by $\sin x$: that throws away the three solutions where $\sin x = 0$. Factorise instead.

Solving with a graph

Some equations, such as $\sin x = 0.3x$, cannot be solved by algebra. On a calculator paper, graph both sides and find the x-coordinates of the intersection points in the interval, giving them to 3 significant figures.

AHL: Further trigonometry

Reciprocal functions and more identities

$$\sec\theta = \frac{1}{\cos\theta}, \qquad \csc\theta = \frac{1}{\sin\theta}, \qquad \cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$$

Dividing $\sin^2\theta + \cos^2\theta = 1$ by $\cos^2\theta$, and then by $\sin^2\theta$, gives

$$1 + \tan^2\theta = \sec^2\theta, \qquad 1 + \cot^2\theta = \csc^2\theta$$

Worked example. Solve $\sec^2 x = 3 + \tan x$ for $0 \le x \le \pi$.

- $1 + \tan^2 x = 3 + \tan x$, so $\tan^2 x - \tan x - 2 = 0$ and $(\tan x - 2)(\tan x + 1) = 0$.
- $\tan x = 2$ gives $x \approx 1.11$; $\tan x = -1$ gives $x = \frac{3\pi}{4}$.

Inverse trigonometric functions

Sine, cosine and tangent are many-to-one, so their **inverse functions** 反函数 use a restricted domain:

Function	Domain	Range
$y = \arcsin x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$y = \arccos x$	$-1 \le x \le 1$	$0 \le y \le \pi$
$y = \arctan x$	all real x	$-\frac{\pi}{2} < y < \frac{\pi}{2}$

For example, $\arcsin\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$ and $\arccos\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$. Their graphs are the reflections of the restricted graphs in $y = x$.

Compound angle identities

$$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B, \qquad \cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}, \qquad \tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

Worked example. Find the exact value of $\sin 75°$.

$$\sin(45° + 30°) = \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \frac{\sqrt{6}+\sqrt{2}}{4}$$

Worked example. Prove that $\dfrac{1-\cos 2\theta}{\sin 2\theta} = \tan\theta$.

$$\text{LHS} = \frac{1-(1-2\sin^2\theta)}{2\sin\theta\cos\theta} = \frac{2\sin^2\theta}{2\sin\theta\cos\theta} = \frac{\sin\theta}{\cos\theta} = \text{RHS}$$

Work on one side only, and choose the form of $\cos 2\theta$ that cancels the 1.

AHL: Vectors

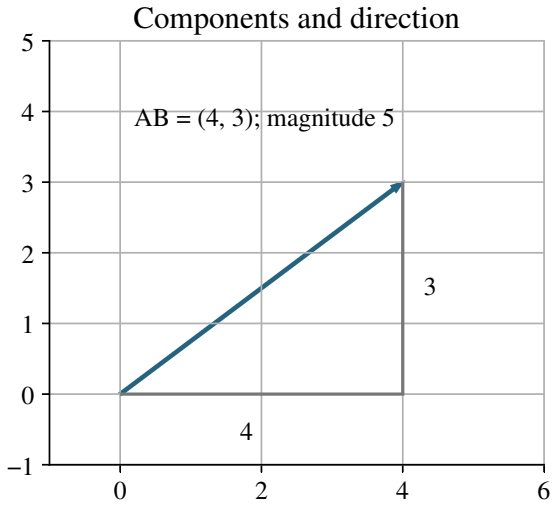
Vectors and their components

A **vector** 向量 has size and direction; a **scalar** 标量 has size only. In components, $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$.



Forces like wind and water are vectors – they have both size and direction.

- **Magnitude** 模: $|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$.
- A **unit vector** 单位向量 has magnitude 1: $\dfrac{\mathbf{v}}{|\mathbf{v}|}$.
- The **position vector** 位置向量 of A is $\mathbf{a} = \overrightarrow{OA}$, and $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$.
- Two vectors are parallel when one is a scalar multiple of the other.



$\overrightarrow{AB}$ has components 4 and 3, so its magnitude is $\sqrt{4^2+3^2}=5$

Vectors also describe motion. A boat starts at $\begin{pmatrix}2\\-1\end{pmatrix}$ km and moves with velocity $\begin{pmatrix}3\\4\end{pmatrix}$ km/h. After t hours it is at $\begin{pmatrix}2+3t\\-1+4t\end{pmatrix}$, and its speed is the magnitude of the velocity, 5 km/h.

The scalar product

$$\mathbf{v}\cdot\mathbf{w}=v_1w_1+v_2w_2+v_3w_3=|\mathbf{v}||\mathbf{w}|\cos\theta$$

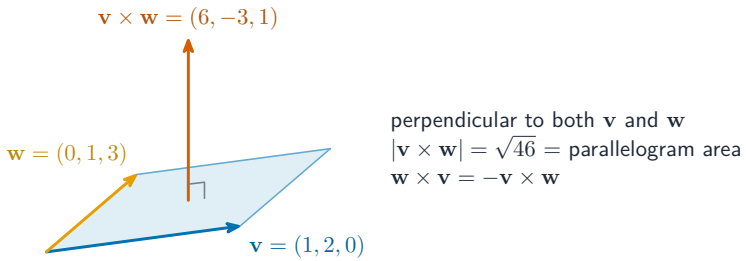
The **scalar product** 数量积 gives a number, and it gives the angle between two vectors. Two non-zero vectors are perpendicular exactly when $\mathbf{v}\cdot\mathbf{w}=0$.

Worked example. $\mathbf{a}=\begin{pmatrix}2\\-1\\2\end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix}3\\0\\4\end{pmatrix}$.

- $|\mathbf{a}|=3$, $|\mathbf{b}|=5$ and $\mathbf{a}\cdot\mathbf{b}=6+0+8=14$.
- $\cos\theta=\frac{14}{15}$, so $\theta\approx 21.0°$.
- The unit vector in the direction of $\mathbf{b}$ is $\begin{pmatrix}0.6\\0\\0.8\end{pmatrix}$.

The vector product

$$\mathbf{v}\times\mathbf{w}=\begin{pmatrix}v_2w_3-v_3w_2\\v_3w_1-v_1w_3\\v_1w_2-v_2w_1\end{pmatrix},\qquad |\mathbf{v}\times\mathbf{w}|=|\mathbf{v}||\mathbf{w}|\sin\theta$$



The vector product is normal to both vectors

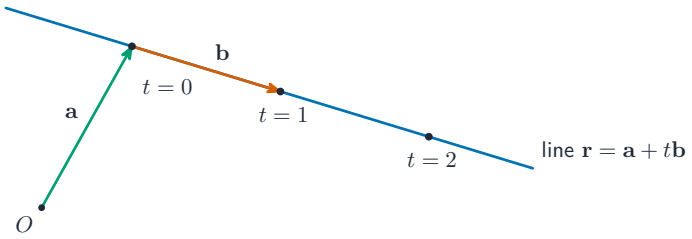
The **vector product** 向量积 gives a vector perpendicular to both $\mathbf{v}$ and $\mathbf{w}$. Its magnitude is the area of the parallelogram they make, so a triangle has half that area. Order matters: $\mathbf{w}\times\mathbf{v}=-\mathbf{v}\times\mathbf{w}$.

Worked example. $\begin{pmatrix}1\\2\\0\end{pmatrix}\times\begin{pmatrix}0\\1\\3\end{pmatrix}=\begin{pmatrix}6-0\\0-3\\1-0\end{pmatrix}=\begin{pmatrix}6\\-3\\1\end{pmatrix}$. Check: its scalar product with each vector is 0. The triangle with these two sides has area $\frac{1}{2}\sqrt{36+9+1}=\frac{\sqrt{46}}{2}\approx 3.39$.

AHL: Lines and planes

Lines in space

The line through the point with position vector $\mathbf{a}$ with **direction vector** 方向向量 $\mathbf{b}$ is



Start at the point $\mathbf{a}$, then add t copies of the direction $\mathbf{b}$ to reach any point on the line.

$$\mathbf{r}=\mathbf{a}+\lambda\mathbf{b}$$

Each value of the parameter λ gives one point. Writing each component separately gives the parametric form, and solving each for λ gives the Cartesian form $\dfrac{x-a_1}{b_1} = \dfrac{y-a_2}{b_2} = \dfrac{z-a_3}{b_3}$.

The angle between two lines is the angle between their direction vectors. Two lines in space are either parallel (direction vectors are multiples), intersecting, or **skew** 异面: not parallel and never meeting.

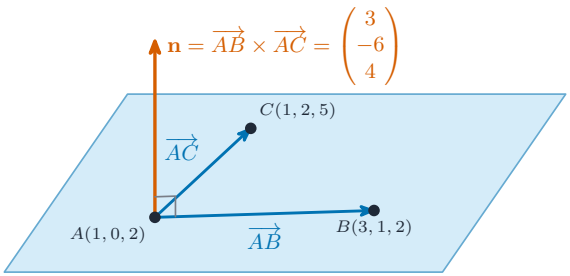
Worked example. $L_1 : \mathbf{r} = \begin{pmatrix}1\\0\\2\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\-1\end{pmatrix}$ and $L_2 : \mathbf{r} = \begin{pmatrix}2\\3\\3\end{pmatrix} + \mu\begin{pmatrix}0\\1\\1\end{pmatrix}$.

- Equate components: $1+\lambda = 2$, $\lambda = 3+\mu$, $2-\lambda = 3+\mu$.
- The first two give $\lambda = 1$, $\mu = -2$. Check the third: $2-1 = 1$ and $3-2 = 1$. It works, so the lines meet at $(2,1,1)$.
- The directions have scalar product $0+1-1 = 0$, so the lines are perpendicular.

If L_2 instead passed through $(2,3,0)$, the third equation would give $1 = -2$: no solution, and the directions are not parallel, so the lines would be skew. Always check the equation you did not use.

Planes

A plane needs a point **a** and two non-parallel directions **b**, **c** in it: $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$. A **normal vector** 法向量 $\mathbf{n} = \mathbf{b} \times \mathbf{c}$ is perpendicular to the plane, giving



$\mathbf{r}\cdot\mathbf{n} = \mathbf{a}\cdot\mathbf{n}$
$3x - 6y + 4z = 11$
check B: $9 - 6 + 8 = 11$

Two directions in the plane give the normal by a cross product, and the normal gives the equation

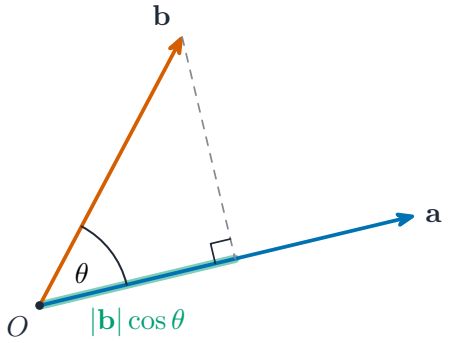
$$\mathbf{r}\cdot\mathbf{n} = \mathbf{a}\cdot\mathbf{n}, \qquad \text{or in Cartesian form } ax+by+cz = d \text{ with } \mathbf{n} = \begin{pmatrix}a\\b\\c\end{pmatrix}$$

Worked example. Find the plane through $A(1,0,2)$, $B(3,1,2)$ and $C(1,2,5)$.

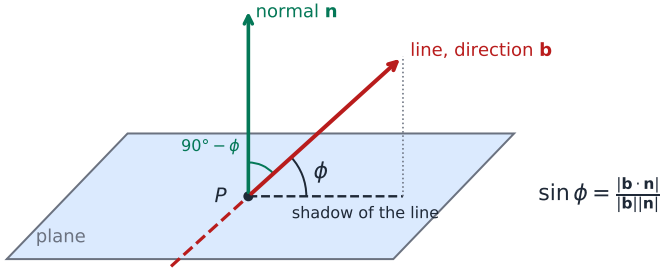
- $\overrightarrow{AB} = \begin{pmatrix}2\\1\\0\end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix}0\\2\\3\end{pmatrix}$, so $\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix}3\\-6\\4\end{pmatrix}$.
- $d = \mathbf{n}\cdot\mathbf{a} = 3+0+8 = 11$, so the plane is $3x - 6y + 4z = 11$.
- Check with B: $9 - 6 + 8 = 11$.

Intersections and angles

- **Line and plane.** Substitute the line's components into the plane's equation and solve for λ.
- **Angle between a line and a plane.** The angle ϕ satisfies $\sin\phi = \dfrac{|\mathbf{b}\cdot\mathbf{n}|}{|\mathbf{b}||\mathbf{n}|}$, because the angle between the line and the normal is $90° - \phi$.
- **Two planes.** Non-parallel planes meet in a line with direction $\mathbf{n}_1 \times \mathbf{n}_2$; the angle between the planes is the angle between their normals.
- **Three planes.** Solve the three equations together: they meet at one point, along a line, coincide in a plane, or have no common point.



The scalar product picks out $|\mathbf{b}|\cos\theta$, how far **b** *reaches along* **a**.



Measure the angle to the plane, not to the normal

Worked example. The line $\mathbf{r} = \begin{pmatrix}1\\1\\0\end{pmatrix} + t\begin{pmatrix}0\\0\\1\end{pmatrix}$ meets the plane $3x - 6y + 4z = 11$.

- Substitute: $3 - 6 + 4t = 11$, so $t = 3.5$ and the point is $(1, 1, 3.5)$.
- $\mathbf{b} \cdot \mathbf{n} = 4$, $|\mathbf{b}| = 1$, $|\mathbf{n}| = \sqrt{61}$, so $\sin\phi = \frac{4}{\sqrt{61}}$ and $\phi \approx 30.8°$.

Worked example. The planes $x + y + z = 6$ and $x - y + 2z = 5$ meet in a line with direction $\begin{pmatrix}1\\1\\1\end{pmatrix} \times \begin{pmatrix}1\\-1\\2\end{pmatrix} = \begin{pmatrix}3\\-1\\-2\end{pmatrix}$. Setting $z = 0$ gives $x = 5.5$, $y = 0.5$, so the line is $\mathbf{r} = \begin{pmatrix}5.5\\0.5\\0\end{pmatrix} + t\begin{pmatrix}3\\-1\\-2\end{pmatrix}$. The angle between the planes is $\arccos\dfrac{2}{\sqrt{3}\sqrt{6}} \approx 61.9°$.

Common mistakes.

- Leaving the calculator in degrees for a radian question, or the other way round.
- Using $s = r\theta$ or $A = \frac{1}{2}r^2\theta$ with θ in degrees. Both formulas need radians.
- Missing the second triangle in the ambiguous case: if $\sin B = 0.6$, then $B \approx 36.9°$ **or** $143.1°$; check the angle sum for each.
- Stopping at the calculator value: $\sin x = 0.5$ for $0 \le x \le 2\pi$ gives $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$.
- Dividing $\sin x \cos x = \sin x$ by $\sin x$. That loses the solutions of $\sin x = 0$; factorise instead.
- Solving $\sin 2x = k$ only on the given interval for x. Solve for $2x$ on the doubled interval, then halve.
- **HL:** reading $\cos^{-1} x$ as $\sec x$. It is the inverse function; $\sec x = \dfrac{1}{\cos x}$.
- **HL:** using $\cos\theta$ with a normal for the angle between a line and a plane. The normal gives the complement, so use $\sin\theta = \dfrac{|\mathbf{b} \cdot \mathbf{n}|}{|\mathbf{b}||\mathbf{n}|}$.
- **HL:** swapping the order in a vector product: $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$.
- **HL:** using the same parameter for two lines when testing whether they meet. Give them different parameters, such as λ and μ.

Practice

1. Find the distance between $P(-1, 4, 2)$ and $Q(5, 1, -4)$, and the midpoint of PQ.
2. A grain store is a cylinder of radius 6 m and height 10 m with a hemisphere on top. Find its volume and its total outside surface area, including the base.
3. A square-based pyramid has base edges 6 m and height 4 m. Find its volume and the angle between a sloping edge and the base.
4. From the top of a cliff 60 m high, the angle of depression of a boat is 18°. Find the horizontal distance from the foot of the cliff to the boat.
5. In triangle ABC, $a = 7$, $b = 9$ and $C = 110°$. Find c and the area of the triangle.
6. A triangle has sides 5, 7 and 8. Without a calculator, show that the angle opposite the side 7 is 60°, and find the exact area.
7. In triangle ABC, $A = 30°$, $a = 6$ and $b = 10$. Find both possible values of B and the two values of c.
8. Convert 135° to radians, and 2.5 rad to degrees.
9. A sector has radius 8 cm and area 40 cm^2. Find its angle and its perimeter.
10. A chord of a circle of radius 6 cm makes an angle of $\frac{\pi}{3}$ at the centre. Find the exact area of the minor segment.
11. $\cos\theta = -\frac{5}{13}$ and $\frac{\pi}{2} < \theta < \pi$. Find the exact values of $\sin\theta$, $\tan\theta$ and $\sin 2\theta$.
12. For $y = 3\sin\left(2(x - \frac{\pi}{4})\right) + 1$, write down the amplitude, the period, the principal axis, and the maximum and minimum values.
13. The depth of water in a harbour is $d = 2.5\cos\frac{\pi t}{6} + 6$ metres, t hours after midnight. Find the greatest and least depths, the times in $0 \le t \le 12$ when the depth is 7.25 m, and for how long the depth is less than 4.75 m in that time.
14. Solve $2\cos x + \sqrt{3} = 0$ for $0 \le x \le 2\pi$.
15. Solve $\tan 3x = 1$ for $0 \le x \le \pi$.
16. Solve $2\sin^2 x = 3\cos x$ for $0 \le x \le 2\pi$.
17. Find the exact value of $\cos\frac{4\pi}{3}$ and of $\tan\frac{7\pi}{4}$.
18. **AHL.** Find the exact values of $\cos 15°$ and $\tan 75°$.
19. **AHL.** Prove that $\dfrac{\sin 2\theta}{1 + \cos 2\theta} = \tan\theta$.
20. **AHL.** Solve $2\sec^2 x - 5\tan x = 0$ for $0 \le x \le \pi$.
21. **AHL.** State the domain and range of $y = \arccos x$, and find the exact values of $\arcsin\left(-\frac{\sqrt{3}}{2}\right)$ and $\arctan\sqrt{3}$.
22. **AHL.** $\mathbf{a} = \begin{pmatrix}1\\2\\-2\end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix}4\\0\\3\end{pmatrix}$. Find $|\mathbf{a}|$, $|\mathbf{b}|$, the angle between them and the unit vector in the direction of $\mathbf{b}$.
23. **AHL.** Find p if $\begin{pmatrix}p\\2\\-1\end{pmatrix}$ is perpendicular to $\begin{pmatrix}3\\-1\\4\end{pmatrix}$.
24. **AHL.** $A(1, 0, 0)$, $B(0, 2, 0)$ and $C(0, 0, 3)$. Use the vector product to find the area of triangle ABC and the Cartesian equation of the plane through A, B and C.
25. **AHL.** Show that $L_1 : \mathbf{r} = \begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda\begin{pmatrix}1\\0\\-1\end{pmatrix}$ and $L_2 : \mathbf{r} = \begin{pmatrix}0\\5\\1\end{pmatrix} + \mu\begin{pmatrix}1\\-1\\0\end{pmatrix}$ intersect, find the point, and find the angle between the lines.

26. **AHL.** Find where the line $\mathbf{r} = \begin{pmatrix}2\\1\\0\end{pmatrix} + t\begin{pmatrix}1\\-1\\2\end{pmatrix}$ meets the plane $x + 2y + z = 10$, and the angle between the line and the plane.

Answers.

1. Differences 6, −3, −6, so $PQ = \sqrt{36 + 9 + 36} = 9$. Midpoint $(2, 2.5, -1)$.
2. Volume $= \pi(6^2)(10) + \frac{2}{3}\pi(6^3) = 360\pi + 144\pi = 504\pi \approx 1580$ m^3. Surface area $= 2\pi(6)(10) + 2\pi(6^2) + \pi(6^2) = 228\pi \approx 716$ m^2.
3. Volume $= \frac{1}{3}(36)(4) = 48$ m^3. Half the base diagonal is $3\sqrt{2} \approx 4.24$, so $\tan\theta = \frac{4}{3\sqrt{2}}$ and $\theta \approx 43.3°$.
4. The angle of elevation of the cliff top from the boat is also 18°, so the distance is $\dfrac{60}{\tan 18°} \approx 185$ m.
5. $c^2 = 49 + 81 - 126\cos 110° \approx 173.1$, so $c \approx 13.2$. Area $= \frac{1}{2}(7)(9)\sin 110° \approx 29.6$.
6. $\cos\theta = \dfrac{5^2 + 8^2 - 7^2}{2(5)(8)} = \dfrac{40}{80} = \frac{1}{2}$, so $\theta = 60°$. Area $= \frac{1}{2}(5)(8)\sin 60° = 10\sqrt{3}$.
7. $\sin B = \dfrac{10\sin 30°}{6} = \frac{5}{6}$, so $B \approx 56.4°$ or $123.6°$; both leave room for C. Then $C \approx 93.6°$, $c \approx 12.0$, or $C \approx 26.4°$, $c \approx 5.34$.
8. $135° = \frac{3\pi}{4}$; 2.5 rad $= 2.5 \times \frac{180}{\pi} \approx 143.2°$.
9. $\frac{1}{2}(8^2)\theta = 40$, so $\theta = 1.25$ rad. Arc $= 8 \times 1.25 = 10$ cm; perimeter $= 10 + 16 = 26$ cm.
10. $\frac{1}{2}(6^2)\left(\frac{\pi}{3} - \sin\frac{\pi}{3}\right) = 18\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right) = 6\pi - 9\sqrt{3} \approx 3.26$ cm^2.
11. $\sin\theta = \frac{12}{13}$ (positive in quadrant 2); $\tan\theta = -\frac{12}{5}$; $\sin 2\theta = 2\left(\frac{12}{13}\right)\left(-\frac{5}{13}\right) = -\frac{120}{169}$.
12. Amplitude 3, period π, principal axis $y = 1$, maximum 4, minimum −2.
13. Greatest 8.5 m, least 3.5 m. $\cos\frac{\pi t}{6} = \frac{1}{2}$ gives $\frac{\pi t}{6} = \frac{\pi}{3}$ or $\frac{5\pi}{3}$, so $t = 2$ or $t = 10$: 02:00 and 10:00. Depth below 4.75 m when $\cos\frac{\pi t}{6} < -\frac{1}{2}$, that is $4 < t < 8$: 4 hours.
14. $\cos x = -\frac{\sqrt{3}}{2}$, so $x = \frac{5\pi}{6}$ or $\frac{7\pi}{6}$.
15. $0 \le 3x \le 3\pi$, so $3x = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}$ and $x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{3\pi}{4}$.
16. $2(1 - \cos^2 x) = 3\cos x$ gives $2\cos^2 x + 3\cos x - 2 = 0$, so $(2\cos x - 1)(\cos x + 2) = 0$. $\cos x = -2$ is impossible, so $\cos x = \frac{1}{2}$ and $x = \frac{\pi}{3}$ or $\frac{5\pi}{3}$.
17. $\cos\frac{4\pi}{3} = -\frac{1}{2}$ (third quadrant); $\tan\frac{7\pi}{4} = -1$ (fourth quadrant).
18. **AHL.** $\cos(45° - 30°) = \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \frac{\sqrt{6}+\sqrt{2}}{4}$. $\tan(45° + 30°) = \dfrac{1 + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}} = \dfrac{\sqrt{3}+1}{\sqrt{3}-1} = 2 + \sqrt{3}$.
19. **AHL.** $\dfrac{2\sin\theta\cos\theta}{1 + 2\cos^2\theta - 1} = \dfrac{2\sin\theta\cos\theta}{2\cos^2\theta} = \dfrac{\sin\theta}{\cos\theta} = \tan\theta$.
20. **AHL.** $2(1 + \tan^2 x) - 5\tan x = 0$ gives $(2\tan x - 1)(\tan x - 2) = 0$, so $\tan x = \frac{1}{2}$ or 2: $x \approx 0.464$ or $x \approx 1.11$.
21. **AHL.** Domain $-1 \le x \le 1$, range $0 \le y \le \pi$. $\arcsin\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$; $\arctan\sqrt{3} = \frac{\pi}{3}$.
22. **AHL.** $|\mathbf{a}| = 3$, $|\mathbf{b}| = 5$, $\mathbf{a}\cdot\mathbf{b} = 4 + 0 - 6 = -2$, so $\cos\theta = -\frac{2}{15}$ and $\theta \approx 97.7°$. Unit vector $\begin{pmatrix}0.8\\0\\0.6\end{pmatrix}$.
23. **AHL.** $3p - 2 - 4 = 0$, so $p = 2$.
24. **AHL.** $\overrightarrow{AB} = \begin{pmatrix}-1\\2\\0\end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix}-1\\0\\3\end{pmatrix}$, $\overrightarrow{AB}\times\overrightarrow{AC} = \begin{pmatrix}6\\3\\2\end{pmatrix}$ with magnitude 7, so the area is 3.5. Plane: $6x + 3y + 2z = 6$.
25. **AHL.** y: $2 = 5 - \mu$, so $\mu = 3$; x: $1 + \lambda = 3$, so $\lambda = 2$; z: $3 - 2 = 1$ and $1 + 0 = 1$. They meet at $(3, 2, 1)$. $\cos\theta = \dfrac{1}{\sqrt{2}\sqrt{2}} = \frac{1}{2}$, so the angle is 60°.
26. **AHL.** $(2 + t) + 2(1 - t) + 2t = 10$ gives $t = 6$, so the point is $(8, -5, 12)$. $\mathbf{b}\cdot\mathbf{n} = 1 - 2 + 2 = 1$, $|\mathbf{b}| = |\mathbf{n}| = \sqrt{6}$, so $\sin\phi = \frac{1}{6}$ and $\phi \approx 9.59°$.

Exam tips

- Draw a clear diagram, and redraw any right-angled triangle from a 3D shape flat on the page.
- Check your calculator is in radians for calculus and arc questions, and in degrees for triangle questions given in degrees.
- After using the sine rule for an angle, ask whether the obtuse angle $180° - B$ also works.
- Bearings are clockwise from north with three figures; draw a north line at every point.
- In trigonometric equations, stretch the interval first for $\sin kx$, find every solution, and never divide by a function that could be zero.
- On Paper 1, keep exact values such as $\frac{\sqrt{3}}{2}$ and π; do not round.
- **HL:** check the equation you did not use when testing whether two lines meet; skew lines are neither parallel nor intersecting.
- **HL:** the angle between a line and a plane uses sin, not cos, with the normal vector.

Statistics and probability

International Baccalaureate · IB Diploma · Mathematics: Analysis and Approaches · HL

Scope and prerequisites

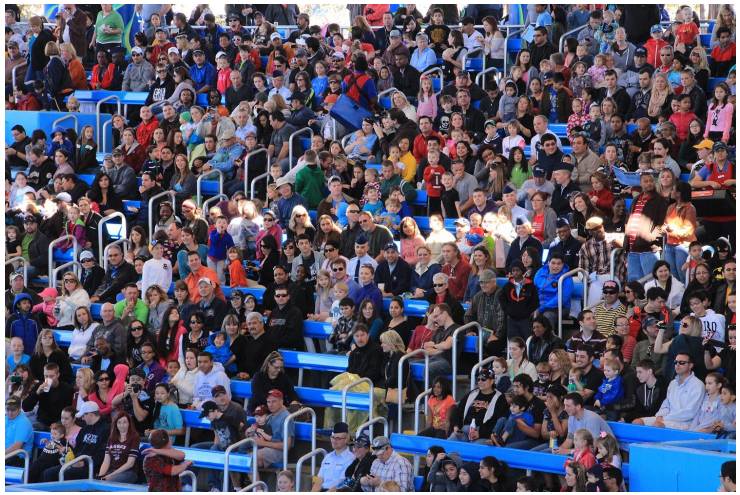
This handout teaches topic 4, Statistics and probability, of Mathematics: Analysis and Approaches, the course first assessed in 2021: sampling and bias, summarising and presenting data, correlation and regression, probability with Venn and tree diagrams, conditional probability and independence, discrete random variables, the binomial distribution and the normal distribution, and, for HL, Bayes' theorem and continuous random variables. It is written for both SL and HL students. HL students also apply the further results in the sections marked AHL.

Prerequisites: fractions, decimals and percentages; substitution into formulae; integration (topic 5) for the HL continuous random variables.

Sampling and data

The biggest poll that failed

In 1936 the American magazine *Literary Digest* mailed about 10 million ballots and received about 2.4 million replies. Its poll predicted that Alf Landon would beat President Franklin Roosevelt easily. Roosevelt won in a landslide, with about 61 % of the vote. A young pollster, George Gallup, predicted the right winner with a sample of only about 50 000. The Digest had taken its names from telephone directories and car registrations, which in 1936 over-represented richer voters, and only the people who chose to reply were counted. A huge biased sample is worse than a small fair one.



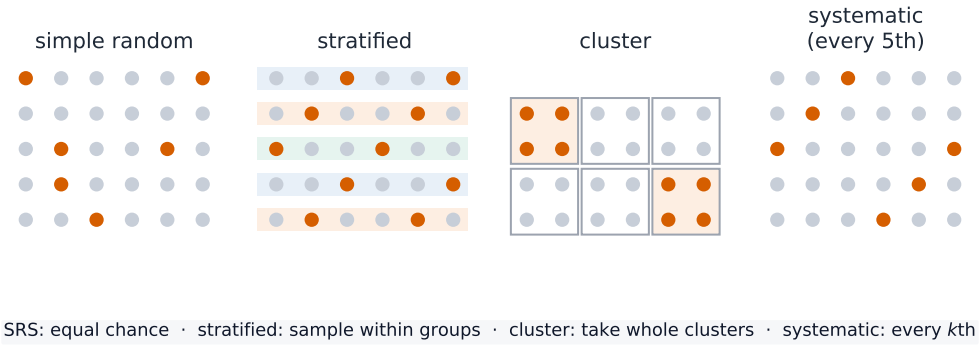
Statistics studies a sample to learn about a whole population.

Populations and samples

- The **population** 总体 is every member of the group being studied; a **sample** 样本 is the part actually measured.
- A sample is **biased** when some members of the population are more likely to be included, so the sample does not represent the population.
- **Discrete data** 离散数据 can take only separate values (number of siblings); **continuous data** 连续数据 can take any value in a range (height, time).

Sampling techniques

Technique	How	Strength	Weakness
Simple random	Every member equally likely, such as random numbers	No selection bias	Needs a full list
Systematic	Every kth member from a list	Quick, spread out	Biased if the list has a pattern
Stratified	Random samples from each group in proportion	Represents every group	Needs group sizes
Quota	Fixed numbers from each group, not random	Quick, cheap	Interviewer chooses: bias
Convenience	Whoever is easy to reach	Easiest	Rarely representative



Four random sampling designs: who gets selected, and how

Worked example. A school has 300 students in Year 10 and 200 in Year 11. A stratified sample of 50 takes $\frac{300}{500} \times 50 = 30$ from Year 10 and 20 from Year 11.

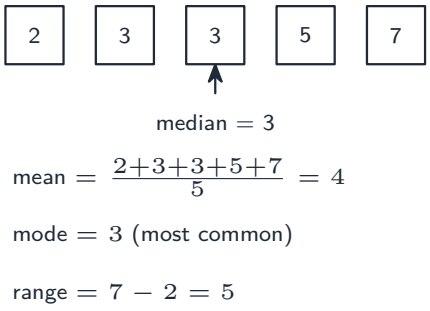
Outliers

An **outlier** 异常值 is a value far from the rest. A common rule: a value is an outlier if it is more than $1.5 \times$ IQR below Q_1 or above Q_3. Outliers should be checked, not simply deleted: a recording error can be corrected, but a genuine extreme value is part of the data.

Summarising data

Measures of centre and spread

For the data $3, 5, 7, 7, 8, 10, 12, 20$ ($n = 8$):



Mean, median, mode and range of a small data set

- **mean** 平均数 $\bar{x} = \dfrac{\sum x}{n} = \dfrac{72}{8} = 9$;
- **median** 中位数 $= \dfrac{7+8}{2} = 7.5$, the middle value of the ordered data;
- **mode** 众数 $= 7$, the most frequent value.

Spread:

- **Quartiles** 四分位数: $Q_1 = 6$ (the median of $3, 5, 7, 7$) and $Q_3 = 11$ (the median of $8, 10, 12, 20$). Technology may use a slightly different method; follow your calculator.
- **Interquartile range** 四分位距 IQR $= Q_3 - Q_1 = 5$. Since $11 + 1.5 \times 5 = 18.5$, the value 20 is an outlier.
- **Standard deviation** 标准差 $\sigma = \sqrt{\dfrac{\sum(x-\bar{x})^2}{n}}$. The squared deviations from 9 add to 192, so the **variance** 方差 is $\sigma^2 = 24$ and $\sigma \approx 4.90$.

The median and IQR resist outliers; the mean and standard deviation are pulled by them. Here the outlier 20 pulls the mean (9) above the median (7.5).

Effect of a change of scale. If every value is multiplied by a and then b is added, the mean and median become $a(\cdot) + b$. Quartiles follow the same rule for $a > 0$; for $a < 0$, the lower and upper quartiles swap, so Q_1 becomes $aQ_3 + b$ and Q_3 becomes $aQ_1 + b$. Meanwhile, the standard deviation and IQR are multiplied by $|a|$ and are not changed by b. Doubling the data above and adding 3 gives mean 21 and standard deviation 9.80.

Grouped data

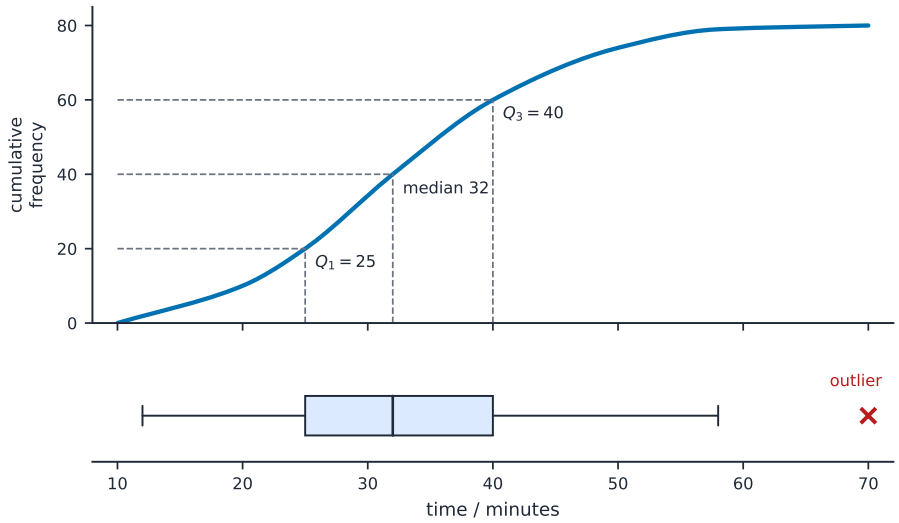
For grouped data, use the **mid-interval values** x and frequencies f: $\bar{x} \approx \dfrac{\sum fx}{\sum f}$.

Time (min)	$0 \le t < 10$	$10 \le t < 20$	$20 \le t < 30$	$30 \le t < 40$
Frequency	4	10	16	10

$\sum fx = 4(5) + 10(15) + 16(25) + 10(35) = 920$, so $\bar{x} \approx \frac{920}{40} = 23$ minutes. The modal class is $20 \le t < 30$. The value is an estimate because the exact data are not known.

Cumulative frequency and box plots

A **cumulative frequency** 累积频率 graph plots the running total against the upper class boundary. Reading across at $\frac{n}{4}$, $\frac{n}{2}$ and $\frac{3n}{4}$ gives estimates of Q_1, the median and Q_3. A **box plot** 箱线图 shows the minimum, Q_1, median, Q_3 and maximum, with outliers marked separately.



Quartiles from a cumulative frequency curve, and the matching box plot

For the 80 students above, the upper fence is $40 + 1.5 \times 15 = 62.5$ minutes, so the

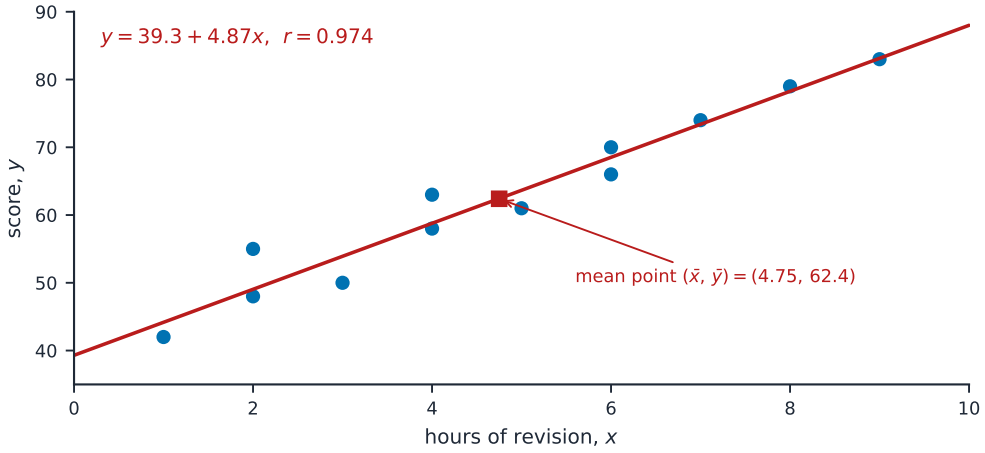
70-minute time is an outlier and the whisker stops at the largest value inside the fence, 58 minutes. When comparing two box plots, compare a measure of centre (median) **and** a measure of spread (IQR) in context.

Correlation and regression

Scatter diagrams and correlation

A **scatter diagram** 散点图 shows paired data (x, y). The **correlation coefficient** 相关系数 r (Pearson's product-moment correlation coefficient), found with technology, measures the strength of the **linear** relationship: $-1 \le r \le 1$.

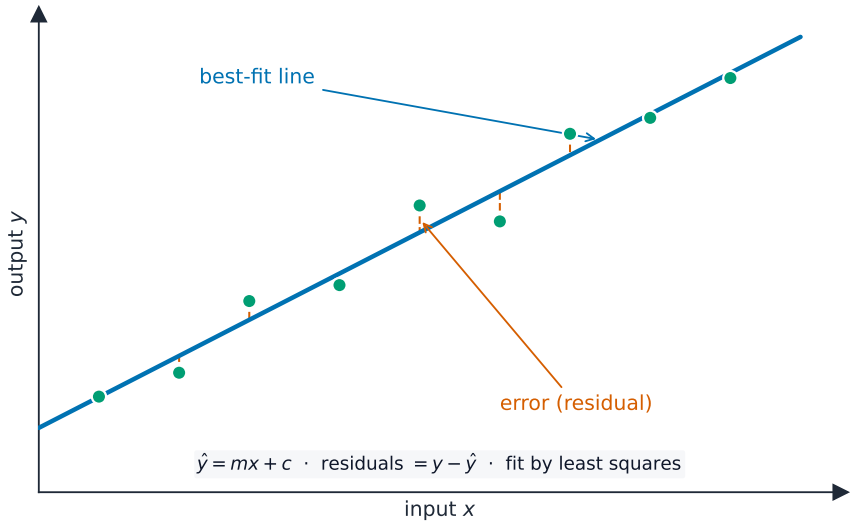
- r close to 1: strong positive linear correlation; close to -1: strong negative; close to 0: little linear correlation.
- Correlation is not causation: ice-cream sales and drowning both rise in hot weather.
- A strong non-linear pattern can still have r close to 0.



Regression line of y on x through the mean point

Regression lines

The **regression line** 回归直线 of y on x, found with technology, minimises the squared vertical distances and always passes through the mean point $(\bar{x}, \bar{y})$. Use it to predict y from x. To predict x from y, use the regression line of x on y, which is a different line unless $r = \pm 1$.



Linear regression fits the line that makes the total squared error (the dashed gaps) as small as possible

> **Worked example.** For the 12 students: $r = 0.974$, $y = 39.3 + 4.87x$.

- Prediction for 5.5 hours: $y = 39.3 + 4.87 \times 5.5 \approx 66$. Reliable: r is strong and 5.5 is within the data (**interpolation** 内插).
- The gradient means each extra hour of revision is associated with about 4.9 more marks.
- A prediction for 15 hours gives 112, more than the maximum mark: **extrapolation** 外推 beyond the data is unreliable.
- To estimate hours from a score of 65, use the x on y line, $x = -7.40 + 0.195y$, giving about 5.3 hours.

Probability

Events and sample spaces

For equally likely outcomes, the **probability** 概率 of event A is $P(A) = \dfrac{n(A)}{n(U)}$, where U is the **sample space** 样本空间. Complementary events: $P(A') = 1 - P(A)$. The **expected number** of times A happens in n trials is $nP(A)$: in 120 rolls of a fair die, expect $120 \times \frac{1}{6} = 20$ sixes.

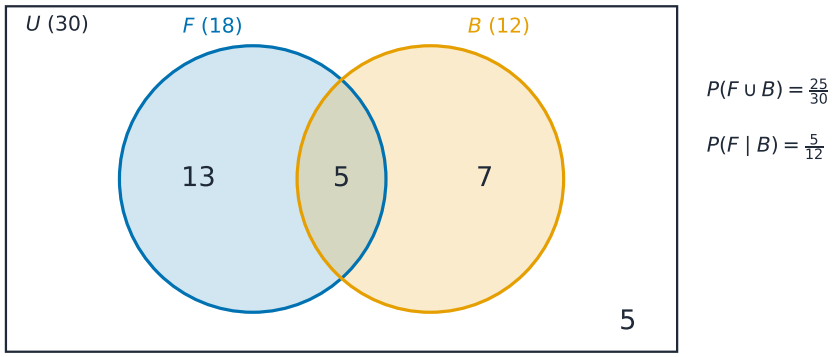


Dice: a familiar starting point for probability.

Combined events and Venn diagrams

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Mutually exclusive 互斥 events cannot both happen: $P(A \cap B) = 0$.



Fill a Venn diagram from the overlap outwards

Worked example. In a class of 30, 18 play football (F), 12 basketball (B) and 5 both.

- Start with the overlap, 5; then F only $= 13$, B only $= 7$; neither $= 30 - 25 = 5$.
- $P(F \cup B) = \frac{18}{30} + \frac{12}{30} - \frac{5}{30} = \frac{25}{30} = \frac{5}{6}$.

Conditional probability and independence

The **conditional probability** 条件概率 of A given B is

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Given that a student plays basketball, the chance they also play football is $P(F \mid B) = \frac{5}{12}$: the sample space shrinks to the 12 basketball players.

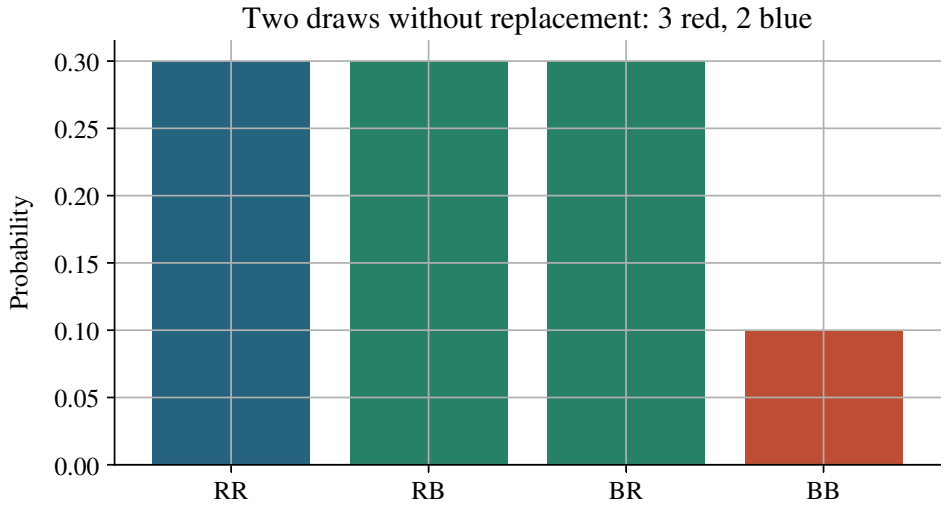
Events are **independent** 独立 if one does not change the probability of the other:

$$P(A \cap B) = P(A)P(B) \quad \text{or equivalently} \quad P(A \mid B) = P(A)$$

Here $P(F)P(B) = 0.6 \times 0.4 = 0.24$, but $P(F \cap B) = \frac{5}{30} \approx 0.167$, so playing football and playing basketball are not independent.

Tree diagrams

A **tree diagram** 树状图 multiplies along branches and adds between branches. Without replacement, the probabilities on the second branches change.



Two draws without replacement from 3 red and 2 blue counters

Worked example. Two counters are taken without replacement from a bag of 3 red and 2 blue.

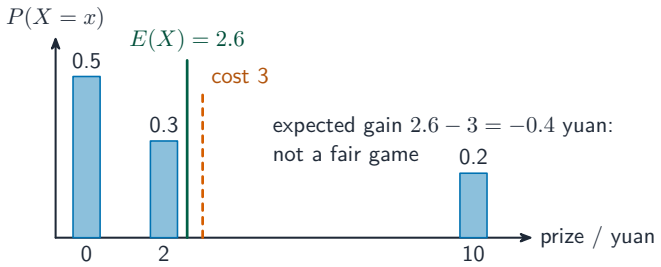
- $P(RR) = \frac{3}{5} \times \frac{2}{4} = 0.3$, $P(BB) = \frac{2}{5} \times \frac{1}{4} = 0.1$.
- $P(\text{same colour}) = 0.3 + 0.1 = 0.4$.
- $P(\text{second is red}) = P(RR) + P(BR) = 0.3 + \frac{2}{5} \times \frac{3}{4} = 0.6$, the same as the first draw, as symmetry suggests.

- $P(\text{first red} \mid \text{second red}) = \dfrac{0.3}{0.6} = 0.5$.

Discrete random variables and the binomial distribution

Probability distributions and expectation

A **discrete random variable** 离散随机变量 X takes separate values x with probabilities $P(X = x)$ that add to 1. Its **expected value** 期望值 is



An unfair game

$$E(X) = \sum x\, P(X = x)$$

> **Worked example.** A game costs ¥3 to play. The prize X is ¥0, ¥2 or ¥10 with probabilities 0.5, 0.3 and 0.2.

- $E(X) = 0(0.5) + 2(0.3) + 10(0.2) = 2.6$.
- The expected gain is $2.6 - 3 = -0.4$ yuan per game: players lose 40 fen per game on average. A game is **fair** if the expected gain is 0.

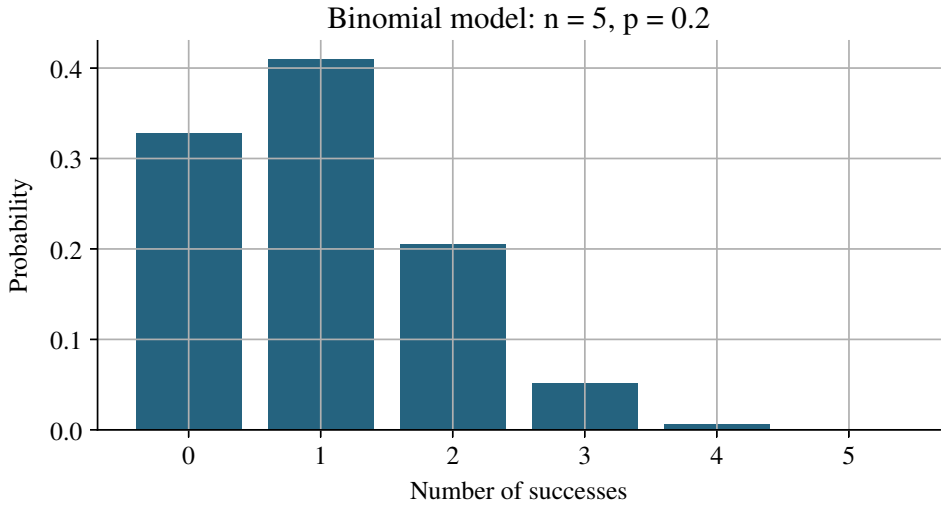
If $P(X = x) = kx$ for $x = 1, 2, 3, 4$, then $k(1 + 2 + 3 + 4) = 1$ gives $k = \frac{1}{10}$.

The binomial distribution

$X \sim B(n, p)$ counts the successes in n independent trials, each with the same probability of success p:

$$P(X = r) = \binom{n}{r} p^r (1-p)^{n-r} \qquad E(X) = np \qquad \operatorname{Var}(X) = np(1-p)$$

Check the conditions in context: a fixed number of trials, two outcomes, independence and constant p.



The binomial distribution B(5, 0.2)

> **Worked example.** 20 % of seeds fail to grow. Five seeds are planted; X is the number that fail, $X \sim B(5, 0.2)$.

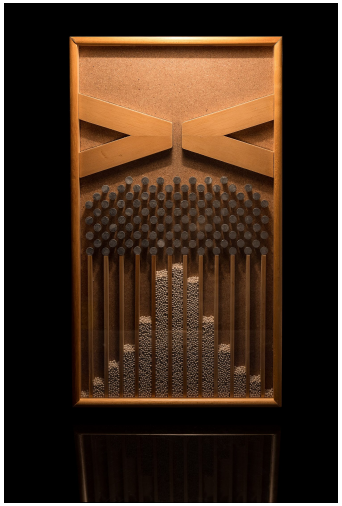
- $P(X = 1) = \binom{5}{1}(0.2)(0.8)^4 = 0.4096$.
- $P(X \geq 1) = 1 - P(X = 0) = 1 - 0.8^5 \approx 0.672$.
- $E(X) = 5 \times 0.2 = 1$ and $\operatorname{Var}(X) = 5 \times 0.2 \times 0.8 = 0.8$.

On calculator papers, use the binomial pdf for $P(X = r)$ and the cdf for $P(X \leq r)$. For $P(X \geq r)$ use $1 - P(X \leq r - 1)$.

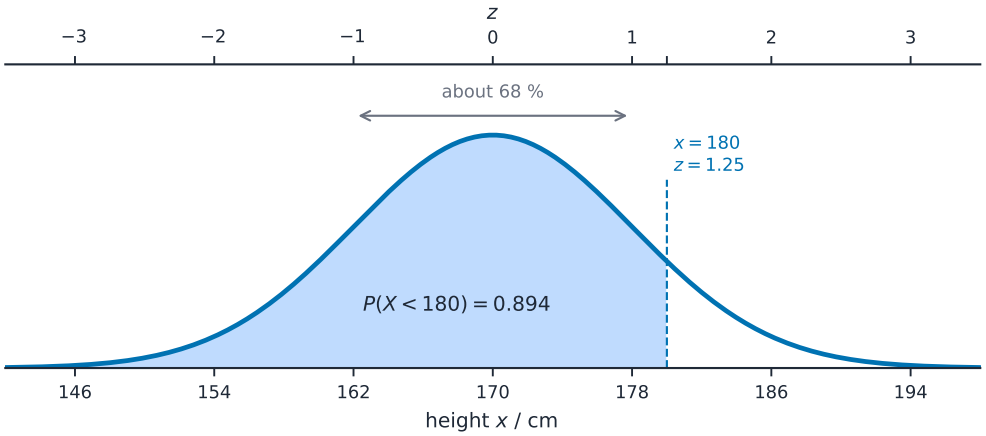
The normal distribution

Properties

Many measurements, such as heights and measurement errors, follow a **normal distribution** 正态分布 $X \sim N(\mu, \sigma^2)$: a symmetric bell-shaped curve centred on the mean μ, with total area 1. About 68 % of values lie within 1σ of the mean, 95 % within 2σ and 99.7 % within 3σ. For a continuous variable, $P(X = a) = 0$, so $P(X < a) = P(X \leq a)$.



A Galton board: balls falling through pins pile up into the bell-shaped normal distribution.



Probability is area under the curve; z counts standard deviations from the mean

Calculations and standardising

The **standardised value** 标准分数

$$z = \frac{x - \mu}{\sigma}$$

counts how many standard deviations x is from the mean. $Z = \dfrac{X - \mu}{\sigma}$ follows $N(0, 1)$.

> **Worked example.** Heights $X \sim N(170, 8^2)$ cm.
>
> - $P(X < 180) = 0.894$ (by calculator; $z = 1.25$).
> - $P(160 < X < 180) = 0.789$.
> - The height exceeded by only 10 % of people: inverse normal, $P(X < h) = 0.9$, gives $h \approx 180.3$ cm.

Unknown mean or standard deviation

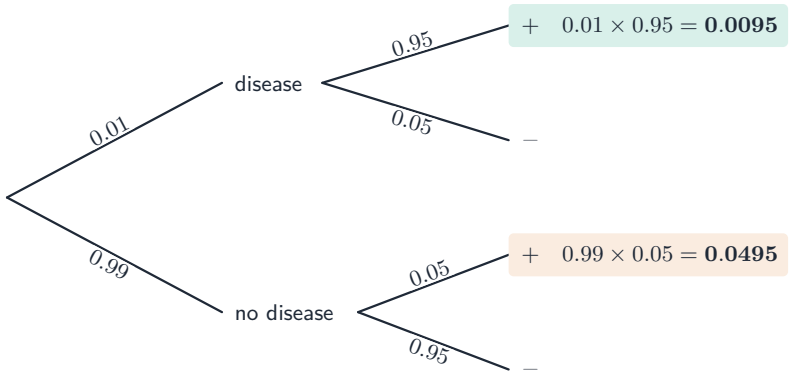
Use the inverse normal on Z to find z, then solve $z = \dfrac{x - \mu}{\sigma}$.

> **Worked example.** $X \sim N(\mu, 4^2)$ and $P(X > 50) = 0.05$. Then $P(Z > z) = 0.05$ gives $z = 1.645$, so $\dfrac{50 - \mu}{4} = 1.645$ and $\mu \approx 43.4$.

> **Worked example.** $X \sim N(25, \sigma^2)$ and $P(X < 20) = 0.2$. Then $z = -0.8416$, so $\dfrac{20 - 25}{\sigma} = -0.8416$ and $\sigma \approx 5.94$.

AHL: Bayes' theorem

When the probability we know runs the wrong way round, **Bayes' theorem** 贝叶斯定理 turns it round:



$$P(D \mid +) = \frac{0.0095}{0.0095 + 0.0495} \approx 0.161$$

Most positive results come from the large group without the disease

$$P(A \mid B) = \frac{P(A)P(B \mid A)}{P(A)P(B \mid A) + P(A')P(B \mid A')}$$

With more than two possible causes A_1, A_2, A_3, the denominator adds $P(A_i)P(B \mid A_i)$ over all of them.

Worked example. A disease affects 1 % of people. A test is positive for 95 % of people with the disease and for 5 % of people without it. Find the probability that a person with a positive test has the disease.

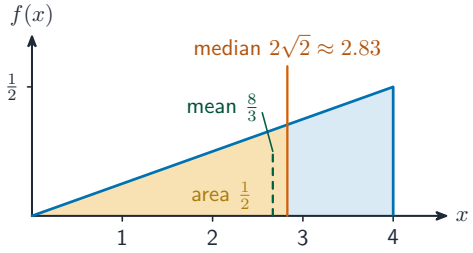
- $P(D)P(+ \mid D) = 0.01 \times 0.95 = 0.0095$.
- $P(D')P(+ \mid D') = 0.99 \times 0.05 = 0.0495$.
- $P(D \mid +) = \dfrac{0.0095}{0.0095 + 0.0495} \approx 0.161$.

Only about 16 % of positive results are true positives, because the disease is rare and false positives from the 99 % without it outnumber the true ones. A tree diagram makes this clear.

AHL: Continuous random variables

Probability density functions

A **continuous random variable** 连续随机变量 X has a **probability density function** 概率密度函数 $f(x) \geq 0$ with total area 1. Probabilities are areas:



Median and mean of $f(x) = x/8$

$$\int f(x)\,dx = 1 \qquad P(a < X < b) = \int_a^b f(x)\,dx$$

$$E(X) = \mu = \int x f(x)\,dx \qquad \operatorname{Var}(X) = \int x^2 f(x)\,dx - \mu^2$$

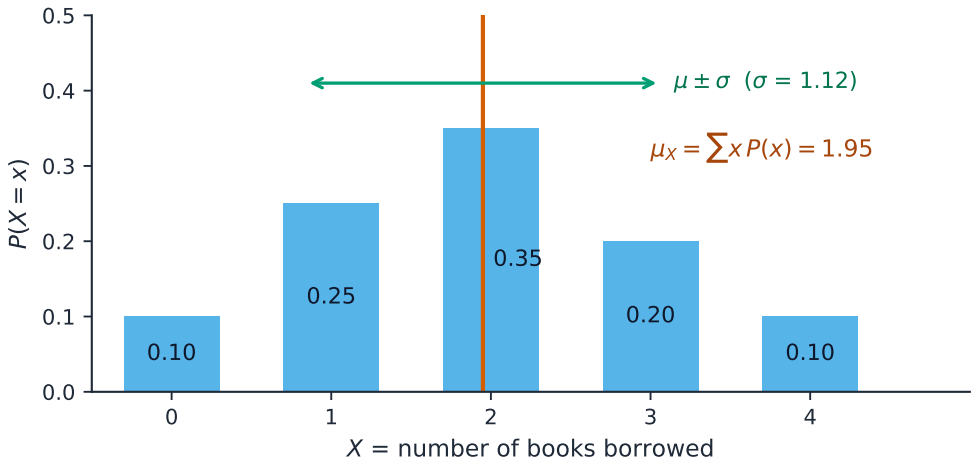
The **median** m satisfies $\displaystyle\int_{-\infty}^{m} f(x)\,dx = \frac{1}{2}$; the **mode** is where $f(x)$ is greatest.

Worked example. $f(x) = \dfrac{x}{8}$ for $0 \leq x \leq 4$, and 0 otherwise.

- Check: $\displaystyle\int_0^4 \frac{x}{8}\,dx = \frac{16}{16} = 1$.
- $E(X) = \displaystyle\int_0^4 \frac{x^2}{8}\,dx = \frac{64}{24} = \frac{8}{3}$.
- $E(X^2) = \displaystyle\int_0^4 \frac{x^3}{8}\,dx = 8$, so $\operatorname{Var}(X) = 8 - \frac{64}{9} = \frac{8}{9}$.
- Median: $\dfrac{m^2}{16} = \dfrac{1}{2}$, so $m = \sqrt{8} \approx 2.83$. Mode $= 4$, where f is largest.

Variance of a discrete random variable and linear changes

$\operatorname{Var}(X) = E(X^2) - [E(X)]^2 \qquad E(aX + b) = aE(X) + b \qquad \operatorname{Var}(aX + b) = a^2 \operatorname{Var}(X)$



A discrete probability distribution with its mean and standard deviation

For the game above, $E(X^2) = 4(0.3) + 100(0.2) = 21.2$, so $\operatorname{Var}(X) = 21.2 - 2.6^2 = 14.44$ and the standard deviation is 3.8 yuan. If the prizes are doubled and ¥1 added to each, the mean becomes $2(2.6) + 1 = 6.2$ and the variance $4 \times 14.44 = 57.76$.

Common mistakes.

- Predicting x from the regression line of y on x. That line is built to predict y only.
- Extrapolating far outside the data, or reading a strong correlation as proof of cause.
- Using $P(A \cap B) = P(A)P(B)$ without knowing the events are independent.
- Dividing by the wrong probability: $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$, not over $P(A)$.
- Writing $P(X \geq 3) = 1 - P(X \leq 3)$. For a discrete variable it is $1 - P(X \leq 2)$.
- Reading the variance as the standard deviation: $X \sim N(50, 16)$ has $\sigma = 4$.
- Using the expected value as a value that must occur: $E(X) = 2.5$ for a count is a long-run mean.
- **HL:** leaving the false positives out of the denominator in Bayes' theorem.
- **HL:** forgetting that the area under a probability density function is 1 when finding a constant k.
- **HL:** writing $\operatorname{Var}(aX + b) = a \operatorname{Var}(X) + b$. It is $a^2 \operatorname{Var}(X)$.

Practice

1. Name each sampling technique and give one weakness: (a) every 10th name on a list; (b) random samples from each year group in proportion to its size; (c) the first 20 people through the gate.
2. For $4, 7, 7, 9, 12, 15, 18, 40$, find the mean, median, quartiles and IQR, and decide whether 40 is an outlier.
3. A data set has mean 20 and standard deviation 3. Each value is multiplied by 3 and then 5 is subtracted. Find the new mean and standard deviation.
4. Estimate the mean height and state the modal class: $150 \le h < 160$: 6; $160 \le h < 170$: 14; $170 \le h < 180$: 15; $180 \le h < 190$: 5.
5. For data with x from 5 to 20, the regression line is $y = 2.4x + 11$ with $r = 0.91$. Predict y when $x = 12$, and comment on using the line when $x = 50$.
6. A study finds $r = -0.35$ between hours of screen time and sleep. Interpret this, and explain why it does not show that screen time causes less sleep.
7. $P(A) = 0.5$, $P(B) = 0.4$ and $P(A \cup B) = 0.7$. Find $P(A \cap B)$ and $P(A \mid B)$, and decide whether A and B are independent.
8. Two fair dice are rolled. Find the probability that the total is 8, and that at least one die shows 6.
9. A bag has 4 green and 6 yellow counters. Two are taken without replacement. Find the probability that both are green, and that they are different colours.
10. $P(X = x) = k(x + 1)$ for $x = 0, 1, 2, 3$. Find k and $E(X)$.
11. A game costs ¥5. A player wins ¥20 with probability 0.1 and ¥5 with probability 0.3, and otherwise nothing. Find the expected gain and decide whether the game is fair.
12. $X \sim B(10, 0.3)$. Find $P(X = 3)$, $P(X \le 2)$, $E(X)$ and $\mathrm{Var}(X)$.
13. $X \sim N(50, 5^2)$. Find $P(X > 58)$ and $P(45 < X < 55)$.
14. For $X \sim N(50, 5^2)$, find the value exceeded by the top 10 %.
15. $X \sim N(\mu, 4^2)$ and $P(X < 30) = 0.2$. Find μ.
16. **AHL.** Machine A makes 60 % of a factory's items, of which 2 % are faulty; machine B makes 40 %, of which 5 % are faulty. An item is faulty. Find the probability it came from machine A.
17. **AHL.** $f(x) = kx^2$ for $0 \le x \le 3$. Find k, $E(X)$, the median and the mode.
18. **AHL.** For the distribution in question 10, find $\mathrm{Var}(X)$ and $\mathrm{Var}(3X - 2)$.
19. **AHL.** $E(X) = 4$ and $\mathrm{Var}(X) = 2$. Find $E(5 - 2X)$ and $\mathrm{Var}(5 - 2X)$.

Answers.

1. (a) Systematic: biased if the list has a repeating pattern. (b) Stratified: needs the size of each group. (c) Convenience: those arriving first may differ from the rest, so it is rarely representative.
2. Mean $= \frac{112}{8} = 14$; median $= \frac{9+12}{2} = 10.5$; $Q_1 = 7$, $Q_3 = 16.5$; IQR $= 9.5$. Upper fence $16.5 + 14.25 = 30.75$, so 40 is an outlier.
3. Mean $3(20) - 5 = 55$; standard deviation $3 \times 3 = 9$.
4. $\sum fx = 6(155) + 14(165) + 15(175) + 5(185) = 6790$, so $\bar{x} \approx \frac{6790}{40} = 169.75$ cm. Modal class $170 \le h < 180$.
5. $y = 2.4(12) + 11 = 39.8$, reliable since r is strong and 12 is within the data. $x = 50$ is far outside the data, so the prediction is extrapolation and unreliable.
6. A weak negative linear correlation: more screen time tends to go with slightly less sleep. Correlation does not show cause: another factor, such as stress, could affect both, and the relationship is weak.
7. $P(A \cap B) = 0.5 + 0.4 - 0.7 = 0.2$; $P(A \mid B) = \frac{0.2}{0.4} = 0.5$. Since $P(A)P(B) = 0.2 = P(A \cap B)$, they are independent.
8. Totals of 8: $(2,6), (3,5), (4,4), (5,3), (6,2)$, so $\frac{5}{36}$. At least one 6: $1 - \left(\frac{5}{6}\right)^2 = \frac{11}{36}$.
9. $P(GG) = \frac{4}{10} \times \frac{3}{9} = \frac{2}{15}$. $P(\text{different}) = 2 \times \frac{4}{10} \times \frac{6}{9} = \frac{8}{15}$.
10. $k(1+2+3+4) = 10k = 1$, so $k = 0.1$. $E(X) = 0(0.1) + 1(0.2) + 2(0.3) + 3(0.4) = 2$.
11. Expected winnings $= 20(0.1) + 5(0.3) = 3.5$, so the expected gain is $3.5 - 5 = -1.5$ yuan. Not fair: players lose on average.
12. $P(X = 3) = \binom{10}{3}(0.3)^3(0.7)^7 \approx 0.267$; $P(X \le 2) \approx 0.383$; $E(X) = 3$; $\mathrm{Var}(X) = 2.1$.
13. $P(X > 58) \approx 0.0548$; $P(45 < X < 55) \approx 0.683$.
14. $P(X < x) = 0.9$ gives $x \approx 56.4$.
15. $z = -0.8416$, so $\frac{30-\mu}{4} = -0.8416$ and $\mu \approx 33.4$.
16. **AHL.** $P(A \mid F) = \dfrac{0.6 \times 0.02}{0.6 \times 0.02 + 0.4 \times 0.05} = \dfrac{0.012}{0.032} = 0.375$.
17. **AHL.** $k \displaystyle\int_0^3 x^2\,dx = 9k = 1$, so $k = \frac{1}{9}$. $E(X) = \displaystyle\int_0^3 \frac{x^3}{9}\,dx = \frac{81}{36} = 2.25$. Median: $\frac{m^3}{27} = \frac{1}{2}$, so $m = \frac{3}{\sqrt[3]{2}} \approx 2.38$. Mode $= 3$.
18. **AHL.** $E(X^2) = 0(0.1) + 1(0.2) + 4(0.3) + 9(0.4) = 5$, so $\mathrm{Var}(X) = 5 - 2^2 = 1$ and $\mathrm{Var}(3X - 2) = 9$.
19. **AHL.** $E(5 - 2X) = 5 - 8 = -3$; $\mathrm{Var}(5 - 2X) = 4 \times 2 = 8$.

Exam tips

- Name the sampling method and say why it might be biased in the context given.
- Compare data sets with one measure of centre **and** one measure of spread, in context.
- Use the y on x regression line only to predict y, only within the data, and only when r is strong.
- State the distribution and its parameters before calculating: $X \sim B(10, 0.3)$, $X \sim N(50, 5^2)$.

- For $P(X \geq r)$ with a binomial, use $1 - P(X \leq r - 1)$.
- Draw a quick normal sketch with the mean and the shaded region before using the calculator.
- Test independence with numbers: compare $P(A \cap B)$ with $P(A)P(B)$.

Calculus

International Baccalaureate · IB Diploma · Mathematics: Analysis and Approaches · HL

Scope and prerequisites

This handout teaches topic 5, Calculus, of Mathematics: Analysis and Approaches, the course first assessed in 2021: the derivative as a gradient and a rate of change, the rules of differentiation, stationary points and optimisation, kinematics, integration and areas, and, for HL, limits and first principles, implicit differentiation and related rates, further integration techniques, volumes of revolution, differential equations and Maclaurin series. It is written for both SL and HL students. HL students also apply the further results in the sections marked AHL.

Prerequisites: functions and their graphs, exponentials and logarithms (topic 2), and trigonometric functions in radians (topic 3). In calculus, angles are always in radians.

Gradients and derivatives

From average to instant

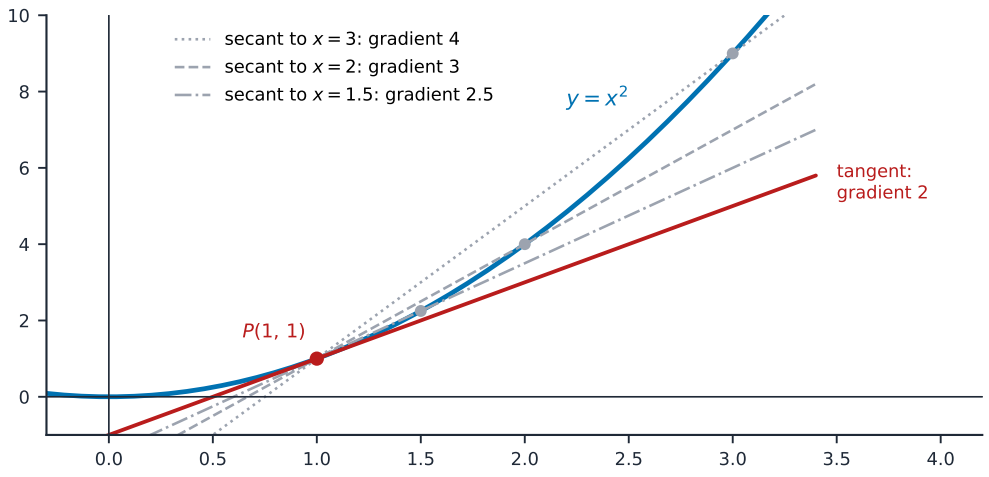
A car's trip computer shows 72 km/h as the average speed of a 36 km journey that took 30 minutes. The speedometer shows something different: the speed at this instant. An average speed is the gradient of a chord on a distance-time graph; an instantaneous speed is the gradient of the tangent. Calculus is the mathematics of getting from the first to the second by making the time interval smaller and smaller.

Secants and the tangent

The gradient of a chord, or **secant** 割线, between $P(x, f(x))$ and a nearby point $Q(x + h, f(x + h))$ is

$$\frac{f(x + h) - f(x)}{h}$$

As Q moves towards P, $h \to 0$ and the secant approaches the **tangent** 切线 at P. The limit of the secant gradients is the **derivative** 导数 $f'(x)$, also written $\frac{dy}{dx}$.



Secant gradients approach the tangent gradient

For $y = x^2$ at $x = 1$ the secant gradients are 4, 3, 2.5, 2.1, 2.01, … as Q comes closer, so the derivative at $x = 1$ is 2.

The derivative has two meanings:

- the **gradient** of the curve $y = f(x)$ at a point;
- the **rate of change** of y with respect to x: if V is a volume in litres and t is time in seconds, $\dfrac{dV}{dt}$ is in litres per second.

The power rule

For any real n, the derivative of x^n is nx^{n-1}. Constants multiply through and terms are differentiated one by one:

$$\frac{d}{dx}(ax^n) = anx^{n-1} \qquad \frac{d}{dx}(c) = 0$$

Worked example. $y = 4x^3 - 5x + \dfrac{2}{x} + 7$. Write $\dfrac{2}{x} = 2x^{-1}$. Then

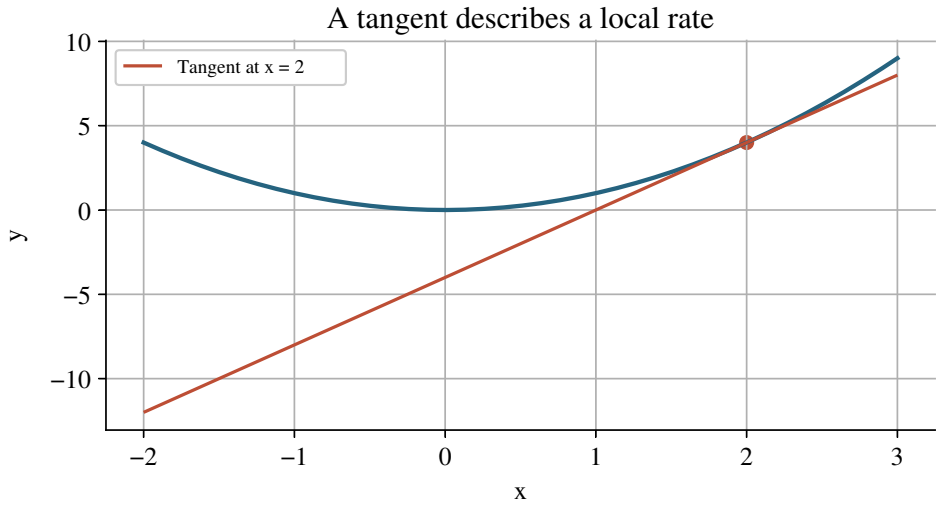
$$\frac{dy}{dx} = 12x^2 - 5 - 2x^{-2} = 12x^2 - 5 - \frac{2}{x^2}$$

Increasing and decreasing

f is **increasing** where $f'(x) > 0$ and **decreasing** where $f'(x) < 0$. For $f(x) = x^3 - 3x$, $f'(x) = 3x^2 - 3 = 3(x-1)(x+1)$, which is negative for $-1 < x < 1$, so f is decreasing there and increasing for $x < -1$ and $x > 1$.

Tangents and normals

At $x = a$, the tangent has gradient $m = f'(a)$ and passes through $(a, f(a))$. The **normal** 法线 is perpendicular to the tangent, with gradient $-\dfrac{1}{m}$ when $m \neq 0$. If the tangent is horizontal, the normal is the vertical line $x = a$.



The tangent to $y = x^2$ at (2, 4)

Worked example. For $y = x^2$ at $x = 2$: $y = 4$ and $\dfrac{dy}{dx} = 2x = 4$.

- Tangent: $y - 4 = 4(x - 2)$, so $y = 4x - 4$.
- Normal: gradient $-\frac{1}{4}$, so $y - 4 = -\frac{1}{4}(x - 2)$, or $x + 4y - 18 = 0$.

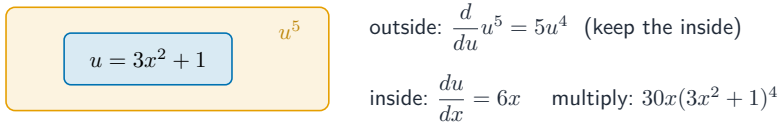
Rules of differentiation

Standard derivatives

$f(x)$	$f'(x)$
x^n	nx^{n-1}
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
e^x	e^x
$\ln x$	$\dfrac{1}{x}$

Chain, product and quotient rules

The **chain rule** 链式法则 differentiates a composite function: if $y = f(u)$ and $u = g(x)$,



Outside, then inside

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Differentiate the outside, keep the inside, then multiply by the derivative of the inside.

The **product rule** 乘积法则 and **quotient rule** 商法则, for u and v functions of x:

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx} \qquad \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Worked examples.

- Chain: $\frac{d}{dx}(3x^2+1)^5 = 5(3x^2+1)^4 \times 6x = 30x(3x^2+1)^4$.
- Chain: $\frac{d}{dx}e^{\sin x} = \cos x\, e^{\sin x}$, and $\frac{d}{dx}\ln(x^2+4) = \frac{2x}{x^2+4}$.
- Product: $\frac{d}{dx}(x^2e^{3x}) = x^2 \times 3e^{3x} + e^{3x} \times 2x = xe^{3x}(3x+2)$.
- Quotient: $\frac{d}{dx}\left(\frac{\ln x}{x}\right) = \frac{x \times \frac{1}{x} - \ln x \times 1}{x^2} = \frac{1 - \ln x}{x^2}$.

The second derivative

Differentiating again gives the **second derivative** 二阶导数 $f''(x)$ or $\frac{d^2y}{dx^2}$: the rate of change of the gradient. Where $f''(x) > 0$ the graph is **concave up** (the gradient is increasing); where $f''(x) < 0$ it is concave down.

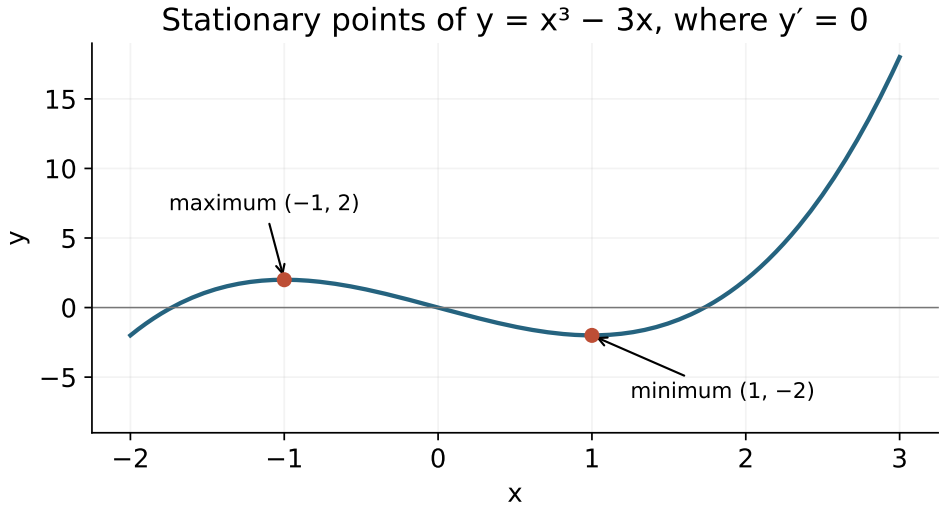
Stationary points and optimisation

Finding and classifying stationary points

A **stationary point** 驻点 is where $f'(x) = 0$: the tangent is horizontal. It can be a local maximum, a local minimum or a stationary point of inflection.



Fenced enclosures: optimization finds the dimensions that maximise area for a fixed fence length



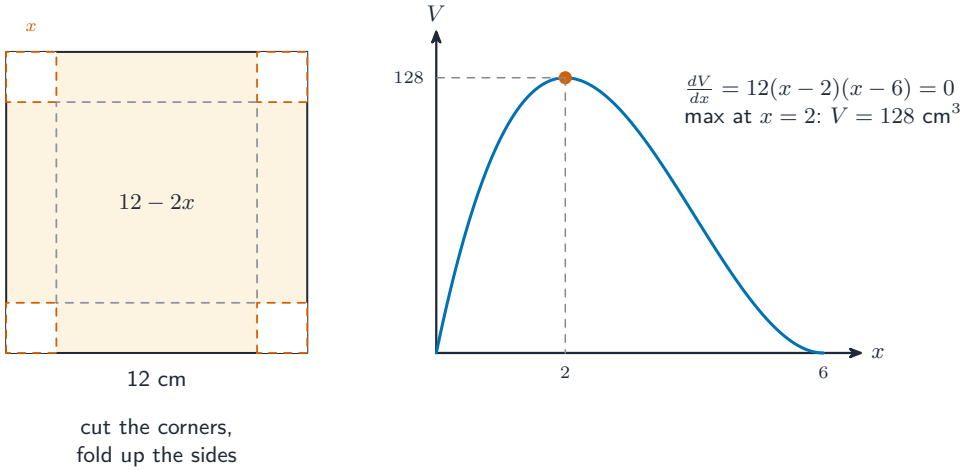
Stationary points of $y = x^3 - 3x$

> **Worked example.** $f(x) = x^3 - 3x$.
>
> - $f'(x) = 3x^2 - 3 = 0$ gives $x = \pm 1$. The points are $(-1, 2)$ and $(1, -2)$.
> - $f''(x) = 6x$. At $x = -1$, $f'' = -6 < 0$: a local **maximum**. At $x = 1$, $f'' = 6 > 0$: a local **minimum**.
> - If $f'' = 0$ the test fails; check the sign of f' on each side instead.

A **point of inflection** 拐点 is where the concavity changes: $f''(x) = 0$ **and** f'' changes sign. For $x^3 - 3x$, $f''(x) = 6x$ changes sign at $x = 0$, so $(0, 0)$ is a point of inflection (not stationary, since $f'(0) = -3$). For $y = x^4$, $f''(0) = 0$ but f'' does not change sign, so the origin is a minimum, not an inflection.

Optimisation

To find the largest or smallest value of a quantity: write it as a function of one variable, differentiate, set the derivative to zero, solve, and check it is a maximum or minimum. Also check the ends of the allowed domain.



Write V in one variable, then the derivative finds the maximum: $x = 2$ gives 128 cm^3

> **Worked example.** Equal squares of side x cm are cut from the corners of a 12 cm by 12 cm sheet, and the sides are folded up to make an open box. Find the largest volume.

- $V = x(12 - 2x)^2 = 4x^3 - 48x^2 + 144x$, with $0 < x < 6$.
- $\dfrac{dV}{dx} = 12x^2 - 96x + 144 = 12(x - 2)(x - 6) = 0$, so $x = 2$ (since $x = 6$ is outside the domain).
- $\dfrac{d^2V}{dx^2} = 24x - 96 = -48 < 0$ at $x = 2$: a maximum.
- $V = 2 \times 8^2 = 128$ cm^3.

Kinematics

Displacement, velocity and acceleration

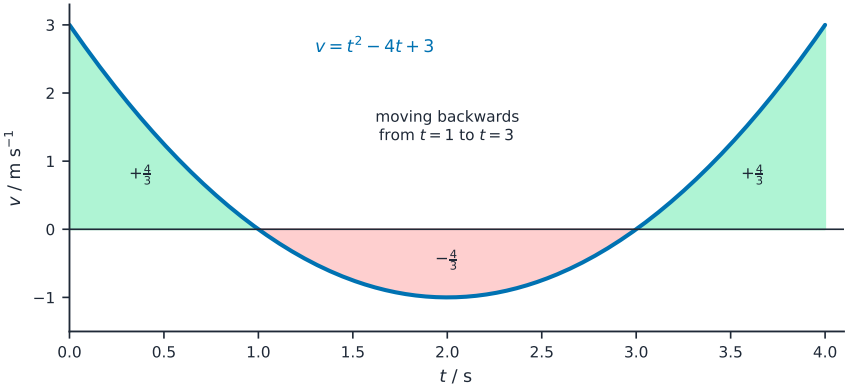
For a particle moving in a straight line, with **displacement** 位移 s from a fixed point at time t:



A roller coaster is calculus you can feel: velocity is the derivative of position, and acceleration is the derivative of velocity

$$v = \frac{ds}{dt} \qquad a = \frac{dv}{dt} = \frac{d^2s}{dt^2} \qquad s = \int v\,dt$$

The **velocity** 速度 v has a sign that shows direction; speed is $|v|$. The particle is at rest when $v = 0$ and changes direction where v changes sign. It is speeding up when v and a have the same sign.



Signed areas give displacement; total area gives distance

Worked example. $v = t^2 - 4t + 3$ m/s for $0 \le t \le 4$.

- $v = (t-1)(t-3) = 0$ at $t = 1$ and $t = 3$: the particle changes direction twice.
- $a = 2t - 4$; at $t = 0$, $a = -4$ m/s^2.
- Displacement from $t = 0$ to 4: $\displaystyle\int_0^4 (t^2 - 4t + 3)\,dt = \left[\frac{t^3}{3} - 2t^2 + 3t\right]_0^4 = \frac{4}{3}$ m.
- Distance travelled: $\displaystyle\int_0^4 |v|\,dt = \frac{4}{3} + \frac{4}{3} + \frac{4}{3} = 4$ m.

Displacement and distance differ whenever the particle turns back.

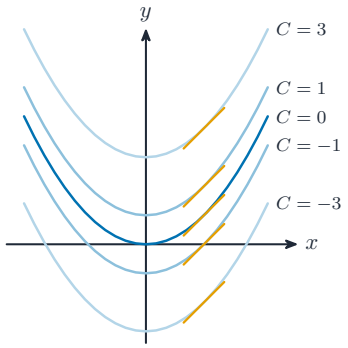
Integration

Antiderivatives

Integration 积分 reverses differentiation. The **indefinite integral** 不定积分 includes a constant C, because constants differentiate to zero:



A water tank: integration accumulates change – total volume from a rate of flow



$\int 2x\,dx = x^2 + C$: every curve has gradient 2 at $x = 1$

Why an antiderivative needs + C

$f(x)$	$\displaystyle\int f(x)\,dx$
x^n, $n \ne -1$	$\dfrac{x^{n+1}}{n+1} + C$
$\dfrac{1}{x}$	$\ln\lvert x\rvert + C$
e^x	$e^x + C$
$\sin x$	$-\cos x + C$
$\cos x$	$\sin x + C$

For a linear inside, $\displaystyle\int f(ax+b)\,dx = \frac{1}{a}F(ax+b) + C$, such as $\displaystyle\int e^{2x}\,dx = \frac{1}{2}e^{2x} + C$.

Worked example. $\displaystyle\int \left(6x^2 - \frac{4}{x} + e^{2x}\right) dx = 2x^3 - 4\ln|x| + \frac{1}{2}e^{2x} + C$.

A boundary condition fixes C: if $\dfrac{dy}{dx} = 6x^2$ and the curve passes through $(1, 5)$, then $y = 2x^3 + C$ and $5 = 2 + C$, so $y = 2x^3 + 3$.

Integration by inspection

When the integrand is a function of $g(x)$ multiplied by $g'(x)$, reverse the chain rule:

$$\int f'(g(x))\,g'(x)\,dx = f(g(x)) + C$$

- $\displaystyle\int 2x(x^2+1)^3\,dx = \frac{(x^2+1)^4}{4} + C.$
- $\displaystyle\int \cos x\, e^{\sin x}\,dx = e^{\sin x} + C.$

- $\displaystyle\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\ln(x^2+1) + C.$

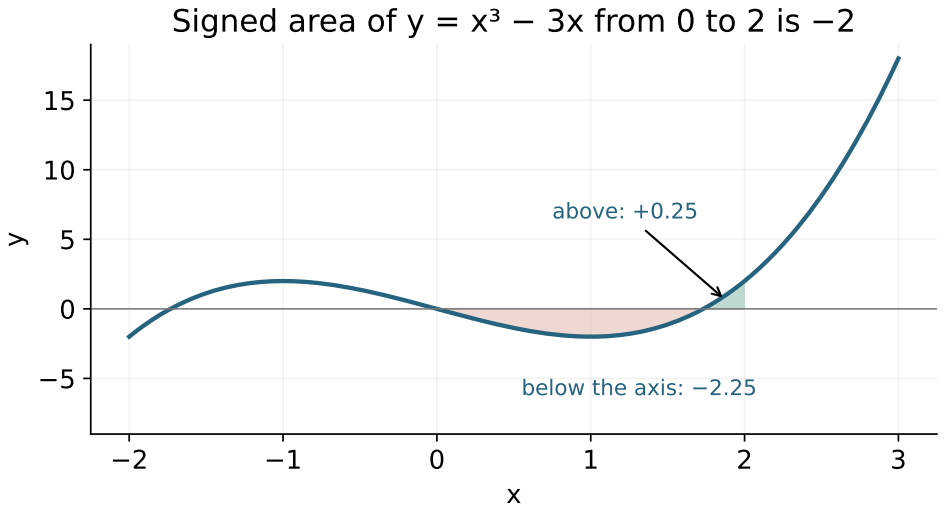
Definite integrals and area

The **definite integral** 定积分 is

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative. Between a curve above the x-axis and the axis, it gives the area. Where the curve is below the axis, the integral is negative.

Worked example. $\displaystyle\int_1^2 (3x^2+1)\,dx = \left[x^3+x\right]_1^2 = 10 - 2 = 8.$



Split the integral where the curve crosses the axis

Worked example. Find the area between $y = x^3 - 3x$ and the x-axis for $0 \le x \le 2$. The curve crosses the axis at $x = \sqrt{3}$.

- $\displaystyle\int_0^{\sqrt{3}} (x^3 - 3x)\,dx = \left[\frac{x^4}{4} - \frac{3x^2}{2}\right]_0^{\sqrt{3}} = \frac{9}{4} - \frac{9}{2} = -\frac{9}{4}$, an area of $\frac{9}{4}$.
- $\displaystyle\int_{\sqrt{3}}^2 (x^3 - 3x)\,dx = (4-6) - \left(-\frac{9}{4}\right) = \frac{1}{4}.$
- Total area $= \frac{9}{4} + \frac{1}{4} = \frac{5}{2}$. The single integral $\int_0^2$ gives -2, which is not the area.

The area between two curves, with $f(x) \ge g(x)$ on $[a, b]$, is

$$A = \int_a^b \big(f(x) - g(x)\big)\,dx$$

For $y = x$ and $y = x^2$, which meet at $x = 0$ and $x = 1$: $A = \displaystyle\int_0^1 (x - x^2)\,dx = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$.

On calculator papers, a definite integral can be evaluated with the graphing calculator; write the integral with its limits first, then the value.

AHL: Limits, first principles and further differentiation

First principles

The derivative from **first principles** 第一原理 is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Worked example. $f(x) = x^3$:

$$\frac{(x+h)^3 - x^3}{h} = \frac{3x^2h + 3xh^2 + h^3}{h} = 3x^2 + 3xh + h^2 \to 3x^2 \text{ as } h \to 0$$

A function is **continuous** at a if $\lim_{x\to a} f(x) = f(a)$, and **differentiable** at a if this derivative limit exists. $|x|$ is continuous at 0 but not differentiable there (the left and right gradients are -1 and 1).

L'Hôpital's rule

Suppose f and g are differentiable near a (except possibly at a), $g'(x) \neq 0$ there, and $\lim f(x) = \lim g(x) = 0$ (or both are infinite). If the limit of $f'(x)/g'(x)$ exists, including an infinite limit, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

- $\displaystyle\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = 1.$
- $\displaystyle\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} = \lim_{x\to 0} \frac{e^x - 1}{2x} = \lim_{x\to 0} \frac{e^x}{2} = \frac{1}{2}$, using the rule twice.

Check the $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form each time before applying it.

Further derivatives

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
$\tan x$	$\sec^2 x$	a^x	$a^x \ln a$
$\sec x$	$\sec x \tan x$	$\log_a x$	$\dfrac{1}{x \ln a}$
$\csc x$	$-\csc x \cot x$	$\arcsin x$	$\dfrac{1}{\sqrt{1-x^2}}$
$\cot x$	$-\csc^2 x$	$\arccos x$	$-\dfrac{1}{\sqrt{1-x^2}}$
		$\arctan x$	$\dfrac{1}{1+x^2}$

Implicit differentiation

When y is not given as a function of x, differentiate both sides with respect to x, using the chain rule on each y term: $\dfrac{d}{dx}(y^2) = 2y\dfrac{dy}{dx}$. This is **implicit differentiation** 隐函数求导.

> **Worked example.** Find the gradient of $x^2 + y^2 = 25$ at $(3, 4)$.

- $2x + 2y\dfrac{dy}{dx} = 0$, so $\dfrac{dy}{dx} = -\dfrac{x}{y}$.
- At $(3, 4)$: $\dfrac{dy}{dx} = -\dfrac{3}{4}$.

Products need the product rule: $\dfrac{d}{dx}(x^2 y) = 2xy + x^2\dfrac{dy}{dx}$.

Related rates

When two quantities are linked and both change with time, the chain rule links their rates. These are **related rates** 相关变化率.

> **Worked example.** A spherical balloon's radius increases at 0.5 cm/s. How fast is its volume increasing when $r = 10$ cm?

- $V = \frac{4}{3}\pi r^3$, so $\dfrac{dV}{dr} = 4\pi r^2$.
- $\dfrac{dV}{dt} = \dfrac{dV}{dr} \times \dfrac{dr}{dt} = 4\pi(10)^2 \times 0.5 = 200\pi \approx 628$ cm^3/s.

AHL: Further integration and volumes

Integrals from the further derivatives

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\arctan\frac{x}{a} + C \qquad \int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \arcsin\frac{x}{a} + C$$

and $\displaystyle\int \sec^2 x\,dx = \tan x + C$, $\displaystyle\int a^x\,dx = \frac{a^x}{\ln a} + C$.

Integration by substitution

Choose u, write du in terms of dx, and change everything (including the limits) to u.

> **Worked example.** $\displaystyle\int x\sqrt{x-1}\,dx$. Let $u = x - 1$, so $x = u + 1$ and $dx = du$.

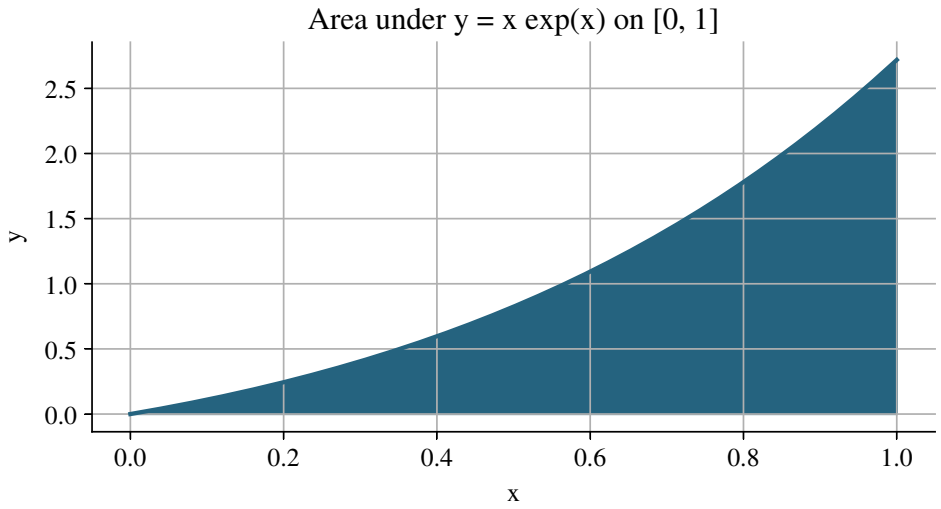
$$\int (u+1)u^{\frac{1}{2}}\,du = \frac{2}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}} + C = \frac{2}{5}(x-1)^{\frac{5}{2}} + \frac{2}{3}(x-1)^{\frac{3}{2}} + C$$

Integration by parts

From the product rule:

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

Choose u to be the factor that gets simpler when differentiated (a power of x, or $\ln x$).



The area under $y = xe^x$ from 0 to 1 is exactly 1

> **Worked example.** $\displaystyle\int_0^1 xe^x\,dx$. Let $u = x$, $\frac{dv}{dx} = e^x$, so $\frac{du}{dx} = 1$, $v = e^x$.

$$\int_0^1 xe^x\,dx = [xe^x]_0^1 - \int_0^1 e^x\,dx = e - (e-1) = 1$$

Repeated parts gives $\int x^2 e^x\,dx = e^x(x^2-2x+2)+C$. With $u=\ln x$ and $\frac{dv}{dx}=1$: $\int \ln x\,dx = x\ln x - x + C$.

Partial fractions

A rational function with a factorised denominator can be split into **partial fractions** 部分分式 and integrated term by term.

Worked example. $\displaystyle\int \frac{3x+5}{(x+1)(x+3)}\,dx$.

- $\dfrac{3x+5}{(x+1)(x+3)} = \dfrac{A}{x+1}+\dfrac{B}{x+3}$, so $3x+5 = A(x+3)+B(x+1)$.
- $x=-1$: $2=2A$, $A=1$. $x=-3$: $-4=-2B$, $B=2$.
- $\int = \ln|x+1| + 2\ln|x+3| + C$.

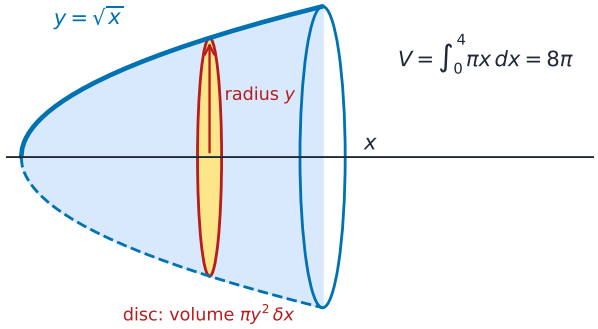
Areas with the y-axis and volumes of revolution

The area between a curve and the y-axis is $\displaystyle\int_c^d x\,dy$. When a region is rotated 2π about an axis, the **volume of revolution** 旋转体体积 is made of thin discs:

$$V = \pi\int_a^b y^2\,dx \text{ (about the } x\text{-axis)} \qquad V = \pi\int_c^d x^2\,dy \text{ (about the } y\text{-axis)}$$



A potter shaping a vessel: solids of revolution are volumes formed by rotating a plane region



A volume of revolution is a stack of thin discs

Worked example. The region under $y=\sqrt{x}$ for $0\le x\le 4$ is rotated about the x-axis: $V = \pi\displaystyle\int_0^4 x\,dx = \pi\left[\frac{x^2}{2}\right]_0^4 = 8\pi$.

AHL: Differential equations and Maclaurin series

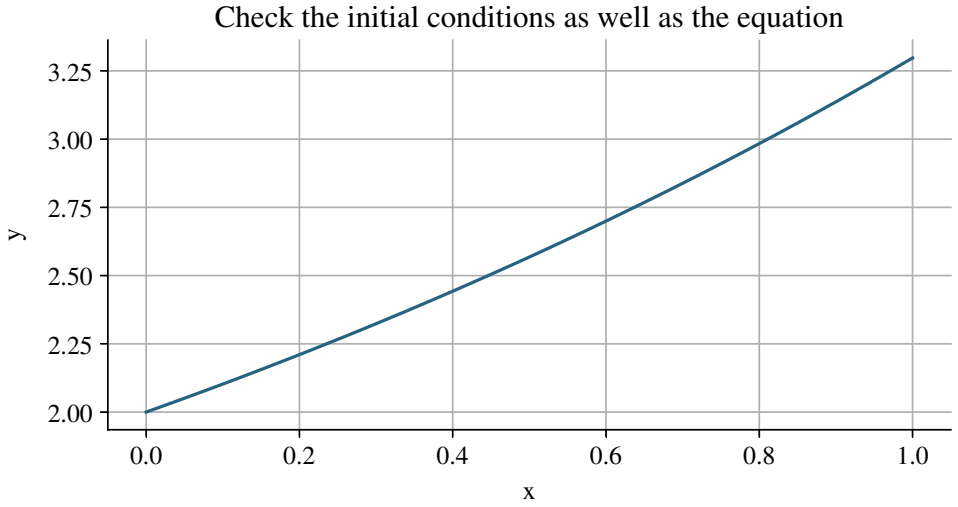
Differential equations and Euler's method

A **differential equation** 微分方程 links a function with its derivatives. Under suitable continuity and uniqueness conditions on f, a first-order equation $\dfrac{dy}{dx}=f(x,y)$ with an initial value has one local solution curve; an initial value alone does not guarantee existence or uniqueness. When it cannot be solved exactly, **Euler's method** 欧拉法 steps along the tangent:



A cooling cup of coffee: Newton's law of cooling is a classic differential equation model

$$x_{n+1} = x_n + h \qquad y_{n+1} = y_n + h \times f(x_n, y_n)$$



The exact solution of $dy/dx = 0.5y$ with $y(0) = 2$

Worked example. $\dfrac{dy}{dx} = 0.5y$, $y(0) = 2$, step $h = 0.5$.

- $y_1 = 2 + 0.5 \times 0.5 \times 2 = 2.5$ at $x = 0.5$.
- $y_2 = 2.5 + 0.5 \times 0.5 \times 2.5 = 3.125$ at $x = 1$.
- The exact value is $2e^{0.5} \approx 3.297$. Euler underestimates here because the curve bends upwards, away from each tangent; a smaller h reduces the error.

Exact methods

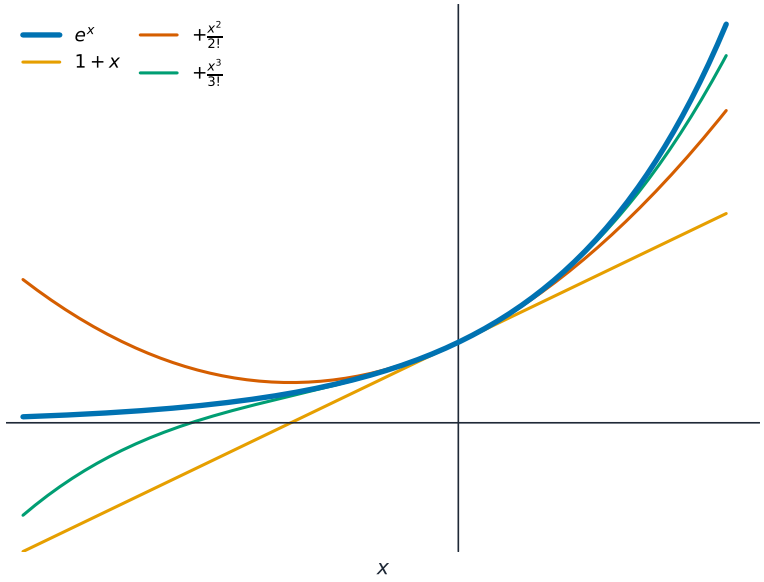
- **Separation of variables** 分离变量法, for $\dfrac{dy}{dx} = f(x)g(y)$: $\displaystyle\int \frac{1}{g(y)}\,dy = \int f(x)\,dx$. For $\dfrac{dy}{dx} = 0.5y$: $\ln|y| = 0.5x + c$, so $y = Ae^{0.5x}$, and $y(0) = 2$ gives $y = 2e^{0.5x}$.
- **Homogeneous** equations $\dfrac{dy}{dx} = f\left(\dfrac{y}{x}\right)$: substitute $y = vx$, so $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$, then separate. For $\dfrac{dy}{dx} = 1 + \dfrac{y}{x}$: $v + x\dfrac{dv}{dx} = 1 + v$, so $v = \ln|x| + c$ and $y = x\ln|x| + cx$.
- **Integrating factor** 积分因子, for $\dfrac{dy}{dx} + P(x)y = Q(x)$: multiply by $e^{\int P\,dx}$, so the left side becomes $\dfrac{d}{dx}\left(ye^{\int P\,dx}\right)$.

Worked example. $\dfrac{dy}{dx} + 2y = e^{-x}$. The integrating factor is e^{2x}:

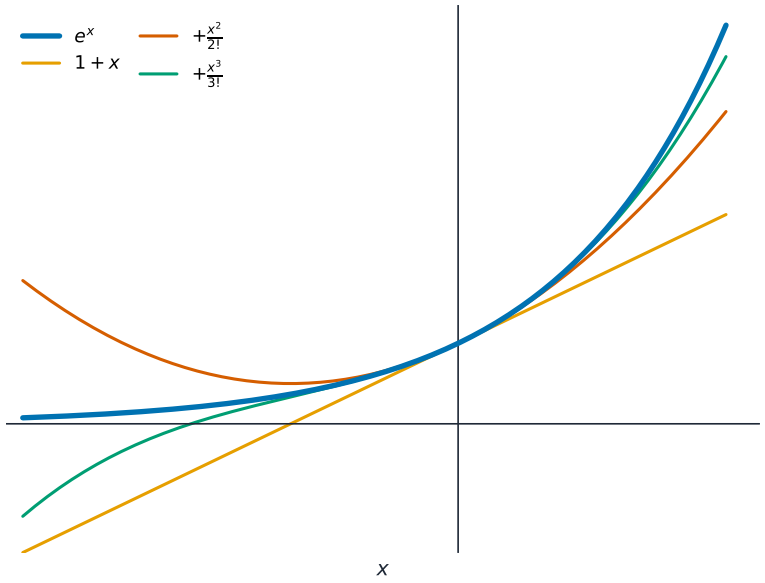
$$\frac{d}{dx}(ye^{2x}) = e^x \quad \Rightarrow \quad ye^{2x} = e^x + C \quad \Rightarrow \quad y = e^{-x} + Ce^{-2x}$$

Maclaurin series

A function can be approximated near $x = 0$ by a polynomial, its **Maclaurin series** 麦克劳林级数:



The linear, quadratic and cubic sums share the exponential function's value at the center.



The linear, quadratic and cubic sums share the exponential function's value at the center.

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \dots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \qquad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \qquad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots \qquad (1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \dots$$

New series come from substitution, multiplication, differentiation and integration of known ones: $e^x \sin x \approx \left(1 + x + \frac{x^2}{2}\right)\left(x - \frac{x^3}{6}\right) \approx x + x^2 + \frac{x^3}{3}$.

Worked example: a series from a differential equation. $\frac{dy}{dx} = x + y$, $y(0) = 1$.

- $y'(0) = 0 + 1 = 1$.
- $y'' = 1 + y'$, so $y''(0) = 2$; $y''' = y''$, so $y'''(0) = 2$.
- $y \approx 1 + x + \frac{2}{2!}x^2 + \frac{2}{3!}x^3 = 1 + x + x^2 + \frac{x^3}{3}$.

Common mistakes.

- Forgetting $+C$ on an indefinite integral, then losing marks when a point is given to find C.
- Missing the inner derivative: $\frac{d}{dx}(3x+1)^4 = 12(3x+1)^3$, not $4(3x+1)^3$.
- Multiplying derivatives for a product: $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$.
- Using the tangent gradient m for a normal. The normal has gradient $-\frac{1}{m}$.
- Taking $f''(x) = 0$ as proof of a point of inflection. Check that f'' changes sign there.
- Integrating straight across a root. Area below the x-axis gives a negative integral, so split at the roots and add the sizes.
- Confusing displacement with distance: distance travelled is $\int |v|\,dt$.
- **HL:** applying L'Hôpital's rule to a limit that is not of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.
- **HL:** writing $\frac{d}{dx}(y^2) = 2y$. Implicit differentiation gives $2y\frac{dy}{dx}$.
- **HL:** keeping the x-limits after a substitution in a definite integral. Change them to the new variable.
- **HL:** using $\pi\int (R-r)^2\,dx$ for a washer. The volume is $\pi\int (R^2 - r^2)\,dx$.

Practice

1. Differentiate $y = 3x^4 - \frac{2}{x^3} + 5\sqrt{x}$.
2. Find the equations of the tangent and the normal to $y = x^3 - 2x$ at $x = 1$.
3. Find the interval on which $f(x) = x^3 - 6x^2 + 9x$ is decreasing.
4. Differentiate: (a) $(2x-5)^6$, (b) e^{x^2}, (c) $\ln(3x+1)$, (d) $\sin^2 x$.
5. Differentiate: (a) $x^3\cos x$, (b) $\frac{e^x}{x^2+1}$.
6. Find and classify the stationary points of $y = 2x^3 - 3x^2 - 12x + 4$.
7. Find the point of inflection of $y = x^3 - 6x^2 + 5$, and show that the concavity changes there.
8. A rectangle has perimeter 40 cm. Find the dimensions that give the largest area, and justify that it is a maximum.
9. A particle has velocity $v = 6t - 3t^2$ m/s and starts at $s = 2$ m. Find s and a in terms of t, and the times when it is at rest.
10. For the particle in question 9, find the displacement and the distance travelled from $t = 0$ to $t = 3$.
11. Find $\int \left(4x^3 + \frac{3}{x} - 2\sin x\right) dx$.
12. $\frac{dy}{dx} = 3x^2 - 4x$ and the curve passes through $(2, 5)$. Find y.
13. Find $\int 3x^2(x^3+2)^4\,dx$ by inspection.

14. Evaluate $\displaystyle\int_0^{\pi/2} \cos x\,dx$ and $\displaystyle\int_1^e \frac{2}{x}\,dx$.
15. Find the area enclosed by $y = 4 - x^2$ and the x-axis.
16. Find the area between $y = x^2$ and $y = 2x$.
17. **AHL.** Differentiate $f(x) = x^2$ from first principles.
18. **AHL.** Find $\displaystyle\lim_{x\to 0} \frac{1-\cos x}{x^2}$.
19. **AHL.** Find $\dfrac{dy}{dx}$ for $x^2y + y^3 = 10$ in terms of x and y.
20. **AHL.** The radius of a circle increases at 2 cm/s. Find the rate of increase of the area when $r = 5$ cm.
21. **AHL.** Find $\displaystyle\int x\cos x\,dx$ and $\displaystyle\int \frac{1}{4+x^2}\,dx$.
22. **AHL.** Find the volume when the region under $y = e^x$ for $0 \le x \le 1$ is rotated 2π about the x-axis.
23. **AHL.** Solve $\dfrac{dy}{dx} = xy^2$ with $y(0) = 1$.
24. **AHL.** Solve $\dfrac{dy}{dx} + \dfrac{y}{x} = x$ for $x > 0$.
25. **AHL.** Find the Maclaurin series of e^{-x^2} up to the term in x^4, and use it to estimate $\displaystyle\int_0^{0.5} e^{-x^2}\,dx$.

Answers.

1. $y = 3x^4 - 2x^{-3} + 5x^{\frac12}$, so $\dfrac{dy}{dx} = 12x^3 + 6x^{-4} + \dfrac52 x^{-\frac12} = 12x^3 + \dfrac{6}{x^4} + \dfrac{5}{2\sqrt{x}}$.
2. At $x = 1$, $y = -1$ and $\frac{dy}{dx} = 3x^2 - 2 = 1$. Tangent $y = x - 2$; normal $y + 1 = -(x-1)$, so $y = -x$.
3. $f'(x) = 3x^2 - 12x + 9 = 3(x-1)(x-3) < 0$ for $1 < x < 3$.
4. (a) $12(2x-5)^5$; (b) $2xe^{x^2}$; (c) $\dfrac{3}{3x+1}$; (d) $2\sin x\cos x$ (which is $\sin 2x$).
5. (a) $3x^2\cos x - x^3\sin x$; (b) $\dfrac{e^x(x^2+1) - 2xe^x}{(x^2+1)^2} = \dfrac{e^x(x-1)^2}{(x^2+1)^2}$.
6. $y' = 6x^2 - 6x - 12 = 6(x-2)(x+1) = 0$ at $x = -1$ and $x = 2$. $y'' = 12x - 6$: at $x = -1$, $-18 < 0$, maximum $(-1, 11)$; at $x = 2$, $18 > 0$, minimum $(2, -16)$.
7. $y' = 3x^2 - 12x$, $y'' = 6x - 12 = 0$ at $x = 2$, where $y = 8 - 24 + 5 = -11$. y'' is negative for $x < 2$ and positive for $x > 2$, so $(2, -11)$ is a point of inflection.
8. Sides x and $20 - x$; $A = 20x - x^2$, $A' = 20 - 2x = 0$ at $x = 10$, and $A'' = -2 < 0$: a maximum. A 10 cm by 10 cm square, area 100 cm^2.
9. $s = 3t^2 - t^3 + 2$; $a = 6 - 6t$; at rest when $3t(2-t) = 0$, at $t = 0$ and $t = 2$ s.
10. Displacement $= s(3) - s(0) = (27 - 27 + 2) - 2 = 0$ m. Distance: from 0 to 2, s goes from 2 to 6 (4 m); from 2 to 3, from 6 to 2 (4 m). Distance = 8 m.
11. $x^4 + 3\ln|x| + 2\cos x + C$.
12. $y = x^3 - 2x^2 + C$; $5 = 8 - 8 + C$, so $y = x^3 - 2x^2 + 5$.
13. $\dfrac{(x^3+2)^5}{5} + C$.
14. $[\sin x]_0^{\pi/2} = 1$; $[2\ln x]_1^e = 2$.
15. Roots ± 2: $\displaystyle\int_{-2}^{2}(4 - x^2)\,dx = \left[4x - \frac{x^3}{3}\right]_{-2}^{2} = \frac{16}{3} + \frac{16}{3} = \frac{32}{3}$.
16. They meet at $x = 0$ and $x = 2$, with $2x \ge x^2$ between: $\displaystyle\int_0^2 (2x - x^2)\,dx = 4 - \frac{8}{3} = \frac{4}{3}$.
17. **AHL.** $\dfrac{(x+h)^2 - x^2}{h} = \dfrac{2xh + h^2}{h} = 2x + h \to 2x$ as $h \to 0$.
18. **AHL.** Form $\frac00$: $\displaystyle\lim \frac{\sin x}{2x}$, again $\frac00$: $\displaystyle\lim \frac{\cos x}{2} = \frac12$.
19. **AHL.** $2xy + x^2\dfrac{dy}{dx} + 3y^2\dfrac{dy}{dx} = 0$, so $\dfrac{dy}{dx} = -\dfrac{2xy}{x^2 + 3y^2}$.
20. **AHL.** $A = \pi r^2$, $\dfrac{dA}{dt} = 2\pi r\dfrac{dr}{dt} = 2\pi(5)(2) = 20\pi \approx 62.8$ cm^2/s.
21. **AHL.** Parts with $u = x$: $x\sin x - \displaystyle\int \sin x\,dx = x\sin x + \cos x + C$. And $\dfrac12\arctan\dfrac{x}{2} + C$.
22. **AHL.** $V = \pi\displaystyle\int_0^1 e^{2x}\,dx = \frac{\pi}{2}(e^2 - 1) \approx 10.0$.
23. **AHL.** $\displaystyle\int y^{-2}\,dy = \int x\,dx$: $-\dfrac1y = \dfrac{x^2}{2} + c$; $y(0) = 1$ gives $c = -1$, so $y = \dfrac{2}{2 - x^2}$.
24. **AHL.** Integrating factor $e^{\int \frac1x dx} = x$: $\dfrac{d}{dx}(xy) = x^2$, so $xy = \dfrac{x^3}{3} + C$ and $y = \dfrac{x^2}{3} + \dfrac{C}{x}$.
25. **AHL.** Substitute $-x^2$ into e^u: $e^{-x^2} \approx 1 - x^2 + \dfrac{x^4}{2}$. $\displaystyle\int_0^{0.5}\left(1 - x^2 + \frac{x^4}{2}\right)dx = 0.5 - \frac{0.125}{3} + \frac{0.03125}{10} \approx 0.4615$.

Exam tips

- Rewrite roots and fractions as powers before differentiating or integrating.
- Use radians in all calculus with trigonometric functions.
- Classify stationary points with the second derivative or a sign table, and give both coordinates.
- For optimisation, state the domain, show the derivative is zero and justify maximum or minimum.
- Split area integrals where the curve crosses the axis; the area is never negative.
- Distance travelled is $\int |v|\,dt$, not $\int v\,dt$.
- Always add $+C$ to an indefinite integral, and use a boundary condition when one is given.

- **HL:** change the limits in a definite substitution, check the $\frac{0}{0}$ form before L'Hôpital, and state the integrating factor before using it.

School-supervised mathematical exploration

International Baccalaureate · IB Diploma · Mathematics: Analysis and Approaches · HL

Scope and prerequisites

This handout teaches the internal assessment of Mathematics: Analysis and Approaches, the course first assessed in 2021: what the mathematical exploration is, how to choose a topic and a research question, the five assessment criteria, how to communicate mathematics in writing, how to model data and reflect on a model, the working process and academic integrity, and, for HL, what criterion E expects at HL. It is written for both SL and HL students. HL students also apply the further results in the sections marked AHL.

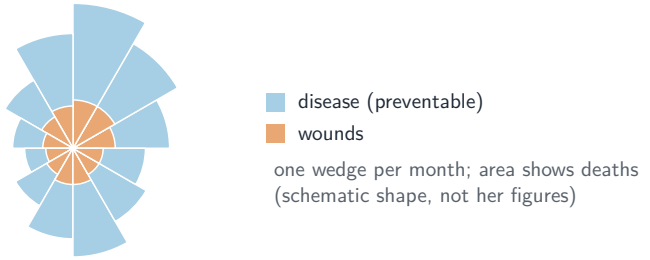
Prerequisites: none beyond the course itself. The exploration uses whatever mathematics fits your question, from any topic.

The worked example in this handout uses original, fictional data. It is not a real student's exploration and must not be copied as one.

What the exploration is

A picture that changed hospitals

During the Crimean War of the 1850s, the nurse Florence Nightingale and the statistician William Farr studied the British army's records of deaths. Nightingale presented the figures in her "rose" diagrams, a kind of circular graph in which the area of each wedge showed the number of deaths in one month. The diagrams made one fact impossible to miss: far more soldiers died from preventable diseases than from wounds in battle. They helped persuade the government to improve sanitation in army hospitals. Nightingale did not invent new mathematics. She chose a real question, used mathematics she understood well, and communicated it so clearly that people acted on it. That is exactly what the exploration asks of you.

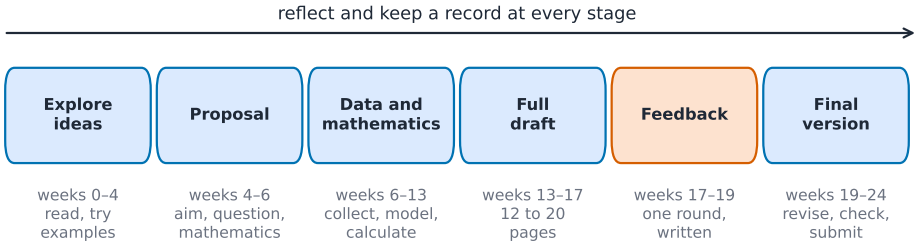


The idea of Nightingale's rose (schematic)

The task

The **mathematical exploration** 数学探究 is a piece of written work, about 12 to 20 pages long, in which you investigate an area of mathematics that interests you. It is the course's **internal assessment** 内部评估: your teacher marks it, and the IB checks a sample of marking from each school (**moderation** 评分审核). It is worth 20 % of the final grade at both SL and HL.

The course sets aside class time for it, shared with the mathematical toolkit, and your teacher gives you a timetable of deadlines. The exploration is not a timed exam and not a research report copied from books: it is your own investigation, written for a reader who is one of your classmates.



Start early: the mathematics takes longer than the writing

The five criteria

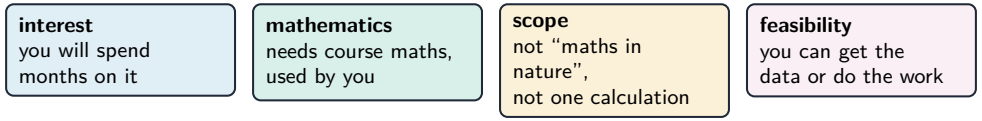
Criterion	Marks	What it rewards
A **Presentation** 表达	4	Organised, coherent and concise: an introduction with an aim, a clear structure and a conclusion
B **Mathematical communication** 数学交流	4	Correct notation, symbols and terms; defined variables; labelled graphs and tables
C **Personal engagement** 个人投入	3	Your own thinking and approach, shown throughout the work
D **Reflection** 反思	3	Critical comments on your results, methods, assumptions and limitations
E **Use of mathematics** 数学运用	6	Relevant, correct mathematics at the level of the course, with clear understanding

The total is 20 marks. Criterion E carries the most marks, but the other four criteria together carry more. A brilliant calculation that is badly explained loses marks under A and B, and a polished essay with little mathematics loses under E.

Choosing a topic and a research question

What makes a good topic

A good topic passes four tests.



Four tests for a topic

- **Interest.** You will spend months on it. Choose something you genuinely want to know, from a hobby, a subject you study, or a question you have wondered about.
- **Mathematics.** The question must need mathematics from the course, used and explained by you. A topic about mathematicians' lives, or about the history of π, is not an exploration.
- **Scope.** A question that is too broad ("mathematics in nature") cannot be answered in 20 pages; one that is too narrow gives one calculation and stops.

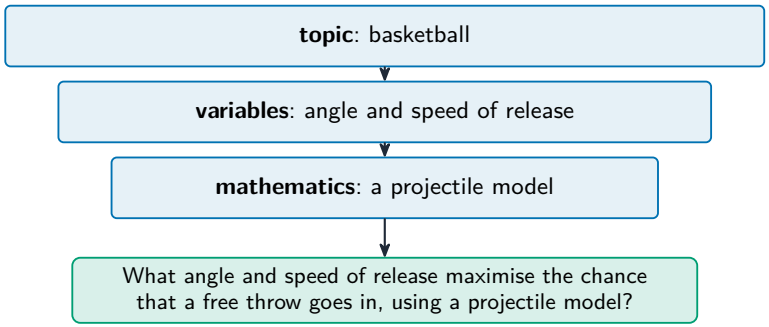
- **Feasibility.** You must be able to get the data, or do the calculation, in the time available. Check this before you commit.

From topic to research question

A **research question** 研究问题 turns an interest into something you can answer. It names the variables and the mathematics.

Weak	Better
Maths in music	How well does a geometric sequence model the frequencies of a piano's keys, and how far is equal temperament from simple whole-number ratios?
Basketball	What angle and speed of release maximise the chance that a free throw goes in, using a projectile model?
Population growth	Does a logistic model describe the growth of the yeast in my experiment better than an exponential model?
The Pythagorean theorem	How many essentially different proofs of Pythagoras' theorem are there, and can the theorem be extended to a tetrahedron with three right angles at one corner?

The better questions are specific, contain a mathematical method, and leave room for you to make decisions. Notice that the last one does not just "show" a famous result: copying a known proof gives no personal engagement and little use of mathematics.



From a topic to a research question

The proposal

Most schools ask for a short **proposal** 研究计划 before you start: your aim, your research question, the mathematics you expect to use, where data will come from, and any safety or consent issues. Writing it forces you to check that the topic passes the four tests. Expect your teacher to suggest changes: a narrower question is usually a better one.

Communicating mathematics

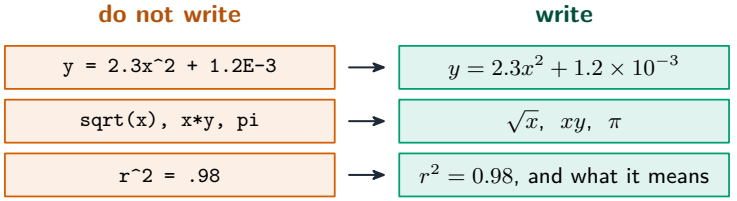
Writing for a classmate

Your reader is a classmate who knows the course but not your topic. They should be able to follow every step without asking you a question.

- State the aim at the start and answer it at the end.
- Explain why each method is chosen before you use it.
- Put long tables of raw data in an appendix and keep a summary in the text.
- Every graph, table and diagram needs a title or caption, labelled axes and units, and a sentence in the text that says what it shows.
- Cite every source, including images, data sets and software, in a bibliography.

Correct notation

Criterion B rewards correct mathematical notation. Calculator and spreadsheet notation is not acceptable in the written work.



Replace calculator notation

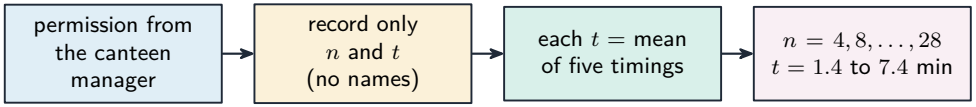
Do not write	Write
`y = 2.3x^2 + 1.2E-3`	$y = 2.3x^2 + 1.2 \times 10^{-3}$
`sqrt(x)`, `x*y`, `pi`	$\sqrt{x}$, xy or $x \times y$, π
`r^2 = .98`, so it fits	$r^2 = 0.98$, and explain what it means
the graph goes up	t increases as n increases

Define every variable when it first appears, with its units: "Let t be the waiting time in minutes and n the number of people ahead in the queue." Use the same letter for the same quantity all the way through, and write $=$ only between things that are equal.

Modelling and reflecting: a worked case

The question and the data

A student, whom we will call Lin, notices that the queue at the school canteen seems to move more slowly when it is long. Lin's research question: *How does the waiting time depend on the number of people ahead, and can a model predict the wait from the queue length?*



Collecting the data ethically

Lin gets permission from the canteen manager, decides to record only queue lengths and times (no names), and times the wait for different numbers of people ahead. Each value of t is the mean of five timings. These are fictional data for this handout:

n (people ahead)	4	8	12	16	20	24	28
t (minutes)	1.4	2.0	2.9	3.7	4.9	5.9	7.4

A first model

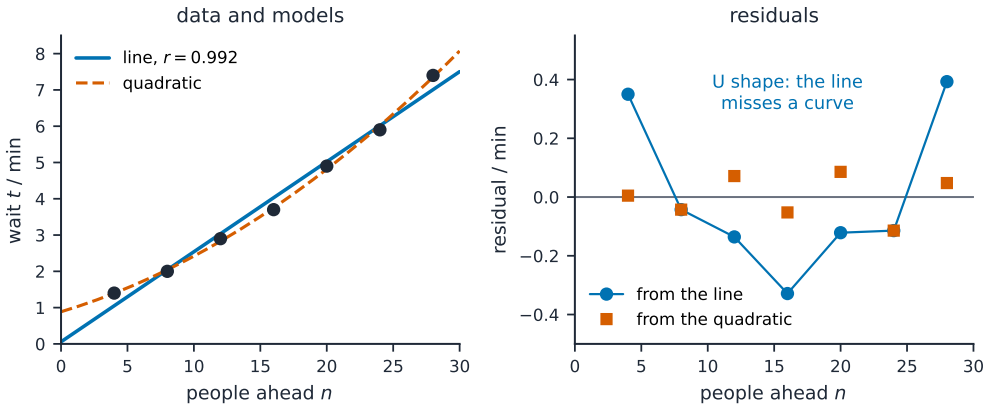
The points look roughly linear, so Lin fits a least-squares **regression line** 回归直线 (topic 4):

$$t = 0.057 + 0.248n, \qquad r = 0.992$$

The gradient means each extra person adds about 0.25 minutes, or 15 seconds, to the wait. The intercept is close to 0, which makes sense: with nobody ahead, there is no wait.

Testing the model with residuals

A strong correlation does not prove that a straight line is the right model. Lin calculates the **residuals** 残差, the observed value minus the predicted value. At $n = 4$ the line predicts $0.057 + 0.248 \times 4 \approx 1.05$, so the residual is $1.4 - 1.05 \approx 0.35$.

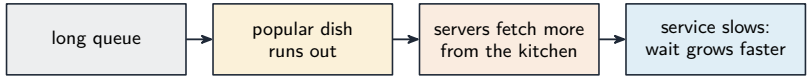


A pattern in the residuals means the model has missed something

The residuals from the line are positive at both ends and negative in the middle: a U shape. If the line were a good model, they would be small and scattered with no pattern. The pattern shows that the true relationship curves upward, even though $r = 0.992$.

A refined model

A curve that bends upward suggests a quadratic. Lin fits



$t = 0.886 + 0.110n + 0.00432n^2$: residuals below 0.12 min;
the observed reason, not the better fit, makes the curve believable

A reason for the curve

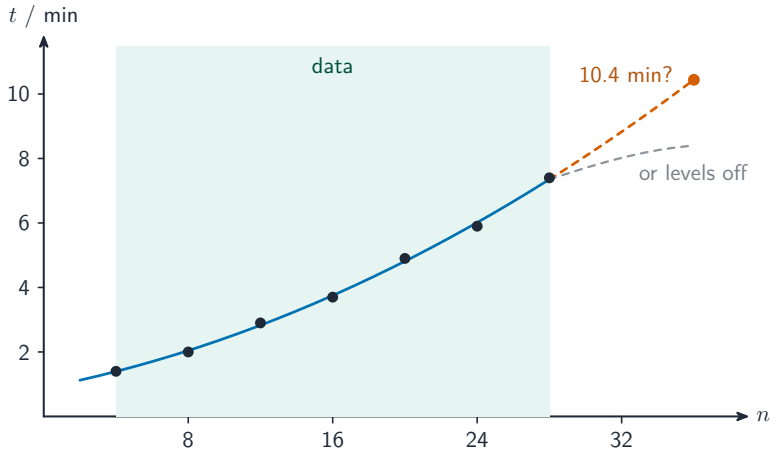
$$t = 0.886 + 0.110n + 0.00432n^2$$

Its residuals are all less than 0.12 minutes and show no clear pattern. The sum of the squared residuals falls from about 0.43 for the line to about 0.03 for the quadratic.

But a better fit is not automatically a better model. Lin asks *why* the wait might grow faster than the queue, and finds a reason by watching: when the queue is long, the most popular dish runs out, and the servers stop to bring more from the kitchen. That explanation is what makes the quadratic believable. It also shows the limits: the model is for this canteen, at lunchtime, with this menu.

Reflection

Reflection is not a summary at the end. It is critical thinking about the results and the method, written wherever it is relevant.



Beyond $n = 28$ the model is a guess

Descriptive (weak)	Critical (strong)
The model worked well.	The quadratic's residuals are all below 0.12 minutes, so its fitted waits differ from the observations by at most about 7 seconds.
My data could have errors.	Each time is a mean of five, but the five varied by up to 40 seconds at $n = 28$, so the last point is the least certain.
I could collect more data.	I have no data beyond $n = 28$. The quadratic predicts 10.4 minutes at $n = 36$, but if the dish runs out only once, the real curve may level off: an extrapolation I cannot trust.
I learned a lot.	Assuming the servers' speed was constant was wrong: watching them explained the curve that the residuals revealed.

Good reflection names an **assumption** 假设, says what effect it has, and suggests what would test it.

Integrity and the working process

Working steadily

- Keep a record: dated notes of what you tried, what failed and why you changed direction. Failed attempts are evidence of personal engagement and can be reflected on.
- Do the mathematics before you polish the writing.
- Your teacher gives written feedback on **one** draft. Use it fully: after that, the final version is yours alone.
- Check the length: an exploration that is far too long is not concise, and one that is far too short rarely has enough mathematics.

Academic integrity

The work must be your own, and anything that is not must be cited.

- Never invent data, change results to make them look better, or copy a sample exploration.
- Cite sources of ideas, data, images and software, in the text and in the bibliography.
- If you use an AI tool, cite it as you would any other source, following your school's policy. Presenting its output as your own work is academic misconduct.
- Collect data about people only with their consent, and keep it anonymous.

Your teacher must confirm that the work is authentic, so expect to be asked to explain any part of it. If you cannot explain a step, it should not be in the exploration.

AHL: What criterion E expects at HL

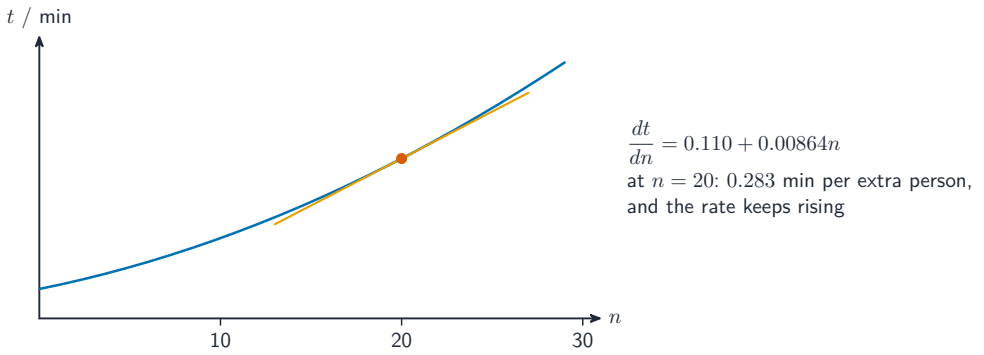
Level of the course

Criterion E is judged against the level of your course. An SL student who uses SL mathematics thoroughly and correctly can reach the top level. At HL, the top levels also expect the mathematics to be **sophisticated and rigorous**:

- **Sophisticated**: the mathematics is chosen well and goes beyond a routine procedure, such as setting up and solving a differential equation instead of fitting a curve with technology.
- **Rigorous**: the reasoning is logically complete. Assumptions and domains are stated, results are derived or proved instead of only checked with numbers, and technology outputs are justified.

Raising Lin's exploration to HL

An HL version of the queue exploration could:



Calculus on the model: the marginal wait

- treat the waiting time as a function and use calculus: $\dfrac{dt}{dn} = 0.110 + 0.00864n$, so at $n = 20$ each extra person adds about 0.28 minutes, and explain why this rate increases;
- build a model from first principles, for example a differential equation for the queue length with arrivals and service, solve it, and compare its predictions with the data;
- prove a general result, such as when a quadratic model predicts more than a linear one, instead of checking a few values.

Using HL mathematics only for decoration does not help. Criterion E rewards mathematics that answers the research question.

Common mistakes.

- Choosing a topic so wide that no mathematics is done, such as *maths in music*. Narrow it to one question you can answer.
- Copying a textbook proof or a website method. Show your own working, choices and checks.
- Using calculator notation in the report: write 2.4×10^{-5}, not 2.4E-05, and x^2, not x^2.
- Graphs without labelled axes, units or a source for the data.
- A reflection that only repeats the results. Say what the results mean, how reliable they are and what limits them.
- Mathematics well below the level of the course, or so far above it that you cannot explain it.
- **HL:** using a result you cannot explain. Rigour at HL means every step is justified, not that the topic is hard.

Practice

1. Rewrite the topic "Maths in football" as a focused research question that names the mathematics.
2. Rewrite in correct notation: `y = 4.1x^3 - 2E-4`, `sqrt(2)*pi`, `r = .87`.
3. Using $t = 0.057 + 0.248n$, find the predicted wait and the residual for $n = 16$, $t = 3.7$.
4. Explain why $r = 0.992$ does not show that the straight line is a good model for Lin's data.
5. Use the quadratic model to predict the wait for $n = 36$, and comment on how far the prediction can be trusted.
6. Which criterion, A to E, does each feature mainly affect? (a) An unlabelled graph; (b) a conclusion that does not answer the aim; (c) a calculation error; (d) a comment on why an assumption failed; (e) collecting your own data with a method you designed.
7. A student's aim is "to show that the Fibonacci numbers appear in sunflowers". Explain why this is weak and suggest a better approach.
8. Turn this into critical reflection: "My results are not very accurate because of human error."
9. A student uses a graph from a website and an AI tool to check some algebra. What must they do?
10. Explain why it helps to define variables with units at the start, using Lin's exploration as an example.
11. **AHL.** Using the quadratic model, find $\dfrac{dt}{dn}$ and its value at $n = 20$, and interpret it.
12. **AHL.** Explain what would make the mathematics in Lin's exploration more rigorous.

Answers.

1. For example: "How does the angle of a free kick from 20 m affect the range of target positions that clear a 2 m wall and go under the bar, using a projectile model?" Any answer must name variables and a method.
2. $y = 4.1x^3 - 2 \times 10^{-4}$; $\sqrt{2}\,\pi$; $r = 0.87$.
3. $0.057 + 0.248 \times 16 \approx 4.03$, so the residual is $3.7 - 4.03 \approx -0.33$ minutes.
4. r only measures how close the points are to some straight line. The residuals form a U shape, which shows a curve the line misses; a high r is possible even when a curve is the better model.
5. $0.886 + 0.110(36) + 0.00432(36^2) \approx 10.4$ minutes. This is extrapolation beyond $n = 28$, and the reason for the curve (the dish running out) may not continue, so the prediction is unreliable.
6. (a) B; (b) A; (c) E; (d) D; (e) C.

7. Showing a well-known fact gives little personal engagement and may need little mathematics. Better: count spirals in real seed heads and test whether they are Fibonacci numbers, or model seed packing using the golden angle and explore what other angles produce.
8. For example: "The stopwatch times for $n = 28$ varied by up to 40 seconds, about 9 % of the mean, so differences smaller than this between the two models cannot be trusted."
9. Cite both, in the text where they are used and in the bibliography, following the school's policy. The AI tool's output must not be presented as their own work.
10. The reader then knows what every symbol means and its units: "t is the waiting time in minutes and n is the number of people ahead" makes the gradient 0.248 readable as minutes per person.
11. **AHL.** $\dfrac{dt}{dn} = 0.110 + 0.00864n$; at $n = 20$ it is about 0.283 minutes per person, about 17 seconds. Each extra person adds more to the wait when the queue is long.
12. **AHL.** Stating the assumptions and domain ($4 \leq n \leq 28$, one canteen, lunchtime), deriving the model from how the queue works instead of only fitting a curve, and justifying the choice of model with residuals and an explanation, not just a higher r.

Exam tips

- Choose a question you care about, and check it needs mathematics from the course.
- Write for a classmate: define every variable, label every graph, and explain every method.
- Never use calculator notation in the written work.
- Look at residuals, not only r, before trusting a model.
- Reflect critically throughout: name assumptions, their effects and how to test them.
- Use your one round of teacher feedback fully, and keep dated notes of your process.
- Cite every source, including software and AI tools; never invent or alter data.
- **HL:** make the mathematics sophisticated and rigorous: derive and prove, do not only fit and check.