

5.5 Differential equations and numerical solutions

Name: _____ Class: _____ Date: _____ **Total: 13 marks**

KEY WORDS initial condition

Objective

Differential equations and numerical solutions. Supported HL concept preparation; original classroom practice, not an official IB paper or assessed learner work.

1 Key knowledge and worked examples

■ What does the starting value decide?

- A growing population's rate is proportional to its current size. The rate equation describes a whole family until an initial population is supplied.
- This lesson studies **initial condition** : A specified value that selects a particular solution from a family.

■ Choose the mathematical structure

- For $dy/dx=ky$, separate variables: $dy/y=k dx$, giving $y=Ae^{(kx)}$. Use an initial condition to find A. Euler's method takes $y_{next}=y+h f(x,y)$, with a chosen step h.
- State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.

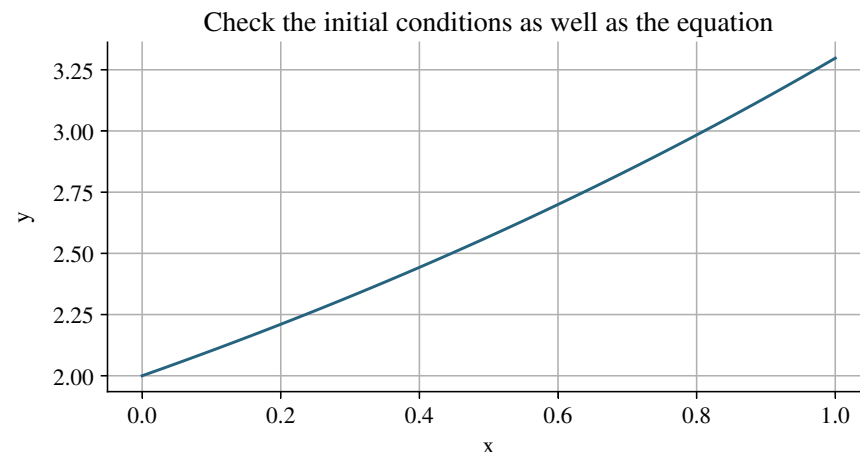
■ Work through a checked case

$$\frac{dy}{dx} = ky, \quad y = Ae^{kx}$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Example:

With $dy/dx=0.5y$ and $y(0)=2$, $y=2e^{(0.5x)}$. Euler with $h=0.2$ gives $y(0.2)\approx 2+0.2\times 1=2.2$ and $y(0.4)\approx 2.2+0.2\times 1.1=2.42$. The exact second value is about 2.4428.



■ Test a tempting shortcut

- An initial condition determines the integration constant; it is not a replacement for integrating. Dividing by y can omit an equilibrium solution $y=0$. Euler accuracy depends on step size and the equation.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Warn:

Every step size gives exactly the same Euler approximation. This claim is false. Explain which definition or assumption it violates.

■ Interpret a new situation

- For second-order linear equations, combine the complementary function with an appropriate particular integral, then apply the required initial conditions. This is Further Pure scope; check the named unit.
- A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.

■ Use this in your course

- Current first-assessment-2021 Analysis and Approaches HL. This is authored concept support; the full guide is needed to certify every objective.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key:

A specified value that selects a particular solution from a family. Choose the relationship, show the method, check its assumptions and interpret the result.

2 Practice

2.1 Using Euler with $h=0.2$ for $y' = 0.5y$, $y_0=2$, find y_1 . Show the relation, substitution and units. [2]

2.2 For that approximation, find y_2 . Show the relation, substitution and units. [2]

2.3 For $y = 2e^{(0.5x)}$, find y at $x=0$. Show the relation, substitution and units. [2]

2.4 Decide whether this claim is true and justify: Every step size gives exactly the same Euler approximation. [2]

2.5 Which description correctly defines initial condition? Choose one answer.

- **A** A specified value that selects a particular solution from a family.
- **B** A numerical answer that needs no assumptions or interpretation
- **C** A rule that remains valid even when its required conditions fail

[1]

3 Exam-style questions

3.1 Use Euler $h=0.5$ for $y' = -y$, $y(0)=2$ to estimate $y(1)$. Compare with $2e^{-1}$ and state a limitation. [4]

Solutions

2.1 Euler uses $y_1 = y_0 + h f(y_0) = 2 + 0.2 \times 0.5 \times 2 = 2.2$.

2.2 Repeat from y_1 : $y_2 = 2.2 + 0.2 \times 0.5 \times 2.2 = 2.42$.

2.3 At $x=0$ the exponential is 1, so $y=2$.

2.4 False. An initial condition determines the integration constant; it is not a replacement for integrating. Dividing by y can omit an equilibrium solution $y=0$. Euler accuracy depends on step size and the equation.

2.5 A A specified value that selects a particular solution from a family.

3.1 Updates $2 \rightarrow 1 \rightarrow 0.5$; exact ≈ 0.735759 . Euler underestimates here; finite steps introduce error. Halve the step and compare convergence, rather than treating the numerical value as exact.