

5.4 Substitution, parts and partial fractions

Name: _____ Class: _____ Date: _____

Total: 13 marks

KEY WORDS integration by parts

Objective

Substitution, parts and partial fractions. Supported HL concept preparation; original classroom practice, not an official IB paper or assessed learner work.

1 Key knowledge and worked examples

■ Can we integrate the two factors separately?

- A simple-looking product such as $x\hat{x}$ cannot be integrated by separately integrating each factor.
- This lesson studies **integration by parts** : An integration method based on the derivative of a product.

■ Choose the mathematical structure

- Use substitution when an inner derivative appears as a factor. By parts, integral $u \text{ v prime} = uv - \text{integral } u \text{ v prime}$. For a rational function, divide first if needed, then decompose into partial fractions.
- State the allowed inputs and units before calculating. An equation should express the relationship, not just record a calculator entry.

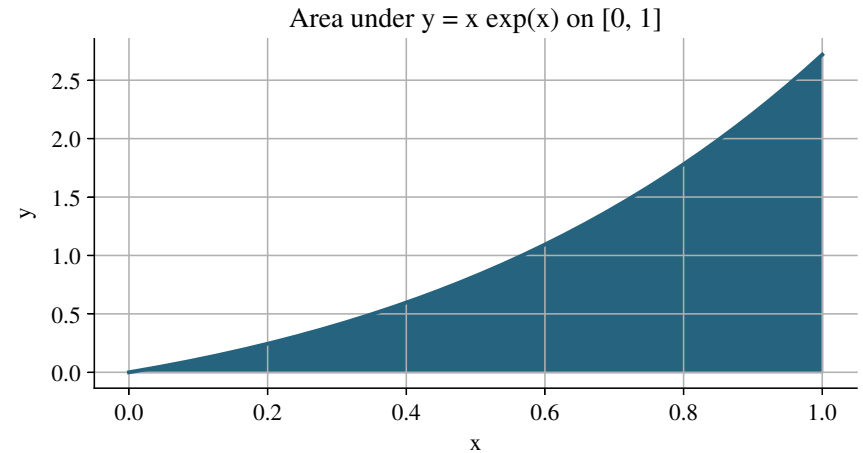
■ Work through a checked case

$$\int uv' dx = uv - \int u'v dx$$

- Check the result against the starting quantities. Substitute into the original relation, or compare the graph and numerical answer where appropriate.

Example:

For integral $x\hat{x}$ dx, take $u=x$ and $v \text{ prime}=\hat{x}$. Then the integral is $x\hat{x}-e^x+C$. For integral $2x/(x^2+1)$ dx, substitute $w=x^2+1$, $dw=2x$ dx; the answer is $\ln(x^2+1)+C$.



■ Test a tempting shortcut

- Choosing u and v prime well matters: the remaining integral should become simpler. In a definite substitution, either change the limits or return to x before applying the original limits.
- When a shortcut fails, identify the assumption it breaks. Keep an exact value until the requested final rounding.

Warn:

The integral of a product equals the product of its separate integrals. This claim is false. Explain which definition or assumption it violates.

■ Interpret a new situation

- Check by differentiating. For volumes of revolution around the x -axis use $V=\pi \text{ integral } y^2 dx$; do not confuse the square of a function with the integral of the function.
- A complete solution gives the mathematical result and explains what it means. Check that it is possible in the stated context.

■ Use this in your course

- Current first-assessment-2021 Analysis and Approaches HL. This is authored concept support; the full guide is needed to certify every objective.
- Give the method before the final answer, and use the paper's calculator and formula rules. Review a wrong answer by locating the first invalid step.

Key:

An integration method based on the derivative of a product. Choose the relationship, show the method, check its assumptions and interpret the result.

2 Practice

2.1 Evaluate the integral of $x e^x$ from 0 to 1. Show the relation, substitution and units.^[2]

2.2 Find the coefficient of $\ln(x^2+1)$ in the integral of $2x/(x^2+1)$. Show the relation, substitution and units.^[2]

2.3 Find the volume divided by π when $y=x$ is revolved about the x-axis from 0 to 3. Show the relation, substitution and units.^[2]

2.4 Decide whether this claim is true and justify: The integral of a product equals the product of its separate integrals.^[2]

2.5 Which description correctly defines integration by parts? Choose one answer.

- **A** An integration method based on the derivative of a product.
- **B** A numerical answer that needs no assumptions or interpretation
- **C** A rule that remains valid even when its required conditions fail^[1]

3 Exam-style questions

3.1 Evaluate integral of $x \ln x$ from 1 to e by parts and differentiate the antiderivative as a check.^[4]

Solutions

2.1 An antiderivative is $(x-1)e^x$. At 1 and 0 it gives 0 and -1, so the integral is 1.

2.2 Substitution $w=x^2+1$ gives $dw=2x \, dx$, so the coefficient of $\ln(x^2+1)$ is 1.

2.3 $V/\pi = \int_0^3 x^2 \, dx = [x^3/3]_0^3 = 9$.

2.4 False. Choosing u and v prime well matters: the remaining integral should become simpler. In a definite substitution, either change the limits or return to x before applying the original limits.

2.5 A An integration method based on the derivative of a product.

3.1 Take $u=\ln x$, $dv=x \, dx$: antiderivative $(x^2/2)\ln x - x^2/4$. Bounds give $(e^2+1)/4$. Differentiation returns $x \ln x$; domain $x>0$ matters.