

Course Companion

International Baccalaureate • IB Diploma • Mathematics: Analysis and Approaches • HL

Every topic handout in one volume

exercises.lol

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Number and algebra

International Baccalaureate · IB Diploma · Mathematics: Analysis and Approaches · HL

Scope and prerequisites

This handout covers selected HL topics for the course first assessed in 2021; it is not the whole syllabus.

Prerequisites: read the stated quantities and units, and use arithmetic and the model conditions below.

The examples are original and fictional: they are not real measurements or real student investigations.

Exact arithmetic and estimation

■ Can we pack without leftovers?

- A supplier packs 72 pencils and 90 pens into identical gift bags. How can we avoid leftovers?
- This lesson studies **prime factor** : A prime number that divides the integer exactly.

■ The method

- Prime factors reveal shared structure.
- Use the smallest common prime powers for the HCF and the largest for the LCM.
- Estimate before calculating; use brackets to preserve the order of operations.

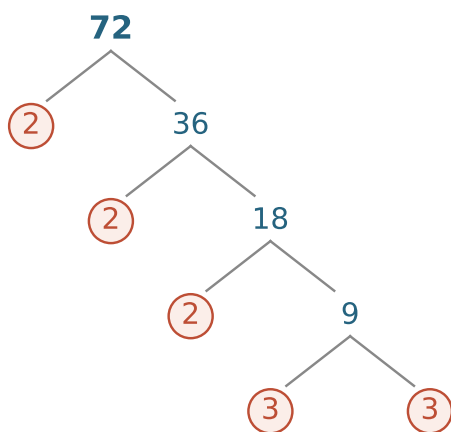
■ A worked case

$$a = \prod p_i^{\alpha_i}, \quad b = \prod p_i^{\beta_i}, \quad \text{HCF}(a, b) = \prod p_i^{\min(\alpha_i, \beta_i)}$$

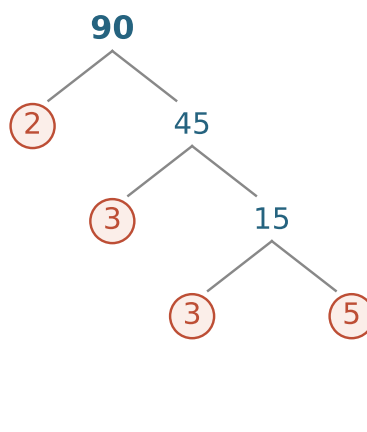
Example:

$72 = 2^3 \times 3^2$ and $90 = 2 \times 3^2 \times 5$. Their HCF is $2 \times 9 = 18$. Make 18 bags with 4 pencils and 5 pens each. Their LCM is $2^3 \times 3^2 \times 5 = 360$.

Factor trees: $\text{HCF}(72, 90) = 2 \times 3^2 = 18$



$$72 = 2^3 \times 3^2$$



$$90 = 2 \times 3^2 \times 5$$

Factor trees: $72 = 2^3 \times 3^2$ and $90 = 2 \times 3^2 \times 5$, so $\text{HCF} = 2 \times 3^2 = 18$

■ Where it goes wrong

- The HCF divides both numbers; the LCM is a multiple of both.
- They answer different questions.
- A decimal estimate is not an exact fraction.

Not true: The HCF of two positive integers is always larger than either integer. The HCF divides both numbers, so it can never be larger than the smaller one: the HCF of 72 and 90 is 18.

■ Applying it

- For a non-calculator paper, keep fractions exact and show cancellation.
- For a calculator paper, enter the full expression and compare with your estimate.

■ In your course

- Analysis and Approaches HL, the course first assessed in 2021.

Key:

- **Prime factor:** A prime number that divides the integer exactly.
- Prime factors reveal shared structure.
- Watch for: the HCF divides both numbers; the LCM is a multiple of both.

■ Write from an HCF of 6

- Find the HCF and LCM of 18 and 30.
- The HCF is smaller than both integers.

Indices, surds and standard form

■ How small is a microscopic length?

- A microscope records a length of 0.000072 metres. A compact representation must keep its size correct.
 - This lesson studies **index** : The power to which a base is raised.
-

■ The method

- For the same positive base, multiplication adds indices and division subtracts them.
 - A negative index means reciprocal; a fractional index represents a root.
 - Standard form has $1 \leq a < 10$.
-

■ A worked case

$$a^m a^n = a^{m+n}, \quad a^{-n} = \frac{1}{a^n}, \quad a^{m/n} = (\sqrt[n]{a})^m$$

Example:

$0.000072 = 7.2 \times 10^{-5}$. Also $16^{3/4} = (16^{1/4})^3 = 2^3 = 8$. Simplify $\sqrt{72} = 6\sqrt{2}$, then rationalise $1/\sqrt{2} = \sqrt{2}/2$.

$0.000072 = 7.2 \times 10^{-5}$: the point moves 5 places

5 jumps right

0 . 0 0 0 0 7 2

$$7.2 \times 10^{-5}$$

coefficient 7.2 is between 1 and 10; the power is -5

Moving the point 5 places right gives 7.2, so $0.000072 = 7.2 \times 10^{-5}$

■ Where it goes wrong

- Index laws do not turn a sum into a single power: $2^3+2^4=24$, not 2^7 .
- Do not round a surd when an exact answer is requested.

Not true: For every positive a , a^2+a^3 equals a^5 . Index laws only add powers when the terms are multiplied: $2^2+2^3=12$, but $2^5=32$.

■ Applying it

- Check powers of ten against the original quantity.
 - Use surds for exact geometry, and round only the final length when the question asks for a decimal.
-

■ In your course

- Analysis and Approaches HL, the course first assessed in 2021.
- Paper 1 is non-calculator: keep fractions, surds and π exact and show each step. On calculator papers, still write the expression before its value.

Key:

- **Index:** The power to which a base is raised.
- For the same positive base, multiplication adds indices and division subtracts them.
- Watch for: index laws do not turn a sum into a single power: $2^3+2^4=24$, not 2^7 .

■ Write from 8 to the power $2/3$

- Evaluate 8 to the power $2/3$.
- A squared plus a cubed is not a to the fifth.

Sequences, series and recurrence

■ Does the change add or multiply?

- A saving plan adds ¥30 more each week; a population model grows by 5% each year. Equal differences and equal ratios need different models.
 - This lesson studies **common ratio** : The constant multiplier between consecutive terms of a geometric sequence.
-

■ The method

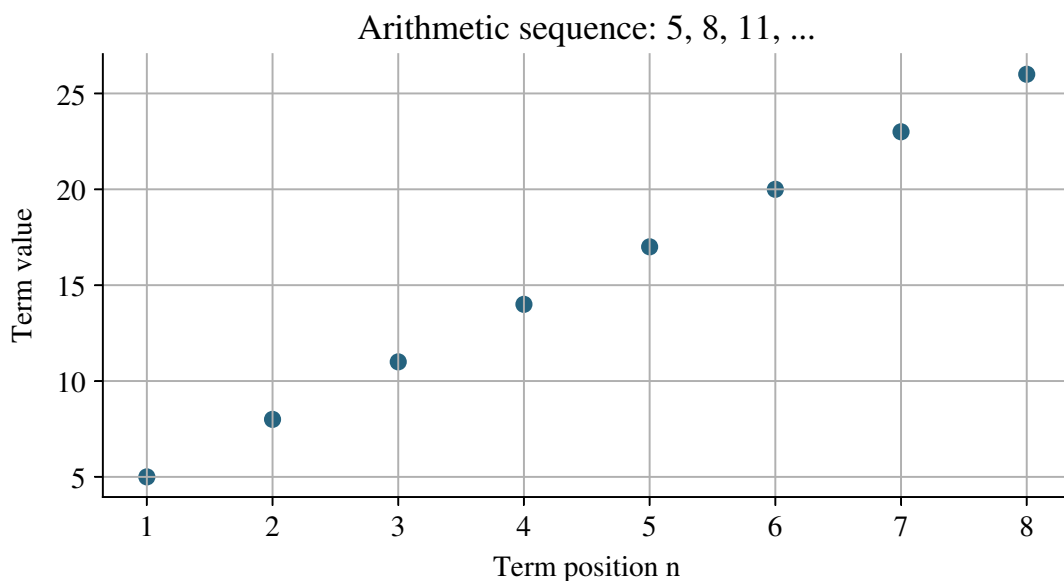
- For an arithmetic progression, $u_n = a + (n-1)d$ and $S_n = n(2a + (n-1)d)/2$.
 - For a geometric progression, $u_n = ar^{n-1}$ and $S_n = a(1-r^n)/(1-r)$.
 - An infinite geometric sum exists only if $|r| < 1$.
-

■ A worked case

$$u_n = a + (n-1)d, \quad S_n = \frac{n}{2} [2a + (n-1)d]$$

Example:

For $a=5, d=3, n=8$, $u_8 = 5 + 7 \times 3 = 26$ and $S_8 = 8(10+21)/2 = 124$. For $a=12, r=1/2$, $S_{\infty} = 12/(1-1/2) = 24$. For $u_{n+1} = 2u_n + 1$ with $u_1 = 1$, the next terms are 3, 7, 15.



$$u_n = 5 + 3(n-1): \text{ equal steps of 3 put the terms on a straight line}$$

■ Where it goes wrong

- The first term has index 1, so the exponent is $n-1$.
- A sequence is a list; a series is a sum.
- A geometric sequence can alternate in sign and still converge.

Not true: Every geometric series has a finite sum to infinity. A sum to infinity exists only when $-1 < r < 1$; for $r=2$ the terms grow without limit.

■ Applying it

- Explain whether the context justifies additive or multiplicative change.
 - In finance, distinguish a single deposit from a stream of deposits before choosing a sum formula.
-

■ In your course

- Analysis and Approaches HL, the course first assessed in 2021.
- Paper 1 is non-calculator: keep fractions, surds and π exact and show each step. On calculator papers, still write the expression before its value.

Key:

- **Common ratio:** The constant multiplier between consecutive terms of a geometric sequence.
- For an arithmetic progression, $u_n = a + (n-1)d$ and $S_n = n(2a + (n-1)d)/2$.
- Watch for: the first term has index 1, so the exponent is $n-1$.

■ Write from the 4th term 17

- An arithmetic sequence has first term 8 and common difference 3.
- Find the 4th term. An infinite geometric series needs a common ratio with size below 1.

Percentages, ratio and proportional reasoning

■ Can we recover the original price?

- A coat is reduced by 20% to ¥240. The discount applies to the original price, not to the sale price.
 - This lesson studies **multiplier**: A factor that performs a percentage change in one multiplication.
-

■ The method

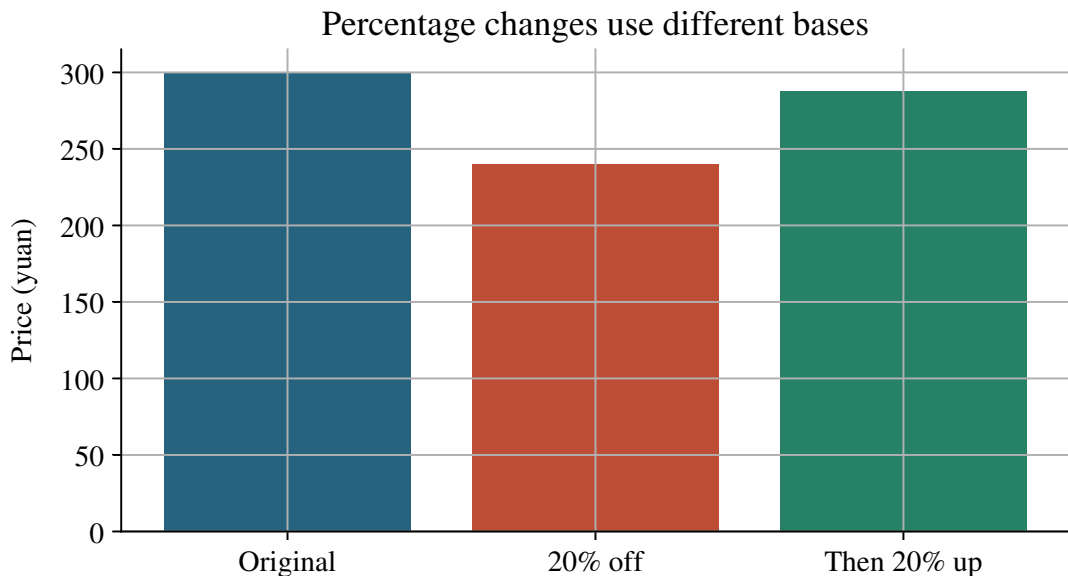
- A $p\%$ increase has multiplier $1+p/100$; a decrease has multiplier $1-p/100$.
 - Reverse a percentage by dividing by the multiplier.
 - In a ratio, first find the total number of parts.
-

■ A worked case

$$P_{\text{new}} = P_{\text{old}} \left(1 + \frac{r}{100}\right)$$

Example:

Let the original price be P . The model is sale price $= 0.8P$. Hence $P = 240/0.8 = 300$. A later 20% increase gives $240 \times 1.2 = 288$, so the two changes do not cancel.



300 → 240 → 288 yuan: 20% off then 20% up does not return to 300, because the second 20% is of 240

■ Where it goes wrong

- A percentage uses a stated base.
- Subtracting the percentages loses that base.
- For compound change, multiply the multipliers; do not add the percentages.

Not true: A 20% decrease followed by a 20% increase restores the starting price. The 20% increase is taken of a smaller amount: $100 \times 0.8 \times 1.2 = 96$, not 100.

■ Applying it

- For direct proportion use $y=kx$; for inverse proportion use $y=k/x$.
- Calculate k from a known pair before using a new value.
- State what you held constant.

■ In your course

- Analysis and Approaches HL, the course first assessed in 2021.
- Paper 1 is non-calculator: keep fractions, surds and π exact and show each step. On calculator papers, still write the expression before its value.

Key:

- **Multiplier:** A factor that performs a percentage change in one multiplication.
- A $p\%$ increase has multiplier $1+p/100$; a decrease has multiplier $1-p/100$.
- Watch for: A percentage uses a stated base.

■ Write from a final price of 240

- A price of 250 falls by 20 percent and then rises by 20 percent.
- Find the final price. The two changes do not cancel.

Complex numbers and roots

■ What if the root is not real?

- The equation $x^2+1=0$ has no real roots. Extending the number system lets us represent and calculate with both roots.
 - This lesson studies **complex conjugate** : The number obtained by reversing the sign of the imaginary part.
-

■ The method

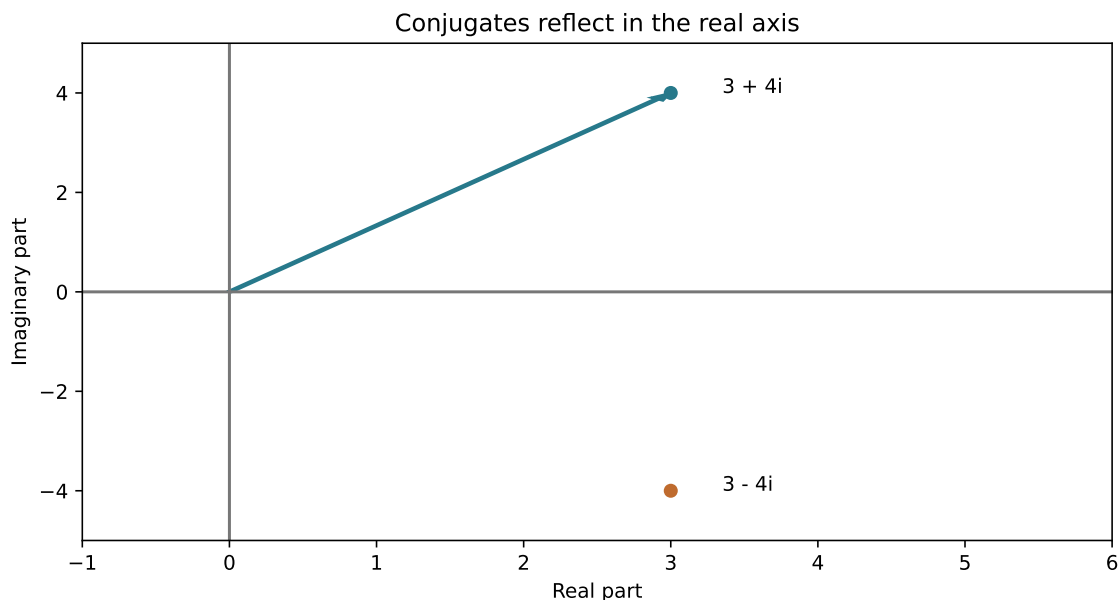
- Write $z=a+bi$ with $i^2=-1$.
 - The conjugate is $a-bi$ and z times its conjugate $=a^2+b^2$.
 - The modulus is $\sqrt{a^2+b^2}$; the argument needs the correct quadrant.
 - Conjugate roots occur for polynomials with real coefficients.
-

■ A worked case

$$z = a + bi, \quad |z| = \sqrt{a^2 + b^2}, \quad z\bar{z} = a^2 + b^2$$

Example:

For $z=3+4i$, $|z|=5$ and z conjugate $=3-4i$. The reciprocal is $(3-4i)/25$. For $x^2-6x+13=0$, the roots are $3\pm 2i$. Multiplying by the conjugate makes a complex denominator real.



The conjugate $3 - 4i$ is the reflection of $3 + 4i$ in the real axis; both have modulus 5

■ Where it goes wrong

- The square root of a sum is not the sum of the square roots.
- An inverse tangent alone may return the wrong quadrant.
- Do not confuse modulus with squared modulus.

Not true: For $a+bi$, its modulus is always $a+b$. The modulus is $\sqrt{a^2+b^2}$: for $3+4i$ it is 5, not 7.

■ Applying it

- Draw the real axis horizontally and imaginary axis vertically.
 - For products, moduli multiply and arguments add; powers extend this geometric pattern through de Moivre's theorem.
-

■ In your course

- Analysis and Approaches HL, the course first assessed in 2021.
- Paper 1 is non-calculator: keep fractions, surds and π exact and show each step. On calculator papers, still write the expression before its value.

Key:

- **Complex conjugate:** The number obtained by reversing the sign of the imaginary part.

- Write $z=a+bi$ with $i^2=-1$.
- Watch for: the square root of a sum is not the sum of the square roots.

■ Write from a modulus of 13

- Find the modulus of 5 plus 12 i.
- The modulus is not the sum of the two parts.

Binomial expansion and valid approximations

■ How does a small change affect a power?

- A small measurement change affects a power of a quantity. An expansion can show the size of first and second effects.
 - This lesson studies **binomial coefficient** : The number of ways to choose a specified number of objects from a set.
-

■ The method

- For positive integer n , $(a+b)^n$ is a finite binomial expansion.
 - For noninteger n , expand $(1+x)^n$ as $1+nx+\frac{n(n-1)}{2}x^2+\dots$ with $|x|<1$.
 - Factor out constants before using this form.
-

■ A worked case

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \dots$$

Example:

$(1+2x)^5=1+10x+40x^2+80x^3+80x^4+32x^5$. For $(1+x)^{-1}$, the first three terms are $1-x+x^2$; at $x=0.1$ this gives 0.91 versus the exact $1/1.1\approx 0.909091$.

(1 + 2x)⁵: Pascal row × powers of 2

Pascal row 5	1	5	10	10	5	1
× 2 ^k	1	2	4	8	16	32
coefficient	1	10	40	80	80	32
	1	x	x ²	x ³	x ⁴	x ⁵

Each coefficient of (1 + 2x)⁵ is a Pascal number times 2^k: 1, 10, 40, 80, 80, 32

■ Where it goes wrong

- An expansion in 2x requires |2x| < 1 for the infinite series, not merely |x| < 1.
- Positive-integer expansions are finite and do not have that convergence restriction.

Not true: Every binomial expansion is a finite polynomial, including powers that are not positive integers. For a negative or fractional power the expansion is an infinite series, valid only for |x| < 1.

■ Applying it

- State the range of validity alongside an approximation.
 - Retain enough terms to support the requested accuracy, and distinguish a coefficient from the whole term containing x.
-

■ In your course

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Key:

- **Binomial coefficient:** The number of ways to choose a specified number of objects from a set.

- For positive integer n , $(a+b)^n$ is a finite binomial expansion.
- Watch for: an expansion in $2x$ requires $|2x| < 1$ for the infinite series, not merely $|x| < 1$.

■ Write from a coefficient of 54

- Find the coefficient of x squared in $(1 + 3x)$ to the power 4.
- A binomial expansion is finite only for a non-negative integer power.

Series, finite differences and Taylor expansions

■ Can a polynomial replace an exponential nearby?

- Near a known point, a complicated function can be replaced by a polynomial approximation. The approximation needs a centre and a valid scale.
 - This lesson studies **Taylor polynomial** : A polynomial formed from a function's derivatives at a chosen centre.
-

■ The method

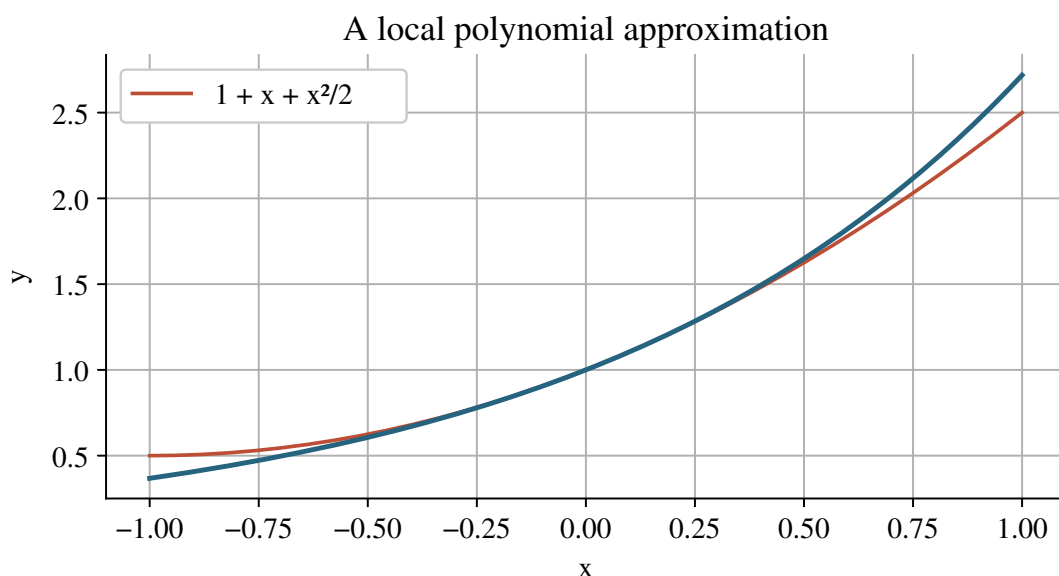
- About $x=a$, $f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\dots$ when the expansion is valid.
 - At $a=0$ it is a Maclaurin expansion.
 - Finite sums may also simplify by cancellation or standard sum formulae.
-

■ A worked case

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots$$

Example:

The first three terms of e^x are $1+x+\frac{x^2}{2}$. At $x=0.1$ this gives 1.105, close to $e^{0.1} \approx 1.105170$. For the sum of k^2 from $k=1$ to 5, $\frac{n(n+1)(2n+1)}{6}$ gives $5 \times 6 \times 11 / 6 = 55$.



$1 + x + x^2/2$ matches e^x closely near $x = 0$ and less well further away

■ Where it goes wrong

- The factorial belongs in the denominator of every Taylor coefficient.
- The expansion centre is not always zero.
- An approximation close to its centre can be poor far away.

Not true: A Taylor polynomial is always exact for the original function at every input. A Taylor polynomial is an approximation; it is usually accurate only near its centre.

■ Applying it

- For a telescoping series, write several terms and identify both surviving ends.
- For a Taylor approximation, state its order and compare with a known value or a remainder estimate where required.

■ In your course

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Key:

- **Taylor polynomial:** A polynomial formed from a function's derivatives at a chosen centre.
- About $x=a$, $f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\dots$ when the expansion is valid.
- Watch for: the factorial belongs in the denominator of every Taylor coefficient.

■ **Write from a square sum of 30**

- Find the sum of k squared from $k = 1$ to 4.
- Use $1 + x + \frac{x^2}{2}$ to estimate e to the power 0.2. A Taylor polynomial is not exact at every input.

■ **Write from a sum of 5050**

- Find the sum of k from $k = 1$ to 100.
- Gauss's pairing: $1 + 100, 2 + 99, \dots$ gives 50 pairs of 101, so 5 050.

■ **Write from a second-order estimate of 1.22**

- Use $1 + x + \frac{x^2}{2}$ at $x = 0.2$.
- The estimate is 1.22 against a true value near 1.2214 – the missing x cubed over 6 (about 0.0013) is the truncation error's size.

Functions

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Domains, inverses and composition

■ Which inputs are allowed?

- A square-root model returns a real output only for some inputs. Its formula alone does not specify a complete function.
 - This lesson studies **domain** : The set of allowed inputs to a function.
-

■ The method

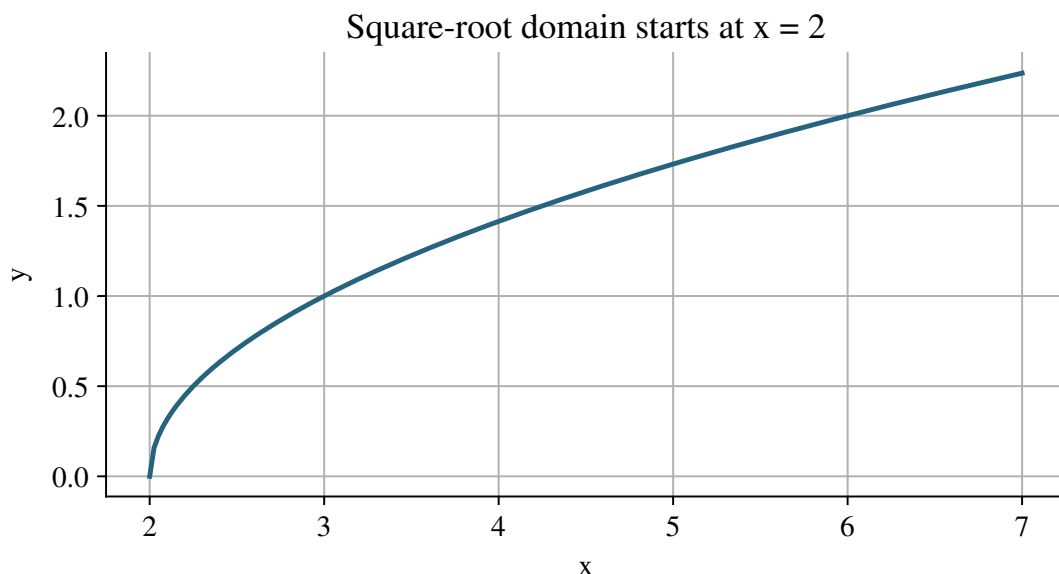
- State the domain and range.
 - For an inverse, first ensure the function is one-to-one on its domain.
 - Composition fg means apply g first, then f ; the intermediate output must be an allowed input to f .
-

■ A worked case

$$f(g(x)) = (f \circ g)(x), \quad f^{-1}(f(x)) = x$$

Example:

For $f(x)=\sqrt{x-2}$, $x \geq 2$ and the range is $y \geq 0$. From $y=\sqrt{x-2}$, $x=y^2+2$. Thus f inverse(x)= x^2+2 with $x \geq 0$. For $g(x)=x+3$, $fg(1)=f(4)=\sqrt{2}$.



$y = \sqrt{x - 2}$ exists only for $x \geq 2$, so its domain starts at 2 and its range starts at 0

■ Where it goes wrong

- Squaring can introduce extraneous solutions.
- Restricting a parabola's domain is essential before claiming an inverse.
- A horizontal translation inside f has the opposite sign to the graph's movement.

Not true: Every quadratic function on all real numbers has an inverse function. A quadratic on all real numbers is many-to-one ($f(1)=f(-1)$ for x^2), so its domain must be restricted first.

■ Applying it

- Check $f(f^{-1}(x))=x$ on the inverse domain.
 - Use a sketch to test whether a horizontal line meets the original graph more than once.
-

■ In your course

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Key:

- **Domain:** The set of allowed inputs to a function.

- State the domain and range.
- Watch for: squaring can introduce extraneous solutions.

■ Write from an inverse value of 6

- The function $6x$ plus 1 sends 6 to 37.
- A quadratic on all real numbers needs a restricted domain before it has an inverse.

Exponentials, logarithms and modelling

■ Why does a decay model stay positive?

- A medicine concentration falls by the same percentage each hour. A constant subtraction would eventually predict a negative amount.
 - This lesson studies **half-life** : The time for a decaying quantity to fall to half its initial value.
-

■ The method

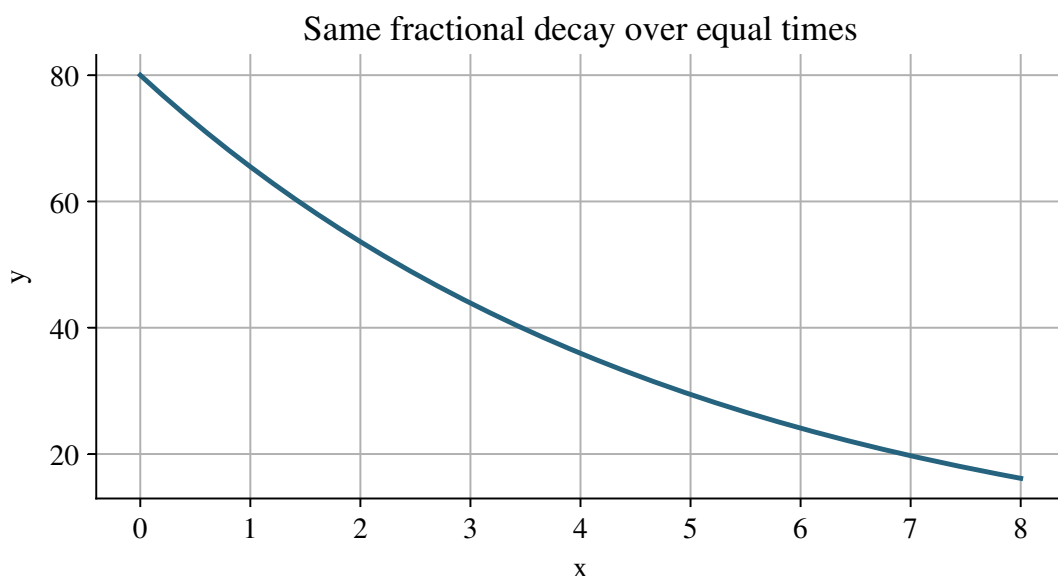
- For $y = Ae^{kt}$, k is a proportional rate.
 - Taking logs gives $\ln y = \ln A + kt$.
 - Logarithms require positive arguments, and $\log(x+y)$ is not $\log x + \log y$.
-

■ A worked case

$$y = Ae^{kt}, \quad \ln y = \ln A + kt$$

Example:

If $y = 80e^{(-0.2t)}$, $y = 40$ gives $e^{(-0.2t)} = 1/2$. Hence $t = \ln 2 / 0.2 \approx 3.466$. For $3^x = 20$, $x = \ln 20 / \ln 3$. Plotting $\ln y$ against t linearises this exponential model.



$y = 80e^{(-0.2x)}$ loses the same fraction, about 18%, in every unit of x

■ Where it goes wrong

- A fitted exponential is a model, not a guarantee.
- Specify the time units and range of use.
- A negative k describes decay; a negative starting amount usually contradicts the context.

Not true: For positive x and y , $\ln(x+y) \neq \ln x + \ln y$. The log of a sum has no simple law; the product rule is $\ln(xy) = \ln x + \ln y$.

■ Applying it

- Compare actual observations with the model.
 - Systematic departures may indicate changing conditions.
 - In a report, explain what the rate and initial value mean, rather than giving a bare equation.
-

■ In your course

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Key:

- **Half-life:** The time for a decaying quantity to fall to half its initial value.
- For $y = Ae^{kt}$, k is a proportional rate.
- Watch for: A fitted exponential is a model, not a guarantee.

■ **Write from 2 to the power 8**

- Solve 2 to the power $x = 256$.
- The sum of $\ln 2$ and $\ln 9$ is $\ln 18$, not $\ln$ of the sum.

■ **Write from the solution 3**

- Solve 5 to the power $x = 125$.
- Both sides as powers of 5: $x = 3$ – matching bases beat log rules here.

■ **Write from a half-life of 10 days**

- 200 g of a substance decays to 50 g in 20 days.
- Two halvings in 20 days, so the half-life is 10 days – count the halvings, not the subtraction.

Geometry and trigonometry

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Angle reasoning, similarity and mensuration

■ Does volume scale like length?

- A model has lengths one third of the real object. How much smaller are its area and volume?
- This lesson studies **scale factor** : The multiplier that relates corresponding lengths in similar shapes.

■ The method

- For similar shapes with length scale factor k , areas scale by k^2 and volumes by k^3 .
- State angle reasons explicitly.
- A circle's tangent is perpendicular to the radius at the contact point.

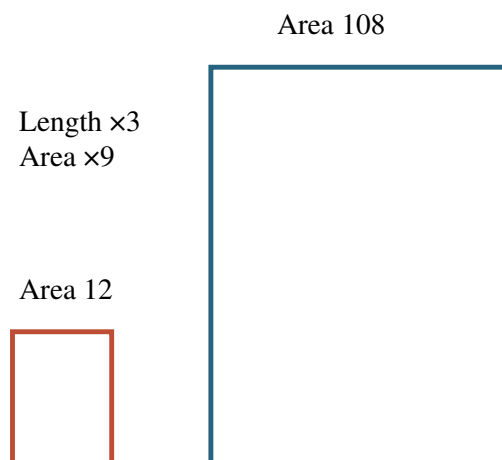
■ A worked case

$$\frac{A_2}{A_1} = k^2, \quad \frac{V_2}{V_1} = k^3$$

Example:

If model-to-real length factor is 3, a model area of 12 cm^2 gives $12 \times 3^2 = 108 \text{ cm}^2$ and a model volume of 8 cm^3 gives $8 \times 3^3 = 216 \text{ cm}^3$. A cylinder with $r=3, h=5$ has volume $\pi r^2 h = 45\pi$.

A length scale factor is squared for area



Length scale factor 3 gives area scale factor $3^2 = 9$: $12 \times 9 = 108$

■ Where it goes wrong

- Equal angles alone establish similarity, not equal size.
- Use corresponding lengths in the same order.
- Convert linear units before calculating area or volume, or square/cube the conversion factor correctly.

Not true: Doubling every length of a solid doubles its volume. Volume scales by the cube of the length factor: doubling every length multiplies the volume by 8.

■ Applying it

- A geometric proof should name the relevant theorem, identify the equal angle or ratio, and draw the conclusion.
 - A scale drawing is evidence only when the task permits measurement.
-

■ In your course

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Key:

- **Scale factor:** The multiplier that relates corresponding lengths in similar shapes.

- For similar shapes with length scale factor k , areas scale by k^2 and volumes by k^3 .
- Watch for: equal angles alone establish similarity, not equal size.

■ Write from a volume factor of 125

- A length scale factor is 5.
- Find the area factor and the volume factor. Doubling lengths does not double the volume.

Right triangles and non-right triangles

■ Which side does the ladder need?

- A ladder reaches a height of 4 m while its foot is 3 m from a wall. Which sides are known, and which angle do we need?
 - This lesson studies **hypotenuse** : The side opposite the right angle in a right-angled triangle.
-

■ The method

- In a right triangle $a^2 + b^2 = c^2$; $\sin\theta = \text{opposite}/\text{hypotenuse}$, $\cos\theta = \text{adjacent}/\text{hypotenuse}$ and $\tan\theta = \text{opposite}/\text{adjacent}$.
 - For other triangles, use the sine or cosine rule, or $\text{area} = ab \sin C/2$.
-

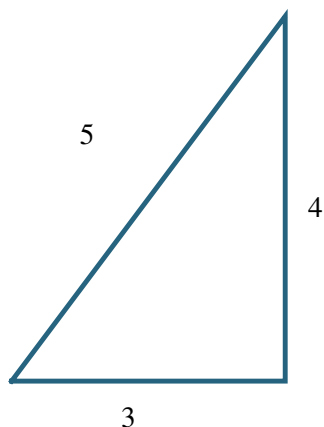
■ A worked case

$$a^2 + b^2 = c^2, \quad \tan \theta = \frac{\text{opposite}}{\text{adjacent}}$$

Example:

The ladder length is $c = \sqrt{3^2 + 4^2} = 5$ m. Its angle to the ground satisfies $\tan\theta = 4/3$, so $\theta \approx 53.1^\circ$. With two sides 6 and 8 enclosing 60° , $c^2 = 6^2 + 8^2 - 2 \times 6 \times 8 \cos 60^\circ = 52$.

Label the known sides before choosing a rule



Label the sides first: here 3, 4 and hypotenuse 5, so $\sin = 4/5$, $\cos = 3/5$, $\tan = 4/3$ at the bottom-left angle

■ Where it goes wrong

- Label sides relative to the chosen angle.
- Pythagoras needs a right angle.
- A calculator angle mode error can produce a plausible but wrong result.
- Keep unrounded values for later steps.

Not true: Pythagoras applies to every triangle, including triangles without a right angle. Pythagoras needs a right angle; for other triangles use the cosine rule.

■ Applying it

- Use a plan or elevation for a three-dimensional problem before applying a triangle rule.
 - Explain why the chosen triangle contains the required length or angle.
-

■ In your course

- Analysis and Approaches HL, the course first assessed in 2021.
- Paper 1 is non-calculator: keep fractions, surds and π exact and show each step. On calculator papers, still write the expression before its value.

Key:

- **Hypotenuse:** The side opposite the right angle in a right-angled triangle.
- In a right triangle $a^2 + b^2 = c^2$; $\sin\theta = \text{opposite/hypotenuse}$, $\cos\theta = \text{adjacent/hypotenuse}$ and $\tan\theta = \text{opposite/adjacent}$.
- Watch for: label sides relative to the chosen angle.

■ Write from a hypotenuse of 25

- A right triangle has legs 7 and 24.
- Find the hypotenuse and the area. Pythagoras is only for right triangles.

Radians, identities and trigonometric equations

■ How far does the rim travel?

- A rotating wheel sweeps an arc. Radians let its angle connect directly to the distance travelled along the rim.
 - This lesson studies **radian** : The angle subtended by an arc whose length equals the radius.
-

■ The method

- For θ in radians, arc length $s=r\theta$ and sector area $A=r^2\theta/2$.
 - Use $\sin^2\theta + \cos^2\theta = 1$ and angle identities to simplify or solve.
 - List every solution in the stated interval.
-

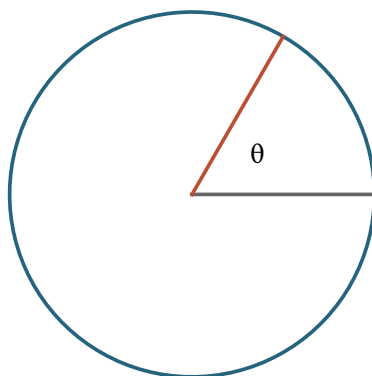
■ A worked case

$$s = r\theta, \quad A = \frac{1}{2}r^2\theta, \quad \sin^2\theta + \cos^2\theta = 1$$

Example:

For $r=4$ and $\theta=\pi/3$, $s=4\pi/3$ and $A=8\pi/3$. Solving $\sin x=1/2$ on $0 \leq x < 2\pi$ gives $x=\pi/6$ and $5\pi/6$. For $\sin 2x=1/2$, solve first on $0 \leq 2x < 4\pi$ before dividing by 2.

A radius fixes distance from the centre



radius $\times$ angle gives arc length

Arc length = $r\theta$ when θ is in radians

■ Where it goes wrong

- A calculator's inverse sine gives one principal value.
- It does not give every solution.
- Dividing by $\sin x$ can discard solutions where $\sin x=0$; check that case first.

Not true: The principal inverse sine value is always the only solution in a full turn. Sine takes most values twice in a full turn: $\sin x=0.5$ gives 30° and 150° .

■ Applying it

- Choose an identity that reduces the equation to one trigonometric function.
- Keep exact familiar angles where possible, and check each solution against the original equation.

■ In your course

- Analysis and Approaches HL, the course first assessed in 2021.
- Paper 1 is non-calculator: keep fractions, surds and π exact and show each step. On calculator papers, still write the expression before its value.

Key:

- **Radian:** The angle subtended by an arc whose length equals the radius.
- For θ in radians, arc length $s=r\theta$ and sector area $A=r^2\theta/2$.

- Watch for: A calculator's inverse sine gives one principal value.

■ Write from an arc length of 3

- A circle has radius 6 and the angle is 0.5 radians.
- Find the arc length and the sector area. The principal inverse sine is not the only solution in a full turn.

Vectors and transformation geometry

■ Why is displacement shorter than the walk?

- Walking 4 m east and 3 m north gives a displacement of 5 m, even though the travelled distance is 7 m.
 - This lesson studies **resultant** : The vector sum representing the combined displacement or force.
-

■ The method

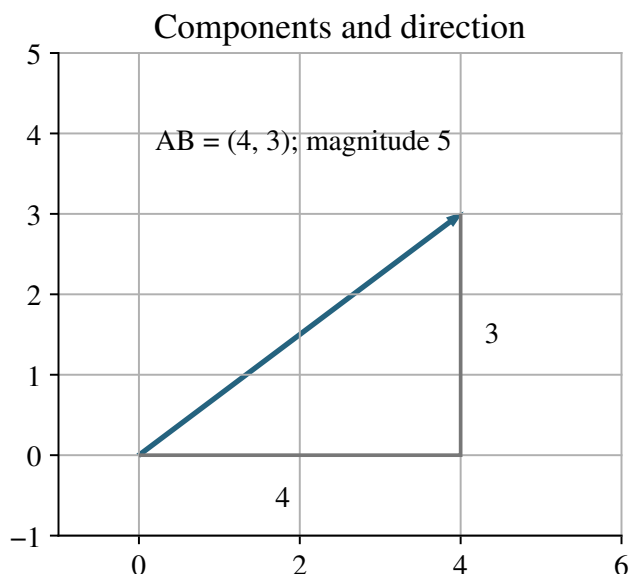
- Add corresponding vector components.
 - A displacement from A to B is OB-OA.
 - Parallel vectors are scalar multiples.
 - A vector has magnitude and direction; a distance is a scalar.
-

■ A worked case

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}, \quad |\mathbf{v}| = \sqrt{v_x^2 + v_y^2}$$

Example:

With $\mathbf{a}=(4,1)$ and $\mathbf{b}=(1,3)$, $\mathbf{a}+\mathbf{b}=(5,4)$. Their scalar product is $\mathbf{a} \cdot \mathbf{b}=4 \times 1+1 \times 3=7$. The magnitude of $\mathbf{a}$ is $\sqrt{17}$. For $\mathbf{A}=(1,2)$, $\mathbf{B}=(5,5)$, $\mathbf{AB}=(4,3)$ and $|\mathbf{AB}|=5$.



$$AB = (4, 3) \text{ has magnitude } \sqrt{4^2 + 3^2} = 5$$

■ Where it goes wrong

- The order of subtraction matters: $BA = -AB$.
- Proving parallelism needs a scalar-multiple relation; a sketch alone is insufficient.
- Negative enlargement reverses position about its centre.

Not true: AB and BA always have the same components. BA goes the opposite way to AB , so $BA = -AB$: every component changes sign.

■ Applying it

- For a translation, every point uses the same displacement.
 - For advanced line problems, distinguish the position vector from a direction vector and solve parameter equations consistently.
-

■ In your course

- Analysis and Approaches HL, the course first assessed in 2021.
- Paper 1 is non-calculator: keep fractions, surds and π exact and show each step. On calculator papers, still write the expression before its value.

Key:

- **Resultant:** The vector sum representing the combined displacement or force.
- Add corresponding vector components.

- Watch for: the order of subtraction matters: $\vec{BA} = -\vec{AB}$.

■ **Write from an x-component of 9**

- Add the vectors $(3, 1)$ and $(6, 2)$.
- Find the magnitude of $(6, 8)$ and the scalar product. $\vec{AB}$ and $\vec{BA}$ do not have the same components.

■ **Write from a magnitude of 5**

- Find the magnitude of the vector $(-3, 4)$.
- The magnitude is Pythagoras on the components: $\sqrt{9 + 16} = 5$.

■ **Write from a zero scalar product**

- Find the scalar product of $(2, 3)$ and $(-6, 4)$.
- $2 \times (-6) + 3 \times 4 = 0$ – a zero scalar product means the vectors are perpendicular.

Statistics and probability

International Baccalaureate · IB Diploma · Mathematics: Analysis and Approaches · HL

Scope and prerequisites

This handout covers selected HL topics for the course first assessed in 2021; it is not the whole syllabus.

Prerequisites: read the stated quantities and units, and use arithmetic and the model conditions below.

The examples are original and fictional: they are not real measurements or real student investigations.

Data summaries, histograms and interpretation

■ Can one average tell the whole story?

- Two groups have the same median but different spread. One summary cannot describe both location and consistency.
 - This lesson studies **frequency density** : Frequency divided by class width, used as histogram height.
-

■ The method

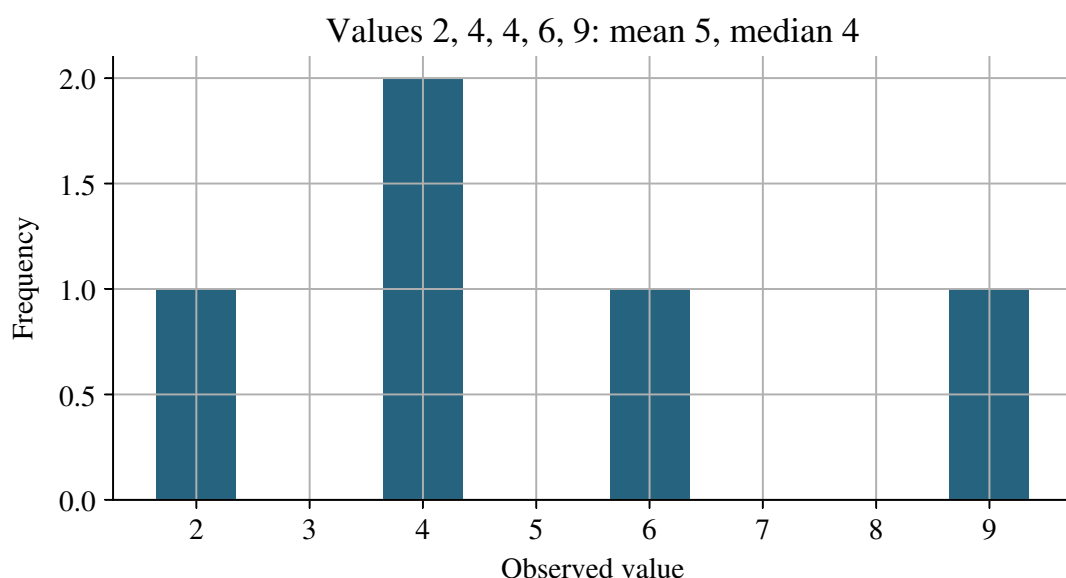
- Compare an appropriate average and spread in context.
 - A histogram uses area for frequency, so height=frequency/class width.
 - Grouped estimates assume representative values within intervals.
-

■ A worked case

$$\bar{x} = \frac{\sum x_i}{n}, \quad \text{density} = \frac{\text{frequency}}{\text{class width}}$$

Example:

A class from 10 to 20 with frequency 30 has density $30/10=3$. A class from 20 to 40 with frequency 20 has density $20/20=1$. Its wider bar must not be mistaken for a larger density. For values 2,4,4,6,9, the median is 4 and mean is 5.



For 2, 4, 4, 6, 9: mean = $25 \div 5 = 5$, median = 4

■ Where it goes wrong

- The tallest histogram bar need not contain the most observations.
- A grouped mean is an estimate.
- Correlation does not prove causation, and extrapolation extends beyond the observed range.

Not true: A histogram bar's height always equals its frequency, even with unequal class widths. With unequal class widths the height is frequency density; frequency is the area of the bar.

■ Applying it

- Choose a display that fits the data type.
- Give both a numerical comparison and what it means for the population; do not infer more precision than the sample supports.

■ In your course

- Analysis and Approaches HL, the course first assessed in 2021.
- Paper 1 is non-calculator: keep fractions, surds and π exact and show each step. On calculator papers, still write the expression before its value.

Key:

- **Frequency density:** Frequency divided by class width, used as histogram height.
- Compare an appropriate average and spread in context.
- Watch for: the tallest histogram bar need not contain the most observations.

■ Write from a mean of 7

- A class of width 4 has frequency 32.
- The values 6, 6, 6, 6 and 11 have a mean and a median. Bar height is not always the frequency.

Probability, trees and conditional reasoning

■ What changes after the first draw?

- A bag has 3 red and 2 blue counters. Taking two without replacement changes the chance of the second colour.
 - This lesson studies **conditional probability** : The probability of an event after restricting the sample space to a stated condition.
-

■ The method

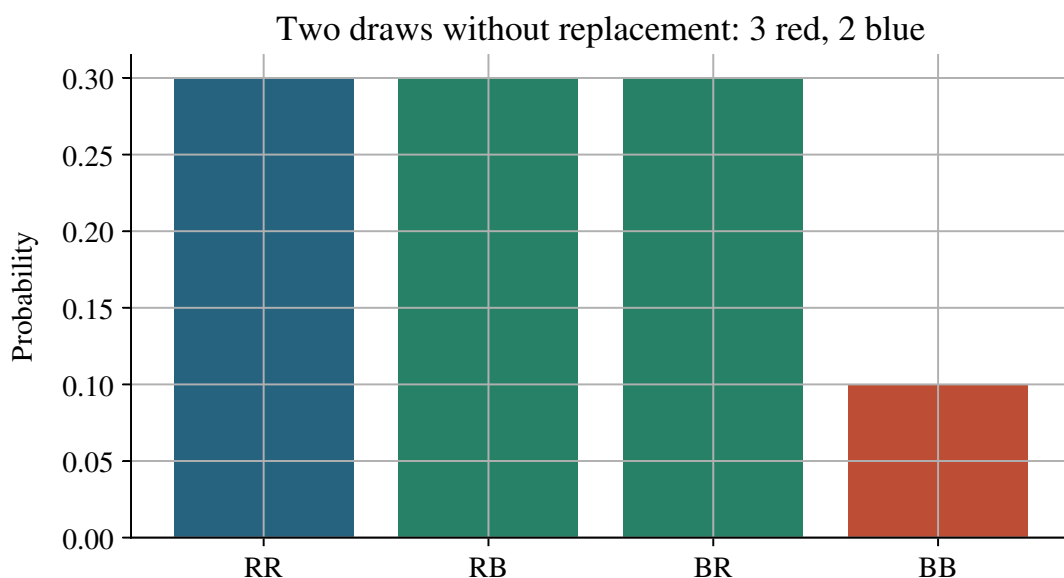
- Multiply along a tree branch and add disjoint branches.
 - With replacement, the composition stays fixed.
 - Conditional probability is $P(A \text{ given } B) = P(A \cap B) / P(B)$, for $P(B) > 0$.
-

■ A worked case

$$P(A | B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) > 0$$

Example:

Without replacement, $P(\text{two red}) = 3/5 \times 2/4 = 3/10$. $P(\text{one of each}) = 3/5 \times 2/4 + 2/5 \times 3/4 = 3/5$.
 If $P(A \cap B) = 0.12$ and $P(B) = 0.3$, $P(A \text{ given } B) = 0.4$.



Without replacement: $P(RR) = \frac{3}{5} \times \frac{2}{4} = 0.3$ and $P(BB) = \frac{2}{5} \times \frac{1}{4} = 0.1$

■ Where it goes wrong

- Mutually exclusive means no overlap; independent means that knowing one event does not change the other's probability.
- Two disjoint events with positive probability are not independent.

Not true: Mutually exclusive events with positive probabilities must be independent. Mutually exclusive events with positive probabilities are dependent: if one happens, the other cannot, so $P(A \text{ and } B) = 0 \neq P(A)P(B)$.

■ Applying it

- A two-way table makes the restricted denominator visible.
- Before using a product $P(A)P(B)$, justify independence from the context or the supplied information.

■ In your course

- Analysis and Approaches HL, the course first assessed in 2021.
- Paper 1 is non-calculator: keep fractions, surds and π exact and show each step. On calculator papers, still write the expression before its value.

Key:

- **Conditional probability:** The probability of an event after restricting the sample space to a stated condition.
- Multiply along a tree branch and add disjoint branches.
- Watch for: mutually exclusive means no overlap; independent means that knowing one event does not change the other's probability.

■ Write from a conditional probability of 0.64

- $P(A \text{ and } B)$ is 0.16 and $P(B)$ is 0.25.
- Find $P(A \text{ given } B)$. Mutually exclusive events that can happen are not independent.

Binomial, normal and Poisson models

■ What makes a count predictable?

- A quality inspector counts defective items. The number is random, but a model can describe its likely range.
 - This lesson studies **expected value** : The probability-weighted mean of a random variable.
-

■ The method

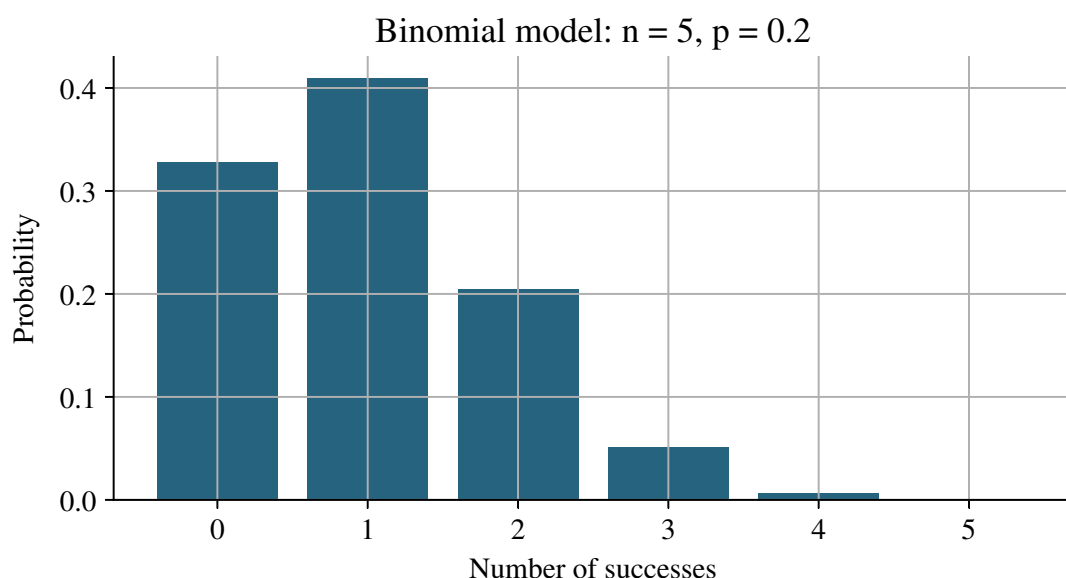
- A binomial model needs fixed n , independent trials, two outcomes and constant p .
 - $E(X)=np$ and $\text{Var}(X)=np(1-p)$.
 - For a normal model use $z=(x-\mu)/\sigma$.
 - A Poisson model describes counts with a constant rate and appropriate independence assumptions.
-

■ A worked case

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}, \quad X \sim B(n, p)$$

Example:

For X binomial(5,0.2), $P(X=0)=0.8^5=0.32768$, $E(X)=1$ and $\text{Var}(X)=0.8$. For a normal quantity with $\mu=100, \sigma=15$, the value 130 has $z=2$. A Poisson mean of 3 per hour gives mean 6 over two hours.



$X \sim B(5, 0.2)$: $P(X = 1) \approx 0.41$ is the largest bar; $P(X = 5)$ is tiny

■ Where it goes wrong

- Not every count is binomial: changing p or dependence can invalidate it.
- For a continuous variable, the probability of one exact value is zero.
- Continuity correction matters when approximating a discrete distribution by a normal one.

Not true: Every count of successes has a binomial distribution regardless of dependence. The binomial model needs independent trials with a fixed probability; dependent trials break it.

■ Applying it

- Write the event as an inequality before using calculator distribution functions.
- Distinguish $P(X < k)$, $P(X \leq k)$ and a tail complement.
- State assumptions in context.

■ In your course

- Analysis and Approaches HL, the course first assessed in 2021.

Key:

- **Expected value:** The probability-weighted mean of a random variable.
- A binomial model needs fixed n , independent trials, two outcomes and constant p .

- Watch for: not every count is binomial: changing p or dependence can invalidate it.
- **Write from a z-score of 3**
- A value 82 comes from a distribution with mean 70 and standard deviation 4.
 - Find the z-score. A binomial count needs independent trials.
- **Write from an expected count of 3**
- A binomial model has $n = 10$ trials with $p = 0.3$.
 - The expected count is $np = 3$ – the probability-weighted mean, not a promise of 3.
- **Write from a z-score of 2**
- A value 60 comes from a distribution with mean 50 and standard deviation 5.
 - The z-score is $10 \div 5 = 2$ – two standard deviations above the mean.

Calculus

International Baccalaureate · IB Diploma · Mathematics: Analysis
and Approaches · HL

Scope and prerequisites

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Prerequisites: read the stated quantities and units, and use arithmetic and the model conditions below.

The examples are original and fictional: they are not real measurements or real student investigations.

Derivatives and stationary points

■ What is the slope at one point?

- A curved road has different slopes at different positions. An average gradient cannot describe every point.
 - This lesson studies **derivative** : The instantaneous rate of change, also the gradient of a tangent.
-

■ The method

- For $y = ax^n$, $dy/dx = anx^{n-1}$.
 - A stationary point satisfies $dy/dx = 0$.
 - Check the sign change of the derivative, or the second derivative when it is nonzero, to classify it.
-

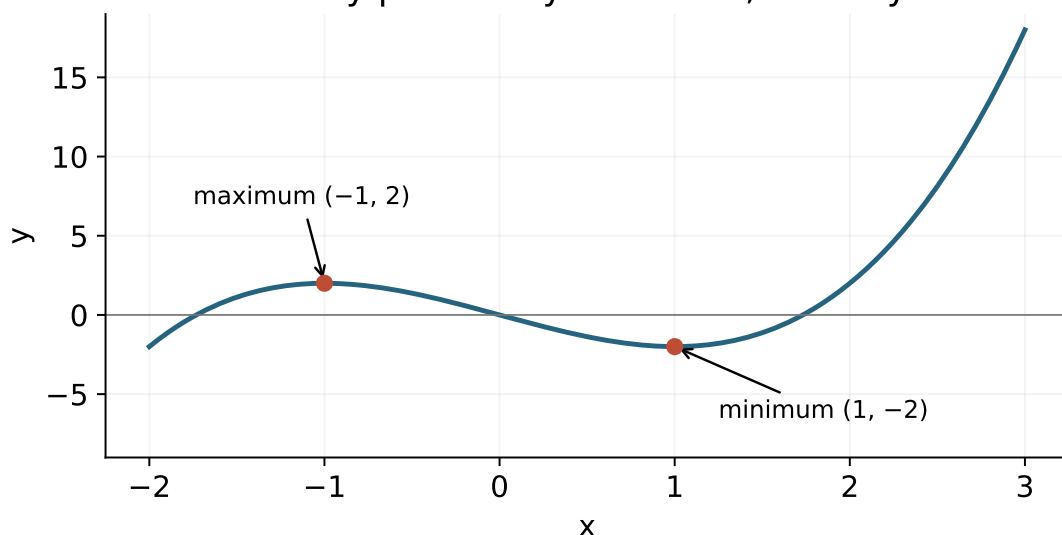
■ A worked case

$$\frac{d}{dx}(ax^n) = anx^{n-1}, \quad f'(x) = 0$$

Example:

For $y = x^3 - 3x$, $dy/dx = 3x^2 - 3$. At $x = 1$, the gradient is 0 and $y = -2$. The second derivative is $6x$, positive at $x = 1$, so this is a local minimum. At $x = -1$, $y = 2$ and the second derivative is negative, giving a local maximum.

Stationary points of $y = x^3 - 3x$, where $y' = 0$



$$y' = 3x^2 - 3 = 0 \text{ at } x = \pm 1: \text{ maximum } (-1, 2), \text{ minimum } (1, -2)$$

■ Where it goes wrong

- A zero derivative does not always mean a maximum or minimum: $y=x^3$ is stationary at 0 but continues increasing.
- An endpoint can also produce an extreme value on a restricted domain.

Not true: Every point with zero derivative is a local maximum or minimum. A zero derivative can also be a point of inflection: $y=x^3$ at $x=0$ is neither a maximum nor a minimum.

■ Applying it

- At GCSE/IGCSE use only the polynomial scope allowed by the tier; do not add chain, product or quotient rules there.
- At advanced level, connect the derivative to rates and optimization with a valid domain.

■ In your course

- Analysis and Approaches HL, the course first assessed in 2021.
- Paper 1 is non-calculator: keep fractions, surds and π exact and show each step. On calculator papers, still write the expression before its value.

Key:

- **Derivative:** The instantaneous rate of change, also the gradient of a tangent.
- For $y = ax^n$, $dy/dx = anx^{n-1}$.
- Watch for: A zero derivative does not always mean a maximum or minimum: $y = x^3$ is stationary at 0 but continues increasing.

■ Write from a derivative of 8

- For $y = x^2 - 10x$, find the derivative at $x = 9$.
- Find the stationary value. A derivative of 0 is not always a turning point.

Integrals, area and accumulation

■ How much change has accumulated?

- A velocity graph tells us motion at an instant. How can we recover the displacement accumulated over time?
 - This lesson studies **antiderivative** : A function whose derivative equals the given integrand.
-

■ The method

- For $n \neq -1$, the integral of ax^n is $ax^{n+1}/(n+1) + C$.
 - A definite integral is signed accumulation.
 - Split at sign changes when total area or distance is required.
-

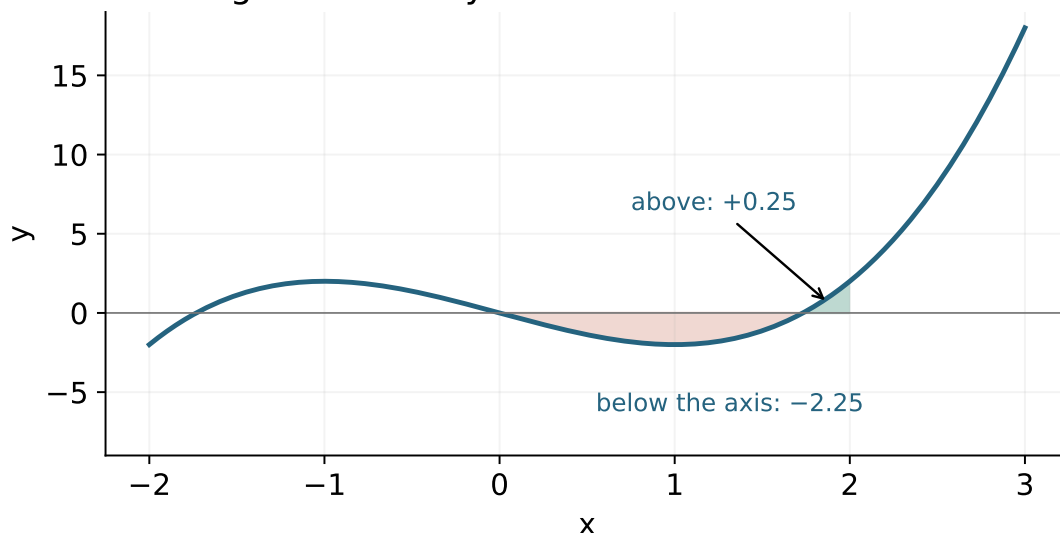
■ A worked case

$$\int ax^n dx = \frac{ax^{n+1}}{n+1} + C \quad (n \neq -1)$$

Example:

For $v(t) = 3t^2 - 3$ over $0 \leq t \leq 2$, displacement = $[t^3 - 3t]_0^2 = 2$. Since v changes sign at $t = 1$, distance = $-[t^3 - 3t]_0^1 + [t^3 - 3t]_1^2 = 2 + 4 = 6$. The constants cancel only for a definite integral.

Signed area of $y = x^3 - 3x$ from 0 to 2 is -2



$$\int_0^2 (x^3 - 3x) dx = -2.25 + 0.25 = -2: \text{ area below the axis counts as negative}$$

■ Where it goes wrong

- The integral of $1/x$ is $\ln|x|+C$, not the power rule with $n=-1$.
- Signed area can be zero even when the total enclosed area is positive.

Not true: A definite integral always equals the total positive area, even when the graph crosses the axis. Area below the axis counts as negative, so a definite integral can be smaller than the total area, or even zero.

■ Applying it

- Identify what the integral means and include the correct units.
- A rate measured per second integrates to the underlying quantity, not to another rate.
- Differentiate an antiderivative to check it.

■ In your course

- Analysis and Approaches HL, the course first assessed in 2021.
- Paper 1 is non-calculator: keep fractions, surds and π exact and show each step. On calculator papers, still write the expression before its value.

Key:

- **Antiderivative:** A function whose derivative equals the given integrand.

- For $n \neq -1$, the integral of ax^n is $ax^{(n+1)}/(n+1)+C$.
- Watch for: the integral of $1/x$ is $\ln|x|+C$, not the power rule with $n=-1$.

■ Write from an integral of 9

- Find the integral of $2x$ from 0 to 3.
- For $v = t$ minus 2, from $t = 0$ to 4, find the displacement and the distance. A signed integral is not the total distance.

Chain, product, quotient and implicit differentiation

■ Does the inside expression change too?

- A cost curve is a power of a changing expression. Differentiating the outer power alone misses the rate of its input.
 - This lesson studies **chain rule** : The rule that multiplies the outer derivative by the inner derivative for a composite function.
-

■ The method

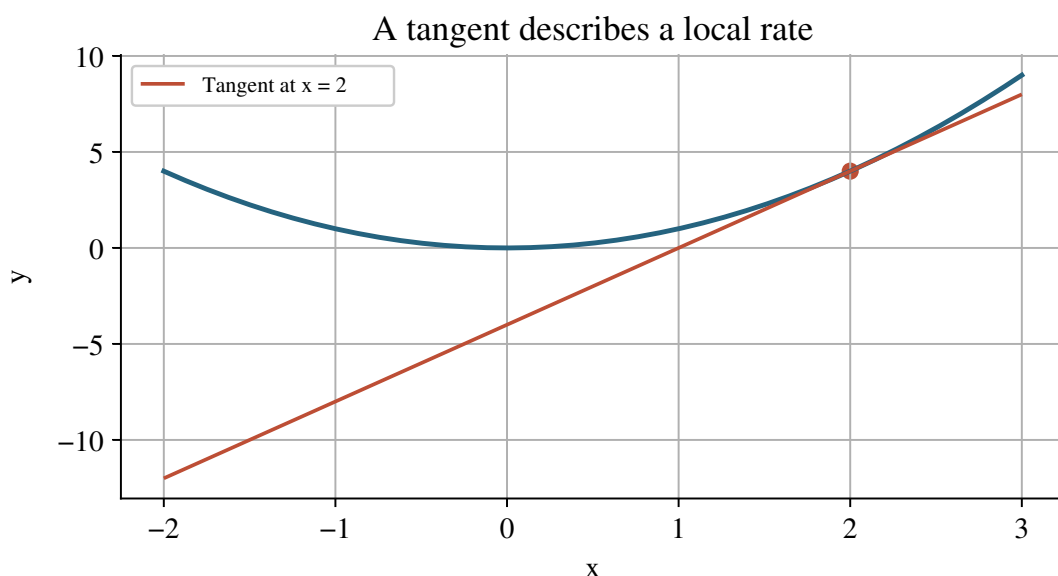
- For $y=f(g(x))$, $y' = f'(g(x))g'(x)$.
 - For uv , differentiate to $u'v + uv'$.
 - For u/v , use $(u'v - uv')/v^2$, where $v \neq 0$.
-

■ A worked case

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$$

Example:

For $y=(3x+1)^4$, $y' = 4(3x+1)^3 \times 3 = 12(3x+1)^3$. At $x=0$, the gradient is 12.
 For $x^2+y^2=25$, differentiate implicitly: $2x+2y y' = 0$, so $y' = -x/y$ when $y \neq 0$.



The tangent to $y = x^2$ at $x = 2$ has gradient $2x = 4$: $y = 4x - 4$

■ Where it goes wrong

- A derivative of a product is not the product of the derivatives.
- In implicit differentiation, every differentiated function of y brings a dy/dx factor.
- A quotient's denominator is squared.

Not true: The derivative of $u(x)v(x)$ is always u prime times v prime. The product rule is $(uv)' = u'v + uv'$: for $x \cdot x$ the derivative is $2x$, not $1 \times 1 = 1$.

■ Applying it

- Choose a useful form before differentiating: expanding a short polynomial may be simpler.
- For related rates, write the relation in symbols, differentiate with respect to time, then substitute measured values.

■ In your course

- Analysis and Approaches HL, the course first assessed in 2021.
- Paper 1 is non-calculator: keep fractions, surds and π exact and show each step. On calculator papers, still write the expression before its value.

Key:

- **Chain rule:** The rule that multiplies the outer derivative by the inner derivative for a composite function.
- For $y=f(g(x))$, $y' = f'(g(x))g'(x)$.
- Watch for: A derivative of a product is not the product of the derivatives.

■ Write from a product derivative of 11

- For $y = (2x + 1)(x + 3)$, find the derivative at $x = 1$.
- The derivative of a product is not the product of the derivatives.

Substitution, parts and partial fractions

■ Can we integrate the two factors separately?

- A simple-looking product such as $x e^x$ cannot be integrated by separately integrating each factor.
 - This lesson studies **integration by parts** : An integration method based on the derivative of a product.
-

■ The method

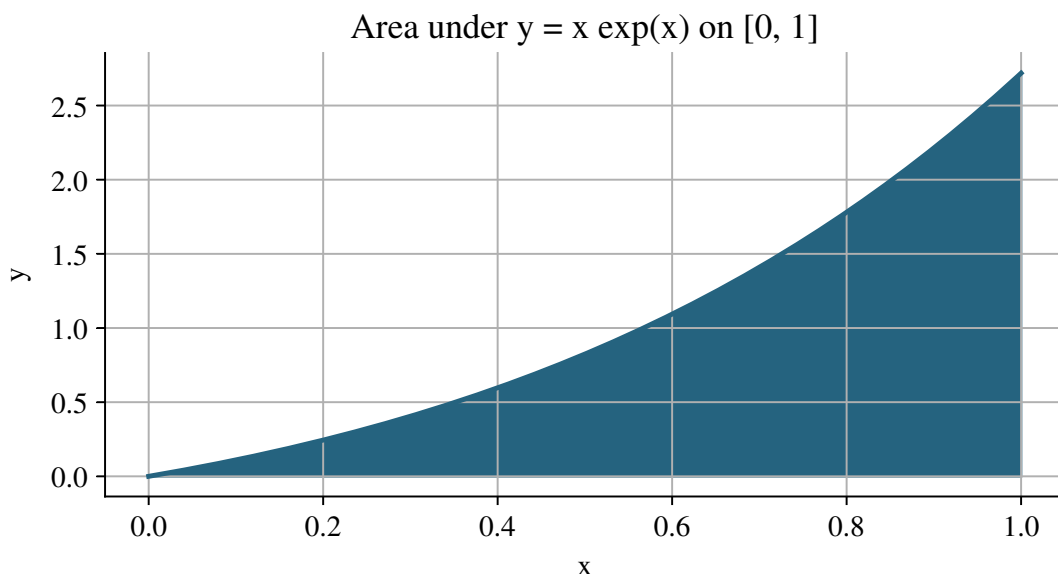
- Use substitution when an inner derivative appears as a factor.
 - By parts, $\int u v' = uv - \int u' v$.
 - For a rational function, divide first if needed, then decompose into partial fractions.
-

■ A worked case

$$\int uv' dx = uv - \int u'v dx$$

Example:

For integral $\int x e^x dx$, take $u=x$ and $v' = e^x$. Then the integral is $x e^x - e^x + C$.
 For integral $\int 2x/(x^2+1) dx$, substitute $w=x^2+1$, $dw=2x dx$; the answer is $\ln(x^2+1) + C$.



$$\text{Integration by parts: } \int_0^1 x e^x dx = [x e^x - e^x]_0^1 = 1$$

■ Where it goes wrong

- Choosing u and v prime well matters: the remaining integral should become simpler.
- In a definite substitution, either change the limits or return to x before applying the original limits.

Not true: The integral of a product equals the product of its separate integrals. A product needs a method such as integration by parts or substitution: $\int x \cdot x dx = x^3/3$, not $(x^2/2)(x^2/2)$.

■ Applying it

- Check by differentiating.
- For volumes of revolution around the x -axis use $V = \pi \int y^2 dx$; do not confuse the square of a function with the integral of the function.

■ In your course

- Analysis and Approaches HL, the course first assessed in 2021.
- Paper 1 is non-calculator: keep fractions, surds and π exact and show each step. On calculator papers, still write the expression before its value.

Key:

- **Integration by parts:** An integration method based on the derivative of a product.
- Use substitution when an inner derivative appears as a factor.
- Watch for: choosing u and v prime well matters: the remaining integral should become simpler.

■ Write from a volume factor of 18

- The line $y = 3$ is revolved about the x -axis from 0 to 2. Find the volume divided by π .
- The integral of a product is not the product of the integrals.

Differential equations and numerical solutions

■ What does the starting value decide?

- A growing population's rate is proportional to its current size. The rate equation describes a whole family until an initial population is supplied.
 - This lesson studies **initial condition** : A specified value that selects a particular solution from a family.
-

■ The method

- For $dy/dx=ky$, separate variables: $dy/y=k dx$, giving $y=Ae^{(kx)}$.
 - Use an initial condition to find A .
 - Euler's method takes $y_{\text{next}}=y+h f(x,y)$, with a chosen step h .
-

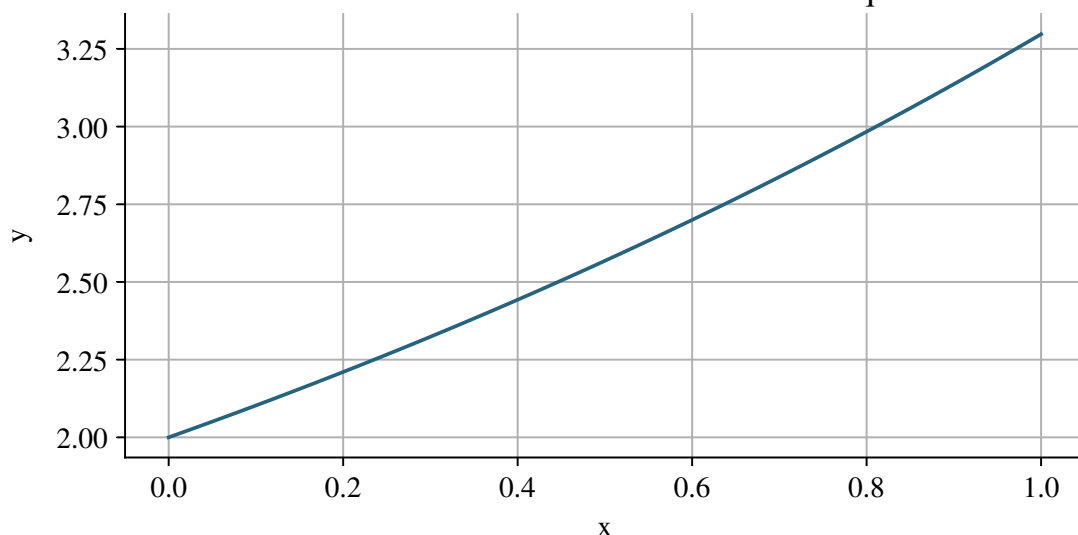
■ A worked case

$$\frac{dy}{dx} = ky, \quad y = Ae^{kx}$$

Example:

With $dy/dx=0.5y$ and $y(0)=2$, $y=2e^{(0.5x)}$. Euler with $h=0.2$ gives $y(0.2)\approx 2+0.2\times 1=2.2$ and $y(0.4)\approx 2.2+0.2\times 1.1=2.42$. The exact second value is about 2.4428.

Check the initial conditions as well as the equation



$$dy/dx = 0.5y \text{ with } y(0) = 2 \text{ gives } y = 2e^{(0.5x)}; y(1) \approx 3.30$$

■ Where it goes wrong

- An initial condition determines the integration constant; it is not a replacement for integrating.
- Dividing by y can omit an equilibrium solution $y=0$.
- Euler accuracy depends on step size and the equation.

Not true: Every step size gives exactly the same Euler approximation. Euler's method depends on the step size; smaller steps usually give a more accurate value.

■ Applying it

- For second-order linear equations, combine the complementary function with an appropriate particular integral, then apply the required initial conditions.
 - This is Further Pure scope; check the named unit.
-

■ In your course

- Analysis and Approaches HL, the course first assessed in 2021.
- Paper 1 is non-calculator: keep fractions, surds and π exact and show each step. On calculator papers, still write the expression before its value.

Key:

- **Initial condition:** A specified value that selects a particular solution from a family.
- For $dy/dx=ky$, separate variables: $dy/y=k dx$, giving $y=Ae^{(kx)}$.
- Watch for: an initial condition determines the integration constant; it is not a replacement for integrating.

■ **Write from a first Euler value of 4.4**

- Use Euler with step 0.2 for $y' = 0.5y$, starting at $y = 4$.
- Find the first two approximations. A different step size gives a different approximation.

■ **Write from a second Euler value of 4.84**

- Continue the same computation one more step.
- Each step multiplies by $1 + 0.5 \times 0.2 = 1.1$, so the values run 4, 4.4, 4.84.

■ **Write from the exact value 4.886**

- Solve $y' = 0.5y$ exactly and evaluate at $x = 0.4$.
- $y = 4e^{0.5 \times 0.4}$ gives 4.886 – Euler's 4.84 undershoots because the curve bends upward.

School-supervised mathematical exploration

International Baccalaureate · IB Diploma · Mathematics: Analysis and Approaches · HL

Scope and prerequisites

This handout covers selected HL topics for the course first assessed in 2021; it is not the whole syllabus.

Prerequisites: read the stated quantities and units, and use arithmetic and the model conditions below.

The examples are original and fictional: they are not real measurements or real student investigations.

Planning a mathematical exploration

■ What makes an investigation mathematical?

- A student wants to predict queue waiting time at the school canteen. A useful exploration needs a question, data and mathematics the student can explain.
- This lesson studies **assumption** : A condition adopted to make a mathematical model possible and whose effect should be evaluated.

■ The method

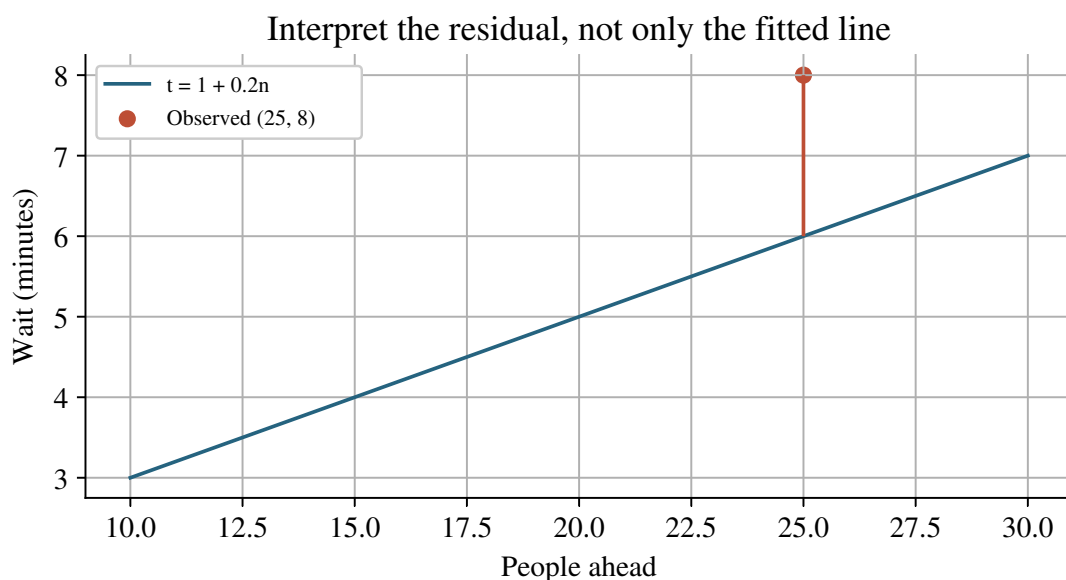
- Frame a focused question with measurable variables and a feasible method.
- Keep a dated record of decisions, sources and mathematical reasoning.
- Use technology to support analysis, and explain every important output in your own words.

■ A worked case

$$\hat{t} = 1 + 0.2n, \quad e = t - \hat{t}$$

Example:

A pilot records mean waiting times of 3,5,7 minutes when 10,20,30 people are ahead. A first model is $t=1+0.2n$. At $n=25$ it predicts 6 minutes. An actual wait of 8 gives residual 2; investigate varying service rates rather than hiding the mismatch.



At $n = 25$ the model predicts 6 minutes but 8 was observed: residual 2

■ Where it goes wrong

- Do not copy a sample exploration, invent data or manufacture personal engagement.
- A prediction is not evidence until it is compared with observations.
- Consent and privacy matter when collecting school data.

Not true: The school-supervised mathematical exploration can be replaced by a timed written mock exam. The exploration is an extended piece of written work assessed on its own criteria; a timed exam tests something different.

■ Applying it

- The school supervises topic approval, checkpoints, authenticity and assessment.
- For the current 2021 course use the current school's guide and criteria; the 2029 inquiry criteria belong to a different assessment version.

■ In your course

- Analysis and Approaches HL, the course first assessed in 2021.
- Paper 1 is non-calculator: keep fractions, surds and π exact and show each step. On calculator papers, still write the expression before its value.

Key:

- **Assumption:** A condition adopted to make a mathematical model possible and whose effect should be evaluated.
- Frame a focused question with measurable variables and a feasible method.
- Watch for: do not copy a sample exploration, invent data or manufacture personal engagement.

■ **Write from a predicted wait of 11**

- The model $t = 3 + 0.4n$ predicts the wait at $n = 20$.
- An observed wait of 15 leaves a residual. A timed mock does not replace the exploration.

■ **Write from a zero residual**

- At $n = 5$ the observed wait equals the model's prediction exactly.
- What is the residual, and does it certify the model? Zero – one perfect point certifies nothing; the PATTERN of all residuals does.

■ **Write from a proportionality check**

- When n doubles from 10 to 20, the observed wait doubles from 5 to 10.
- Which fits, linear or proportional? Proportional – wait = $0.5n$, a line through the origin; plot before trusting any trend.