

# Calculus

International Baccalaureate · IB Diploma · Mathematics: Analysis and Approaches · HL

## Scope and prerequisites

This handout covers selected HL topics for the course first assessed in 2021; it is not the whole syllabus.

Prerequisites: read the stated quantities and units, and use arithmetic and the model conditions below.

The examples are original and fictional: they are not real measurements or real student investigations.

## Derivatives and stationary points

### ■ What is the slope at one point?

- A curved road has different slopes at different positions. An average gradient cannot describe every point.
- This lesson studies **derivative** : The instantaneous rate of change, also the gradient of a tangent.

### ■ The method

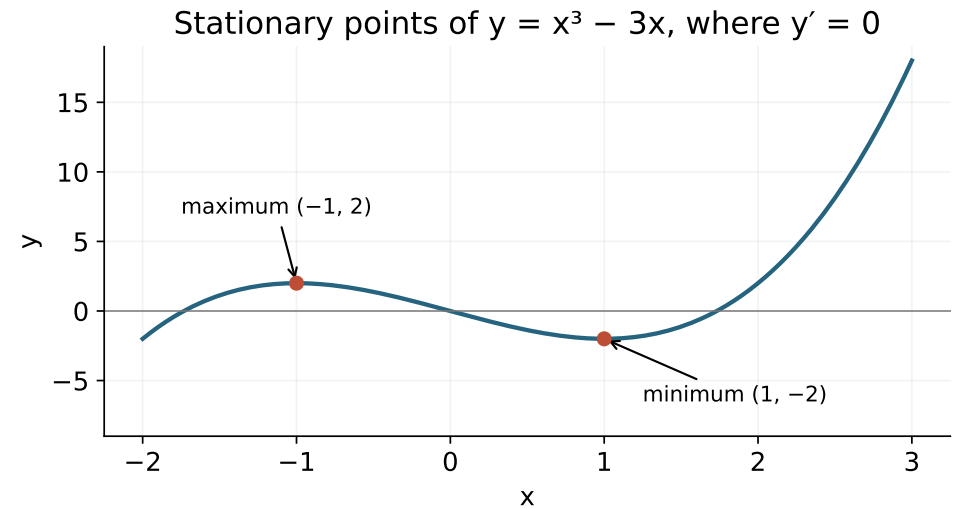
- For  $y = ax^n$ ,  $dy/dx = anx^{n-1}$ .
- A stationary point satisfies  $dy/dx = 0$ .
- Check the sign change of the derivative, or the second derivative when it is nonzero, to classify it.

### ■ A worked case

$$\frac{d}{dx}(ax^n) = anx^{n-1}, \quad f'(x) = 0$$

#### Example:

For  $y = x^3 - 3x$ ,  $dy/dx = 3x^2 - 3$ . At  $x = 1$ , the gradient is 0 and  $y = -2$ . The second derivative is  $6x$ , positive at  $x = 1$ , so this is a local minimum. At  $x = -1$ ,  $y = 2$  and the second derivative is negative, giving a local maximum.



$$y' = 3x^2 - 3 = 0 \text{ at } x = \pm 1: \text{ maximum } (-1, 2), \text{ minimum } (1, -2)$$

### ■ Where it goes wrong

- A zero derivative does not always mean a maximum or minimum:  $y = x^3$  is stationary at 0 but continues increasing.
- An endpoint can also produce an extreme value on a restricted domain.

**Not true:** Every point with zero derivative is a local maximum or minimum. A zero derivative can also be a point of inflection:  $y = x^3$  at  $x = 0$  is neither a maximum nor a minimum.

### ■ Applying it

- At GCSE/IGCSE use only the polynomial scope allowed by the tier; do not add chain, product or quotient rules there.
- At advanced level, connect the derivative to rates and optimization with a valid domain.

### ■ In your course

- Analysis and Approaches HL, the course first assessed in 2021.
- Paper 1 is non-calculator: keep fractions, surds and  $\pi$  exact and show each step. On calculator papers, still write the expression before its value.

**Key:**

- **Derivative:** The instantaneous rate of change, also the gradient of a tangent.
- For  $y = ax^n$ ,  $dy/dx = anx^{n-1}$ .
- Watch for: A zero derivative does not always mean a maximum or minimum:  $y = x^3$  is stationary at 0 but continues increasing.

#### ■ Write from a derivative of 8

- For  $y = x^2 - 10x$ , find the derivative at  $x = 9$ .
- Find the stationary value. A derivative of 0 is not always a turning point.

## Integrals, area and accumulation

#### ■ How much change has accumulated?

- A velocity graph tells us motion at an instant. How can we recover the displacement accumulated over time?
- This lesson studies **antiderivative** : A function whose derivative equals the given integrand.

#### ■ The method

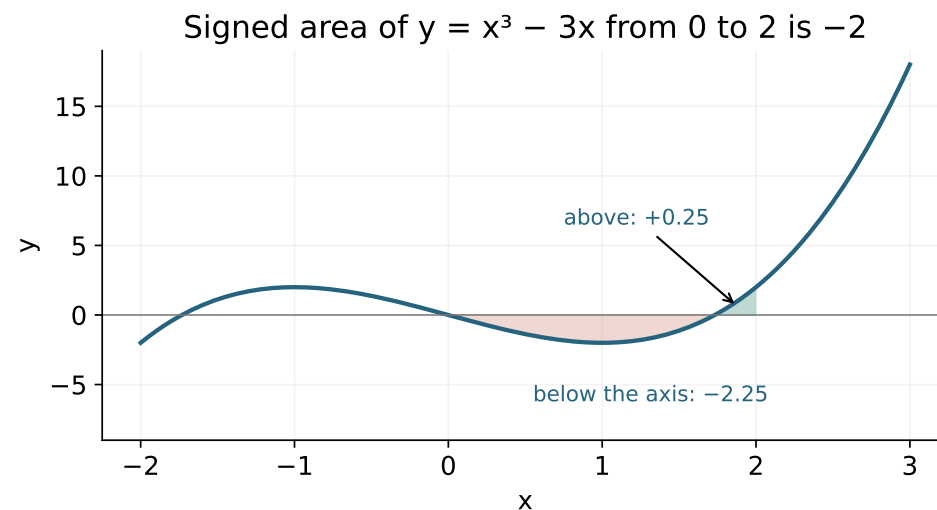
- For  $n \neq -1$ , the integral of  $ax^n$  is  $ax^{n+1}/(n+1) + C$ .
- A definite integral is signed accumulation.
- Split at sign changes when total area or distance is required.

#### ■ A worked case

$$\int ax^n dx = \frac{ax^{n+1}}{n+1} + C \quad (n \neq -1)$$

#### Example:

For  $v(t) = 3t^2 - 3$  over  $0 \leq t \leq 2$ , displacement =  $[t^3 - 3t]_0^2 = 2$ . Since  $v$  changes sign at  $t = 1$ , distance =  $-[t^3 - 3t]_0^1 + [t^3 - 3t]_1^2 = 2 + 4 = 6$ . The constants cancel only for a definite integral.



$$\int_0^2 (x^3 - 3x) dx = -2.25 + 0.25 = -2: \text{ area below the axis counts as negative}$$

#### ■ Where it goes wrong

- The integral of  $1/x$  is  $\ln|x| + C$ , not the power rule with  $n = -1$ .
- Signed area can be zero even when the total enclosed area is positive.

**Not true:** A definite integral always equals the total positive area, even when the graph crosses the axis. Area below the axis counts as negative, so a definite integral can be smaller than the total area, or even zero.

#### ■ Applying it

- Identify what the integral means and include the correct units.
- A rate measured per second integrates to the underlying quantity, not to another rate.
- Differentiate an antiderivative to check it.

#### ■ In your course

- Analysis and Approaches HL, the course first assessed in 2021.
- Paper 1 is non-calculator: keep fractions, surds and  $\pi$  exact and show each step. On calculator papers, still write the expression before its value.

#### Key:

- **Antiderivative:** A function whose derivative equals the given integrand.

- For  $n \neq -1$ , the integral of  $ax^n$  is  $ax^{n+1}/(n+1) + C$ .
- Watch for: the integral of  $1/x$  is  $\ln|x| + C$ , not the power rule with  $n = -1$ .

### ■ Write from an integral of 9

- Find the integral of  $2x$  from 0 to 3.
- For  $v = t$  minus 2, from  $t = 0$  to 4, find the displacement and the distance. A signed integral is not the total distance.

## Chain, product, quotient and implicit differentiation

### ■ Does the inside expression change too?

- A cost curve is a power of a changing expression. Differentiating the outer power alone misses the rate of its input.
- This lesson studies **chain rule**: The rule that multiplies the outer derivative by the inner derivative for a composite function.

### ■ The method

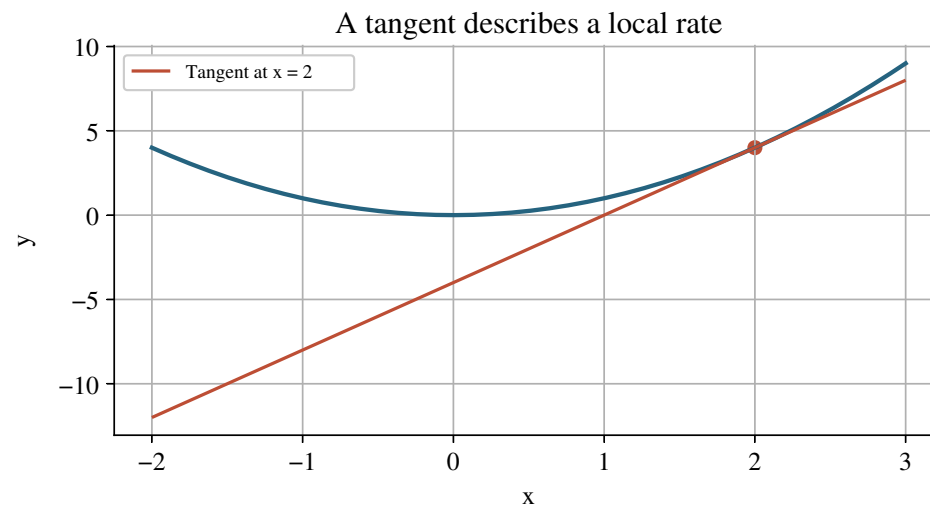
- For  $y = f(g(x))$ ,  $y' = f'(g(x))g'(x)$ .
- For  $uv$ , differentiate to  $u'v + uv'$ .
- For  $u/v$ , use  $(u'v - uv')/v^2$ , where  $v \neq 0$ .

### ■ A worked case

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$$

### Example:

For  $y = (3x+1)^4$ ,  $y' = 4(3x+1)^3 \times 3 = 12(3x+1)^3$ . At  $x=0$ , the gradient is 12.  
 For  $x^2 + y^2 = 25$ , differentiate implicitly:  $2x + 2y y' = 0$ , so  $y' = -x/y$  when  $y \neq 0$ .



*The tangent to  $y = x^2$  at  $x = 2$  has gradient  $2x = 4$ :  $y = 4x - 4$*

### ■ Where it goes wrong

- A derivative of a product is not the product of the derivatives.
- In implicit differentiation, every differentiated function of  $y$  brings a  $dy/dx$  factor.
- A quotient's denominator is squared.

**Not true:** The derivative of  $u(x)v(x)$  is always  $u'$  times  $v$  prime. The product rule is  $(uv)' = u'v + uv'$ : for  $x \cdot x$  the derivative is  $2x$ , not  $1 \times 1 = 1$ .

### ■ Applying it

- Choose a useful form before differentiating: expanding a short polynomial may be simpler.
- For related rates, write the relation in symbols, differentiate with respect to time, then substitute measured values.

### ■ In your course

- Analysis and Approaches HL, the course first assessed in 2021.
- Paper 1 is non-calculator: keep fractions, surds and  $\pi$  exact and show each step. On calculator papers, still write the expression before its value.

**Key:**

- **Chain rule:** The rule that multiplies the outer derivative by the inner derivative for a composite function.
- For  $y=f(g(x))$ ,  $y' = f'(g(x))g'(x)$ .
- Watch for: A derivative of a product is not the product of the derivatives.

#### ■ Write from a product derivative of 11

- For  $y = (2x + 1)(x + 3)$ , find the derivative at  $x = 1$ .
- The derivative of a product is not the product of the derivatives.

## Substitution, parts and partial fractions

#### ■ Can we integrate the two factors separately?

- A simple-looking product such as  $x e^x$  cannot be integrated by separately integrating each factor.
- This lesson studies **integration by parts** : An integration method based on the derivative of a product.

#### ■ The method

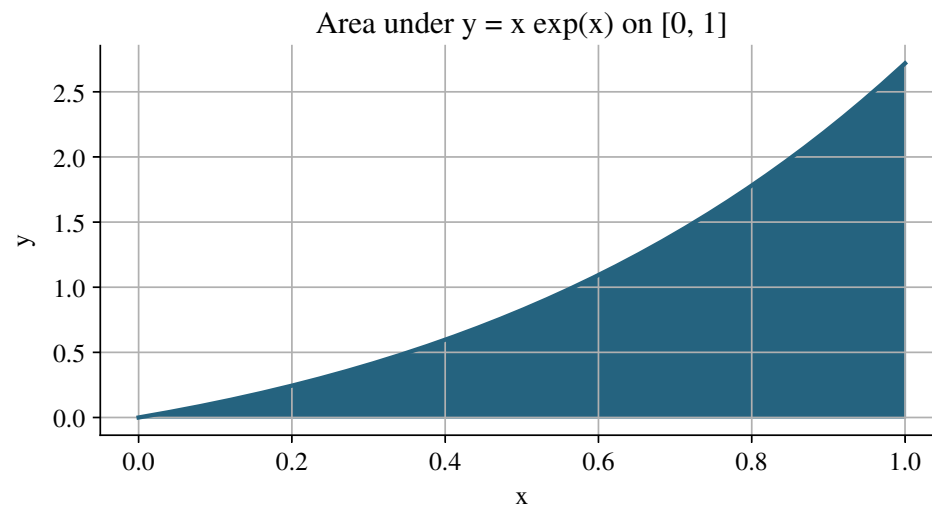
- Use substitution when an inner derivative appears as a factor.
- By parts,  $\int u v' = uv - \int u' v$ .
- For a rational function, divide first if needed, then decompose into partial fractions.

#### ■ A worked case

$$\int u v' dx = uv - \int u' v dx$$

#### Example:

For integral  $x e^x dx$ , take  $u=x$  and  $v' = e^x$ . Then the integral is  $x e^x - e^x + C$ .  
 For integral  $2x/(x^2+1) dx$ , substitute  $w=x^2+1$ ,  $dw=2x dx$ ; the answer is  $\ln(x^2+1)+C$ .



$$\text{Integration by parts: } \int_0^1 x e^x dx = [x e^x - e^x]_0^1 = 1$$

#### ■ Where it goes wrong

- Choosing  $u$  and  $v'$  well matters: the remaining integral should become simpler.
- In a definite substitution, either change the limits or return to  $x$  before applying the original limits.

**Not true:** The integral of a product equals the product of its separate integrals. A product needs a method such as integration by parts or substitution:  $\int x \cdot x dx = x^3/3$ , not  $(x^2/2)(x^2/2)$ .

#### ■ Applying it

- Check by differentiating.
- For volumes of revolution around the  $x$ -axis use  $V = \pi \int y^2 dx$ ; do not confuse the square of a function with the integral of the function.

#### ■ In your course

- Analysis and Approaches HL, the course first assessed in 2021.
- Paper 1 is non-calculator: keep fractions, surds and  $\pi$  exact and show each step. On calculator papers, still write the expression before its value.

**Key:**

- **Integration by parts:** An integration method based on the derivative of a product.
- Use substitution when an inner derivative appears as a factor.
- Watch for: choosing u and v prime well matters: the remaining integral should become simpler.

#### ■ Write from a volume factor of 18

- The line  $y = 3$  is revolved about the x-axis from 0 to 2. Find the volume divided by pi.
- The integral of a product is not the product of the integrals.

## Differential equations and numerical solutions

#### ■ What does the starting value decide?

- A growing population's rate is proportional to its current size. The rate equation describes a whole family until an initial population is supplied.
- This lesson studies **initial condition** : A specified value that selects a particular solution from a family.

#### ■ The method

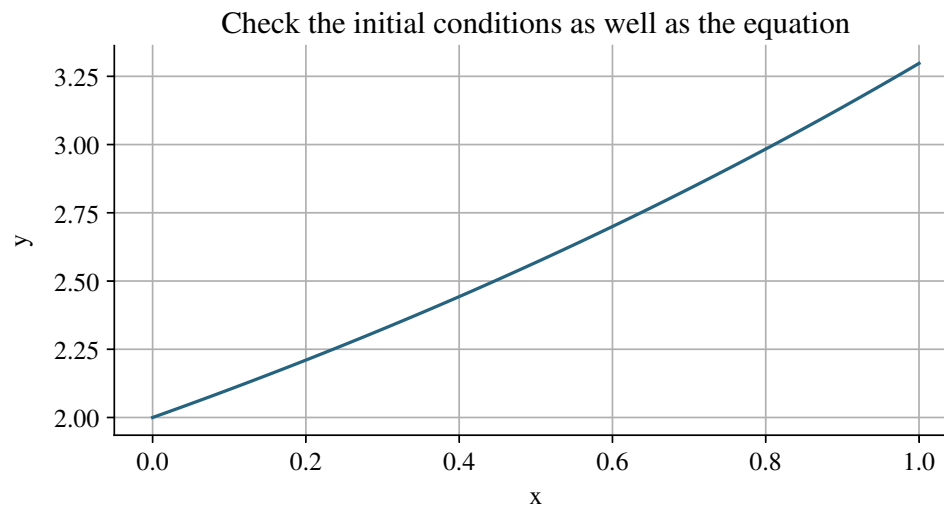
- For  $dy/dx=ky$ , separate variables:  $dy/y=k dx$ , giving  $y=Ae^{(kx)}$ .
- Use an initial condition to find A.
- Euler's method takes  $y_{next}=y+h f(x,y)$ , with a chosen step h.

#### ■ A worked case

$$\frac{dy}{dx} = ky, \quad y = Ae^{kx}$$

#### Example:

With  $dy/dx=0.5y$  and  $y(0)=2$ ,  $y=2e^{(0.5x)}$ . Euler with  $h=0.2$  gives  $y(0.2)\approx 2+0.2\times 1=2.2$  and  $y(0.4)\approx 2.2+0.2\times 1.1=2.42$ . The exact second value is about 2.4428.



$$dy/dx = 0.5y \text{ with } y(0) = 2 \text{ gives } y = 2e^{(0.5x)}; y(1) \approx 3.30$$

#### ■ Where it goes wrong

- An initial condition determines the integration constant; it is not a replacement for integrating.
- Dividing by y can omit an equilibrium solution  $y=0$ .
- Euler accuracy depends on step size and the equation.

**Not true:** Every step size gives exactly the same Euler approximation. Euler's method depends on the step size; smaller steps usually give a more accurate value.

#### ■ Applying it

- For second-order linear equations, combine the complementary function with an appropriate particular integral, then apply the required initial conditions.
- This is Further Pure scope; check the named unit.

#### ■ In your course

- Analysis and Approaches HL, the course first assessed in 2021.
- Paper 1 is non-calculator: keep fractions, surds and  $\pi$  exact and show each step. On calculator papers, still write the expression before its value.

**Key:**

- **Initial condition:** A specified value that selects a particular solution from a family.
- For  $dy/dx=ky$ , separate variables:  $dy/y=k dx$ , giving  $y=Ae^{(kx)}$ .
- Watch for: an initial condition determines the integration constant; it is not a replacement for integrating.

■ **Write from a first Euler value of 4.4**

- Use Euler with step 0.2 for  $y' = 0.5 y$ , starting at  $y = 4$ .
- Find the first two approximations. A different step size gives a different approximation.

■ **Write from a second Euler value of 4.84**

- Continue the same computation one more step.
- Each step multiplies by  $1 + 0.5 \times 0.2 = 1.1$ , so the values run 4, 4.4, 4.84.

■ **Write from the exact value 4.886**

- Solve  $y' = 0.5 y$  exactly and evaluate at  $x = 0.4$ .
- $y = 4 e^{0.2}$  to the 0.5  $x$  gives 4.886 – Euler's 4.84 undershoots because the curve bends upward.