

Calculus

International Baccalaureate · IB Diploma · Mathematics: Analysis
and Approaches · HL

Scope and prerequisites

This handout covers selected HL topics for the course first assessed in 2021; it is not the whole syllabus.

Prerequisites: read the stated quantities and units, and use arithmetic and the model conditions below.

The examples are original and fictional: they are not real measurements or real student investigations.

Derivatives and stationary points

■ What is the slope at one point?

- A curved road has different slopes at different positions. An average gradient cannot describe every point.
- This lesson studies **derivative** : The instantaneous rate of change, also the gradient of a tangent.

■ The method

- For $y = ax^n$, $dy/dx = anx^{n-1}$.
- A stationary point satisfies $dy/dx = 0$.
- Check the sign change of the derivative, or the second derivative when it is nonzero, to classify it.

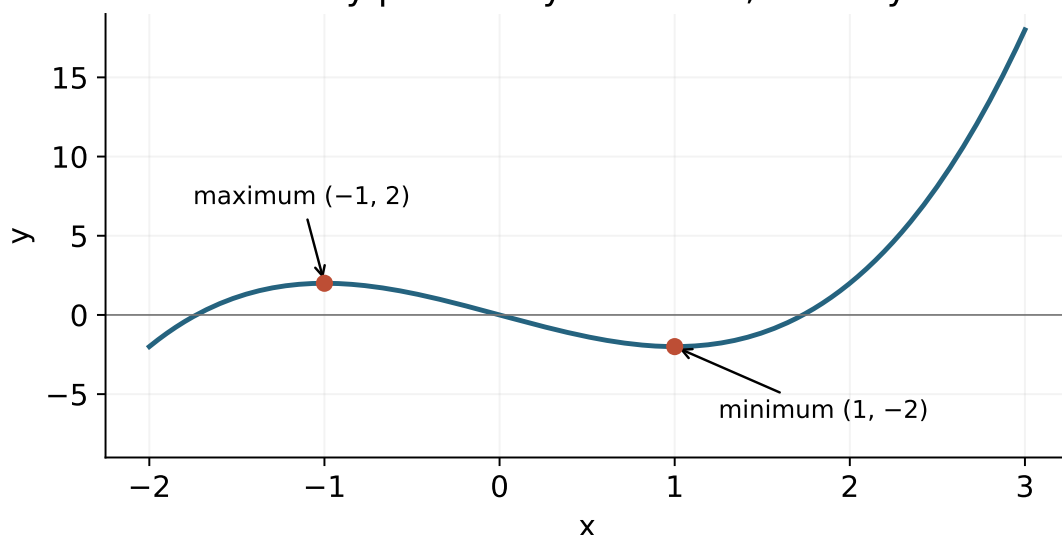
■ A worked case

$$\frac{d}{dx}(ax^n) = anx^{n-1}, \quad f'(x) = 0$$

Example:

For $y = x^3 - 3x$, $dy/dx = 3x^2 - 3$. At $x = 1$, the gradient is 0 and $y = -2$. The second derivative is $6x$, positive at $x = 1$, so this is a local minimum. At $x = -1$, $y = 2$ and the second derivative is negative, giving a local maximum.

Stationary points of $y = x^3 - 3x$, where $y' = 0$



$$y' = 3x^2 - 3 = 0 \text{ at } x = \pm 1: \text{ maximum } (-1, 2), \text{ minimum } (1, -2)$$

■ Where it goes wrong

- A zero derivative does not always mean a maximum or minimum: $y=x^3$ is stationary at 0 but continues increasing.
- An endpoint can also produce an extreme value on a restricted domain.

Not true: Every point with zero derivative is a local maximum or minimum. A zero derivative can also be a point of inflection: $y=x^3$ at $x=0$ is neither a maximum nor a minimum.

■ Applying it

- At GCSE/IGCSE use only the polynomial scope allowed by the tier; do not add chain, product or quotient rules there.
- At advanced level, connect the derivative to rates and optimization with a valid domain.

■ In your course

- Analysis and Approaches HL, the course first assessed in 2021.
- Paper 1 is non-calculator: keep fractions, surds and π exact and show each step. On calculator papers, still write the expression before its value.

Key:

- **Derivative:** The instantaneous rate of change, also the gradient of a tangent.
- For $y = ax^n$, $dy/dx = anx^{n-1}$.
- Watch for: A zero derivative does not always mean a maximum or minimum: $y = x^3$ is stationary at 0 but continues increasing.

■ Write from a derivative of 8

- For $y = x^2 - 10x$, find the derivative at $x = 9$.
- Find the stationary value. A derivative of 0 is not always a turning point.

Integrals, area and accumulation

■ How much change has accumulated?

- A velocity graph tells us motion at an instant. How can we recover the displacement accumulated over time?
 - This lesson studies **antiderivative** : A function whose derivative equals the given integrand.
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■ The method

- For $n \neq -1$, the integral of ax^n is $ax^{n+1}/(n+1) + C$.
 - A definite integral is signed accumulation.
 - Split at sign changes when total area or distance is required.
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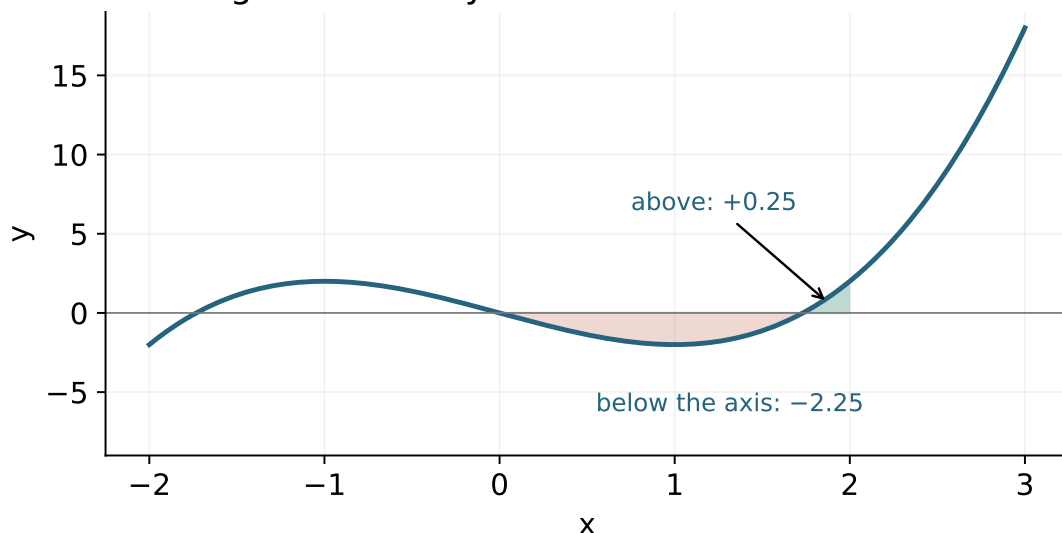
■ A worked case

$$\int ax^n dx = \frac{ax^{n+1}}{n+1} + C \quad (n \neq -1)$$

Example:

For $v(t) = 3t^2 - 3$ over $0 \leq t \leq 2$, displacement = $[t^3 - 3t]_0^2 = 2$. Since v changes sign at $t = 1$, distance = $-[t^3 - 3t]_0^1 + [t^3 - 3t]_1^2 = 2 + 4 = 6$. The constants cancel only for a definite integral.

Signed area of $y = x^3 - 3x$ from 0 to 2 is -2



$$\int_0^2 (x^3 - 3x) dx = -2.25 + 0.25 = -2: \text{ area below the axis counts as negative}$$

■ Where it goes wrong

- The integral of $1/x$ is $\ln|x|+C$, not the power rule with $n=-1$.
- Signed area can be zero even when the total enclosed area is positive.

Not true: A definite integral always equals the total positive area, even when the graph crosses the axis. Area below the axis counts as negative, so a definite integral can be smaller than the total area, or even zero.

■ Applying it

- Identify what the integral means and include the correct units.
- A rate measured per second integrates to the underlying quantity, not to another rate.
- Differentiate an antiderivative to check it.

■ In your course

- Analysis and Approaches HL, the course first assessed in 2021.
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Key:

- **Antiderivative:** A function whose derivative equals the given integrand.

- For $n \neq -1$, the integral of ax^n is $ax^{(n+1)}/(n+1)+C$.
- Watch for: the integral of $1/x$ is $\ln|x|+C$, not the power rule with $n=-1$.

■ Write from an integral of 9

- Find the integral of $2x$ from 0 to 3.
- For $v = t$ minus 2, from $t = 0$ to 4, find the displacement and the distance. A signed integral is not the total distance.

Chain, product, quotient and implicit differentiation

■ Does the inside expression change too?

- A cost curve is a power of a changing expression. Differentiating the outer power alone misses the rate of its input.
 - This lesson studies **chain rule** : The rule that multiplies the outer derivative by the inner derivative for a composite function.
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■ The method

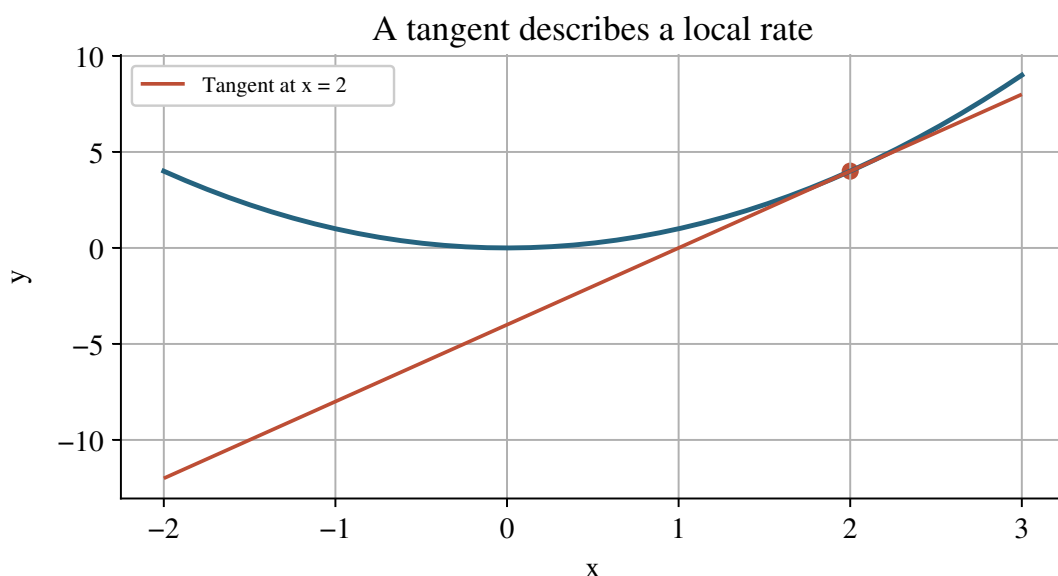
- For $y=f(g(x))$, $y' = f'(g(x))g'(x)$.
 - For uv , differentiate to $u'v + uv'$.
 - For u/v , use $(u'v - uv')/v^2$, where $v \neq 0$.
-

■ A worked case

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$$

Example:

For $y=(3x+1)^4$, $y' = 4(3x+1)^3 \times 3 = 12(3x+1)^3$. At $x=0$, the gradient is 12.
 For $x^2+y^2=25$, differentiate implicitly: $2x+2y y' = 0$, so $y' = -x/y$ when $y \neq 0$.



The tangent to $y = x^2$ at $x = 2$ has gradient $2x = 4$: $y = 4x - 4$

■ Where it goes wrong

- A derivative of a product is not the product of the derivatives.
- In implicit differentiation, every differentiated function of y brings a dy/dx factor.
- A quotient's denominator is squared.

Not true: The derivative of $u(x)v(x)$ is always u prime times v prime. The product rule is $(uv)' = u'v + uv'$: for $x \cdot x$ the derivative is $2x$, not $1 \times 1 = 1$.

■ Applying it

- Choose a useful form before differentiating: expanding a short polynomial may be simpler.
- For related rates, write the relation in symbols, differentiate with respect to time, then substitute measured values.

■ In your course

- Analysis and Approaches HL, the course first assessed in 2021.
- Paper 1 is non-calculator: keep fractions, surds and π exact and show each step. On calculator papers, still write the expression before its value.

Key:

- **Chain rule:** The rule that multiplies the outer derivative by the inner derivative for a composite function.
- For $y=f(g(x))$, $y' = f'(g(x))g'(x)$.
- Watch for: A derivative of a product is not the product of the derivatives.

■ Write from a product derivative of 11

- For $y = (2x + 1)(x + 3)$, find the derivative at $x = 1$.
- The derivative of a product is not the product of the derivatives.

Substitution, parts and partial fractions

■ Can we integrate the two factors separately?

- A simple-looking product such as $x e^x$ cannot be integrated by separately integrating each factor.
 - This lesson studies **integration by parts** : An integration method based on the derivative of a product.
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■ The method

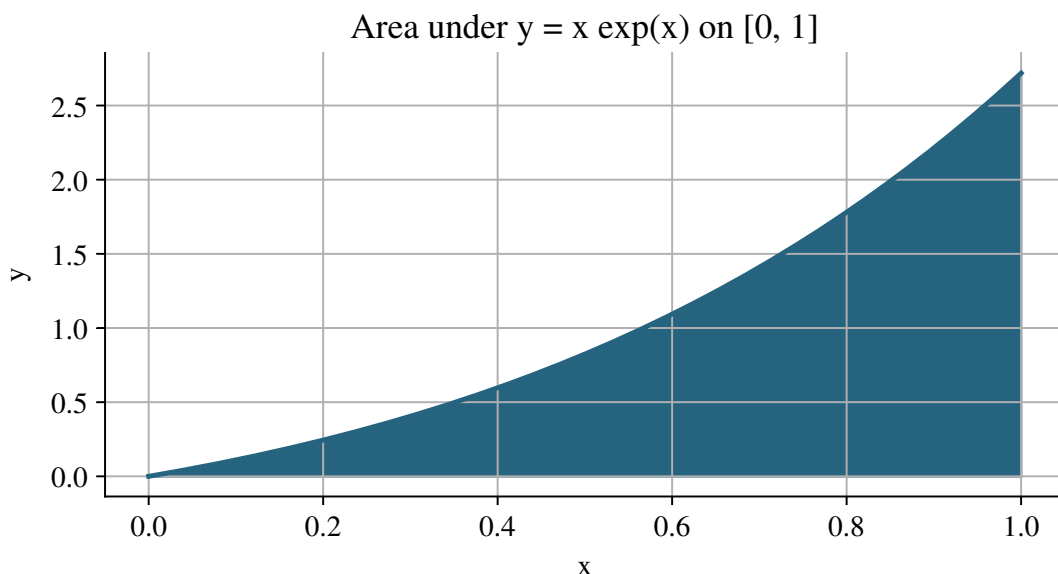
- Use substitution when an inner derivative appears as a factor.
 - By parts, $\int u v' = uv - \int u' v$.
 - For a rational function, divide first if needed, then decompose into partial fractions.
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■ A worked case

$$\int uv' dx = uv - \int u'v dx$$

Example:

For integral $\int x e^x dx$, take $u=x$ and $v' = e^x$. Then the integral is $x e^x - e^x + C$.
 For integral $\int 2x/(x^2+1) dx$, substitute $w=x^2+1$, $dw=2x dx$; the answer is $\ln(x^2+1) + C$.



$$\text{Integration by parts: } \int_0^1 x e^x dx = [x e^x - e^x]_0^1 = 1$$

■ Where it goes wrong

- Choosing u and v prime well matters: the remaining integral should become simpler.
- In a definite substitution, either change the limits or return to x before applying the original limits.

Not true: The integral of a product equals the product of its separate integrals. A product needs a method such as integration by parts or substitution: $\int x \cdot x dx = x^3/3$, not $(x^2/2)(x^2/2)$.

■ Applying it

- Check by differentiating.
- For volumes of revolution around the x -axis use $V = \pi \int y^2 dx$; do not confuse the square of a function with the integral of the function.

■ In your course

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- Paper 1 is non-calculator: keep fractions, surds and π exact and show each step. On calculator papers, still write the expression before its value.

Key:

- **Integration by parts:** An integration method based on the derivative of a product.
- Use substitution when an inner derivative appears as a factor.
- Watch for: choosing u and v prime well matters: the remaining integral should become simpler.

■ Write from a volume factor of 18

- The line $y = 3$ is revolved about the x -axis from 0 to 2. Find the volume divided by π .
- The integral of a product is not the product of the integrals.

Differential equations and numerical solutions

■ What does the starting value decide?

- A growing population's rate is proportional to its current size. The rate equation describes a whole family until an initial population is supplied.
 - This lesson studies **initial condition** : A specified value that selects a particular solution from a family.
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■ The method

- For $dy/dx=ky$, separate variables: $dy/y=k dx$, giving $y=Ae^{(kx)}$.
 - Use an initial condition to find A .
 - Euler's method takes $y_{\text{next}}=y+h f(x,y)$, with a chosen step h .
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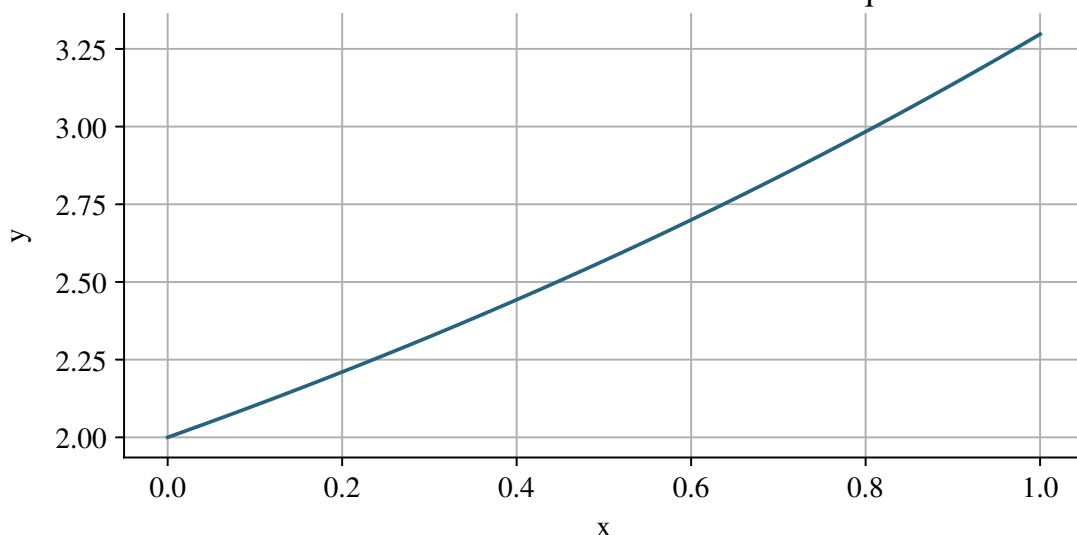
■ A worked case

$$\frac{dy}{dx} = ky, \quad y = Ae^{kx}$$

Example:

With $dy/dx=0.5y$ and $y(0)=2$, $y=2e^{(0.5x)}$. Euler with $h=0.2$ gives $y(0.2)\approx 2+0.2\times 1=2.2$ and $y(0.4)\approx 2.2+0.2\times 1.1=2.42$. The exact second value is about 2.4428.

Check the initial conditions as well as the equation



$$dy/dx = 0.5y \text{ with } y(0) = 2 \text{ gives } y = 2e^{(0.5x)}; y(1) \approx 3.30$$

■ Where it goes wrong

- An initial condition determines the integration constant; it is not a replacement for integrating.
- Dividing by y can omit an equilibrium solution $y=0$.
- Euler accuracy depends on step size and the equation.

Not true: Every step size gives exactly the same Euler approximation. Euler's method depends on the step size; smaller steps usually give a more accurate value.

■ Applying it

- For second-order linear equations, combine the complementary function with an appropriate particular integral, then apply the required initial conditions.
 - This is Further Pure scope; check the named unit.
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■ In your course

- Analysis and Approaches HL, the course first assessed in 2021.
- Paper 1 is non-calculator: keep fractions, surds and π exact and show each step. On calculator papers, still write the expression before its value.

Key:

- **Initial condition:** A specified value that selects a particular solution from a family.
- For $dy/dx=ky$, separate variables: $dy/y=k dx$, giving $y=Ae^{(kx)}$.
- Watch for: an initial condition determines the integration constant; it is not a replacement for integrating.

■ **Write from a first Euler value of 4.4**

- Use Euler with step 0.2 for $y' = 0.5y$, starting at $y = 4$.
- Find the first two approximations. A different step size gives a different approximation.

■ **Write from a second Euler value of 4.84**

- Continue the same computation one more step.
- Each step multiplies by $1 + 0.5 \times 0.2 = 1.1$, so the values run 4, 4.4, 4.84.

■ **Write from the exact value 4.886**

- Solve $y' = 0.5y$ exactly and evaluate at $x = 0.4$.
- $y = 4e^{0.5 \times 0.4}$ gives 4.886 – Euler's 4.84 undershoots because the curve bends upward.