

Number and algebra

International Baccalaureate · IB Diploma · Mathematics: Analysis and Approaches · HL

Scope and prerequisites

This handout covers selected HL topics for the course first assessed in 2021; it is not the whole syllabus.

Prerequisites: read the stated quantities and units, and use arithmetic and the model conditions below.

The examples are original and fictional: they are not real measurements or real student investigations.

Exact arithmetic and estimation

■ Can we pack without leftovers?

- A supplier packs 72 pencils and 90 pens into identical gift bags. How can we avoid leftovers?
- This lesson studies **prime factor** : A prime number that divides the integer exactly.

■ The method

- Prime factors reveal shared structure.
- Use the smallest common prime powers for the HCF and the largest for the LCM.
- Estimate before calculating; use brackets to preserve the order of operations.

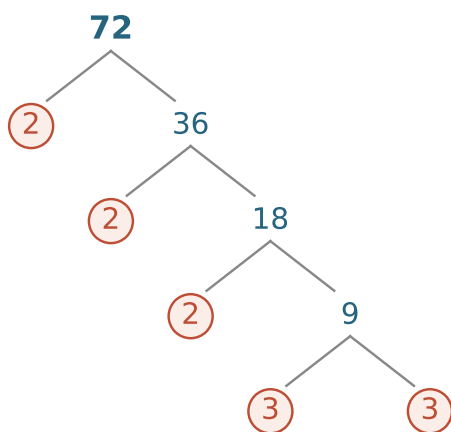
■ A worked case

$$a = \prod p_i^{\alpha_i}, \quad b = \prod p_i^{\beta_i}, \quad \text{HCF}(a, b) = \prod p_i^{\min(\alpha_i, \beta_i)}$$

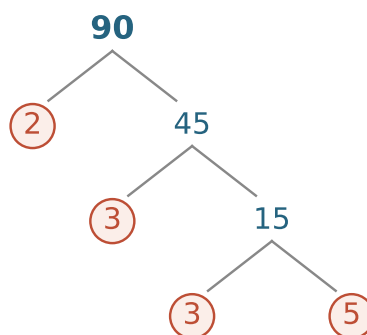
Example:

$72 = 2^3 \times 3^2$ and $90 = 2 \times 3^2 \times 5$. Their HCF is $2 \times 9 = 18$. Make 18 bags with 4 pencils and 5 pens each. Their LCM is $2^3 \times 3^2 \times 5 = 360$.

Factor trees: $\text{HCF}(72, 90) = 2 \times 3^2 = 18$



$$72 = 2^3 \times 3^2$$



$$90 = 2 \times 3^2 \times 5$$

Factor trees: $72 = 2^3 \times 3^2$ and $90 = 2 \times 3^2 \times 5$, so $\text{HCF} = 2 \times 3^2 = 18$

■ Where it goes wrong

- The HCF divides both numbers; the LCM is a multiple of both.
- They answer different questions.
- A decimal estimate is not an exact fraction.

Not true: The HCF of two positive integers is always larger than either integer. The HCF divides both numbers, so it can never be larger than the smaller one: the HCF of 72 and 90 is 18.

■ Applying it

- For a non-calculator paper, keep fractions exact and show cancellation.
- For a calculator paper, enter the full expression and compare with your estimate.

■ In your course

- Analysis and Approaches HL, the course first assessed in 2021.

Key:

- **Prime factor:** A prime number that divides the integer exactly.
- Prime factors reveal shared structure.
- Watch for: the HCF divides both numbers; the LCM is a multiple of both.

■ Write from an HCF of 6

- Find the HCF and LCM of 18 and 30.
- The HCF is smaller than both integers.

Indices, surds and standard form

■ How small is a microscopic length?

- A microscope records a length of 0.000072 metres. A compact representation must keep its size correct.
 - This lesson studies **index** : The power to which a base is raised.
-

■ The method

- For the same positive base, multiplication adds indices and division subtracts them.
 - A negative index means reciprocal; a fractional index represents a root.
 - Standard form has $1 \leq a < 10$.
-

■ A worked case

$$a^m a^n = a^{m+n}, \quad a^{-n} = \frac{1}{a^n}, \quad a^{m/n} = (\sqrt[n]{a})^m$$

Example:

$0.000072 = 7.2 \times 10^{-5}$. Also $16^{3/4} = (16^{1/4})^3 = 2^3 = 8$. Simplify $\sqrt{72} = 6\sqrt{2}$, then rationalise $1/\sqrt{2} = \sqrt{2}/2$.

$0.000072 = 7.2 \times 10^{-5}$: the point moves 5 places

5 jumps right

0 . 0 0 0 0 7 2

$$7.2 \times 10^{-5}$$

coefficient 7.2 is between 1 and 10; the power is -5

Moving the point 5 places right gives 7.2, so $0.000072 = 7.2 \times 10^{-5}$

■ Where it goes wrong

- Index laws do not turn a sum into a single power: $2^3+2^4=24$, not 2^7 .
- Do not round a surd when an exact answer is requested.

Not true: For every positive a , a^2+a^3 equals a^5 . Index laws only add powers when the terms are multiplied: $2^2+2^3=12$, but $2^5=32$.

■ Applying it

- Check powers of ten against the original quantity.
 - Use surds for exact geometry, and round only the final length when the question asks for a decimal.
-

■ In your course

- Analysis and Approaches HL, the course first assessed in 2021.
- Paper 1 is non-calculator: keep fractions, surds and π exact and show each step. On calculator papers, still write the expression before its value.

Key:

- **Index:** The power to which a base is raised.
- For the same positive base, multiplication adds indices and division subtracts them.
- Watch for: index laws do not turn a sum into a single power: $2^3+2^4=24$, not 2^7 .

■ Write from 8 to the power $2/3$

- Evaluate 8 to the power $2/3$.
- A squared plus a cubed is not a to the fifth.

Sequences, series and recurrence

■ Does the change add or multiply?

- A saving plan adds ¥30 more each week; a population model grows by 5% each year. Equal differences and equal ratios need different models.
 - This lesson studies **common ratio** : The constant multiplier between consecutive terms of a geometric sequence.
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■ The method

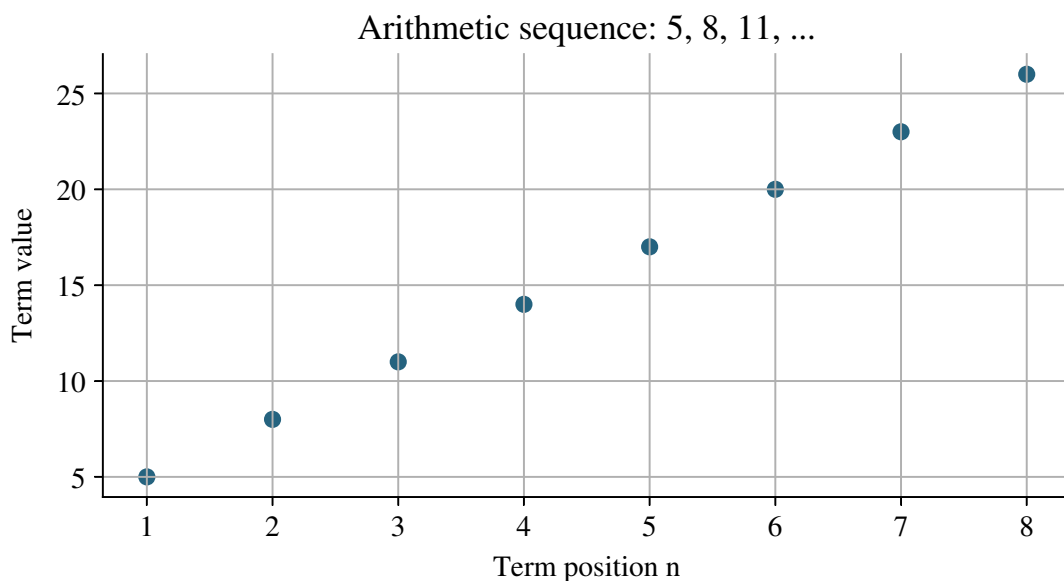
- For an arithmetic progression, $u_n = a + (n-1)d$ and $S_n = n(2a + (n-1)d)/2$.
 - For a geometric progression, $u_n = ar^{n-1}$ and $S_n = a(1-r^n)/(1-r)$.
 - An infinite geometric sum exists only if $|r| < 1$.
-

■ A worked case

$$u_n = a + (n-1)d, \quad S_n = \frac{n}{2} [2a + (n-1)d]$$

Example:

For $a=5, d=3, n=8$, $u_8 = 5 + 7 \times 3 = 26$ and $S_8 = 8(10+21)/2 = 124$. For $a=12, r=1/2$, $S_{\infty} = 12/(1-1/2) = 24$. For $u_{n+1} = 2u_n + 1$ with $u_1 = 1$, the next terms are 3, 7, 15.



$$u_n = 5 + 3(n-1): \text{equal steps of 3 put the terms on a straight line}$$

■ Where it goes wrong

- The first term has index 1, so the exponent is $n-1$.
- A sequence is a list; a series is a sum.
- A geometric sequence can alternate in sign and still converge.

Not true: Every geometric series has a finite sum to infinity. A sum to infinity exists only when $-1 < r < 1$; for $r=2$ the terms grow without limit.

■ Applying it

- Explain whether the context justifies additive or multiplicative change.
 - In finance, distinguish a single deposit from a stream of deposits before choosing a sum formula.
-

■ In your course

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Key:

- **Common ratio:** The constant multiplier between consecutive terms of a geometric sequence.
- For an arithmetic progression, $u_n = a + (n-1)d$ and $S_n = n(2a + (n-1)d)/2$.
- Watch for: the first term has index 1, so the exponent is $n-1$.

■ Write from the 4th term 17

- An arithmetic sequence has first term 8 and common difference 3.
- Find the 4th term. An infinite geometric series needs a common ratio with size below 1.

Percentages, ratio and proportional reasoning

■ Can we recover the original price?

- A coat is reduced by 20% to ¥240. The discount applies to the original price, not to the sale price.
 - This lesson studies **multiplier**: A factor that performs a percentage change in one multiplication.
-

■ The method

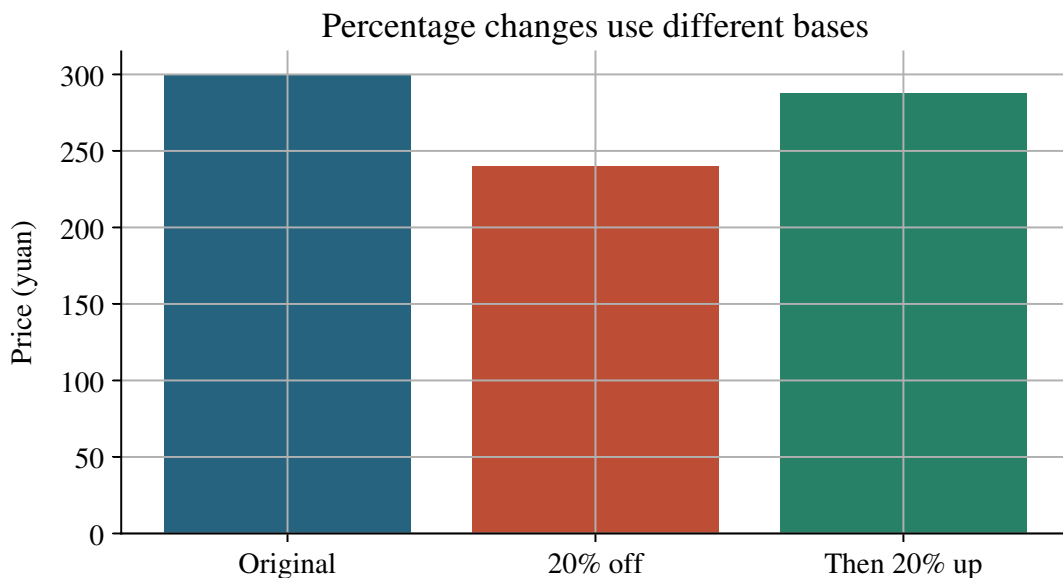
- A $p\%$ increase has multiplier $1+p/100$; a decrease has multiplier $1-p/100$.
 - Reverse a percentage by dividing by the multiplier.
 - In a ratio, first find the total number of parts.
-

■ A worked case

$$P_{\text{new}} = P_{\text{old}} \left(1 + \frac{r}{100}\right)$$

Example:

Let the original price be P . The model is sale price $= 0.8P$. Hence $P = 240/0.8 = 300$. A later 20% increase gives $240 \times 1.2 = 288$, so the two changes do not cancel.



300 → 240 → 288 yuan: 20% off then 20% up does not return to 300, because the second 20% is of 240

■ Where it goes wrong

- A percentage uses a stated base.
- Subtracting the percentages loses that base.
- For compound change, multiply the multipliers; do not add the percentages.

Not true: A 20% decrease followed by a 20% increase restores the starting price. The 20% increase is taken of a smaller amount: $100 \times 0.8 \times 1.2 = 96$, not 100.

■ Applying it

- For direct proportion use $y=kx$; for inverse proportion use $y=k/x$.
- Calculate k from a known pair before using a new value.
- State what you held constant.

■ In your course

- Analysis and Approaches HL, the course first assessed in 2021.
- Paper 1 is non-calculator: keep fractions, surds and π exact and show each step. On calculator papers, still write the expression before its value.

Key:

- **Multiplier:** A factor that performs a percentage change in one multiplication.
- A $p\%$ increase has multiplier $1+p/100$; a decrease has multiplier $1-p/100$.
- Watch for: A percentage uses a stated base.

■ Write from a final price of 240

- A price of 250 falls by 20 percent and then rises by 20 percent.
- Find the final price. The two changes do not cancel.

Complex numbers and roots

■ What if the root is not real?

- The equation $x^2+1=0$ has no real roots. Extending the number system lets us represent and calculate with both roots.
 - This lesson studies **complex conjugate** : The number obtained by reversing the sign of the imaginary part.
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■ The method

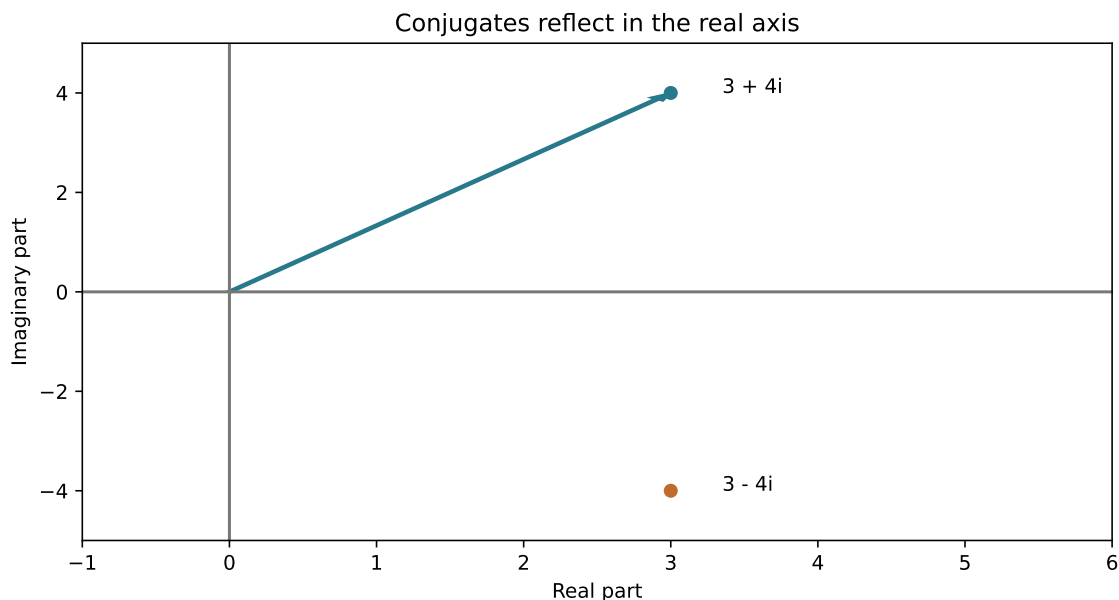
- Write $z=a+bi$ with $i^2=-1$.
 - The conjugate is $a-bi$ and z times its conjugate $=a^2+b^2$.
 - The modulus is $\sqrt{a^2+b^2}$; the argument needs the correct quadrant.
 - Conjugate roots occur for polynomials with real coefficients.
-

■ A worked case

$$z = a + bi, \quad |z| = \sqrt{a^2 + b^2}, \quad z\bar{z} = a^2 + b^2$$

Example:

For $z=3+4i$, $|z|=5$ and z conjugate $=3-4i$. The reciprocal is $(3-4i)/25$. For $x^2-6x+13=0$, the roots are $3\pm 2i$. Multiplying by the conjugate makes a complex denominator real.



The conjugate $3 - 4i$ is the reflection of $3 + 4i$ in the real axis; both have modulus 5

■ Where it goes wrong

- The square root of a sum is not the sum of the square roots.
- An inverse tangent alone may return the wrong quadrant.
- Do not confuse modulus with squared modulus.

Not true: For $a+bi$, its modulus is always $a+b$. The modulus is $\sqrt{a^2+b^2}$: for $3+4i$ it is 5, not 7.

■ Applying it

- Draw the real axis horizontally and imaginary axis vertically.
 - For products, moduli multiply and arguments add; powers extend this geometric pattern through de Moivre's theorem.
-

■ In your course

- Analysis and Approaches HL, the course first assessed in 2021.
- Paper 1 is non-calculator: keep fractions, surds and π exact and show each step. On calculator papers, still write the expression before its value.

Key:

- **Complex conjugate:** The number obtained by reversing the sign of the imaginary part.

- Write $z=a+bi$ with $i^2=-1$.
- Watch for: the square root of a sum is not the sum of the square roots.

■ Write from a modulus of 13

- Find the modulus of 5 plus 12 i.
- The modulus is not the sum of the two parts.

Binomial expansion and valid approximations

■ How does a small change affect a power?

- A small measurement change affects a power of a quantity. An expansion can show the size of first and second effects.
 - This lesson studies **binomial coefficient** : The number of ways to choose a specified number of objects from a set.
-

■ The method

- For positive integer n , $(a+b)^n$ is a finite binomial expansion.
 - For noninteger n , expand $(1+x)^n$ as $1+nx+\frac{n(n-1)}{2}x^2+\dots$ with $|x|<1$.
 - Factor out constants before using this form.
-

■ A worked case

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \dots$$

Example:

$(1+2x)^5=1+10x+40x^2+80x^3+80x^4+32x^5$. For $(1+x)^{-1}$, the first three terms are $1-x+x^2$; at $x=0.1$ this gives 0.91 versus the exact $1/1.1\approx 0.909091$.

(1 + 2x)⁵: Pascal row × powers of 2

Pascal row 5	1	5	10	10	5	1
× 2 ^k	1	2	4	8	16	32
coefficient	1	10	40	80	80	32
	1	x	x ²	x ³	x ⁴	x ⁵

Each coefficient of (1 + 2x)⁵ is a Pascal number times 2^k: 1, 10, 40, 80, 80, 32

■ Where it goes wrong

- An expansion in 2x requires $|2x| < 1$ for the infinite series, not merely $|x| < 1$.
- Positive-integer expansions are finite and do not have that convergence restriction.

Not true: Every binomial expansion is a finite polynomial, including powers that are not positive integers. For a negative or fractional power the expansion is an infinite series, valid only for $|x| < 1$.

■ Applying it

- State the range of validity alongside an approximation.
- Retain enough terms to support the requested accuracy, and distinguish a coefficient from the whole term containing x.

■ In your course

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- Paper 1 is non-calculator: keep fractions, surds and π exact and show each step. On calculator papers, still write the expression before its value.

Key:

- **Binomial coefficient:** The number of ways to choose a specified number of objects from a set.

- For positive integer n , $(a+b)^n$ is a finite binomial expansion.
- Watch for: an expansion in $2x$ requires $|2x| < 1$ for the infinite series, not merely $|x| < 1$.

■ Write from a coefficient of 54

- Find the coefficient of x squared in $(1 + 3x)$ to the power 4.
- A binomial expansion is finite only for a non-negative integer power.

Series, finite differences and Taylor expansions

■ Can a polynomial replace an exponential nearby?

- Near a known point, a complicated function can be replaced by a polynomial approximation. The approximation needs a centre and a valid scale.
 - This lesson studies **Taylor polynomial** : A polynomial formed from a function's derivatives at a chosen centre.
-

■ The method

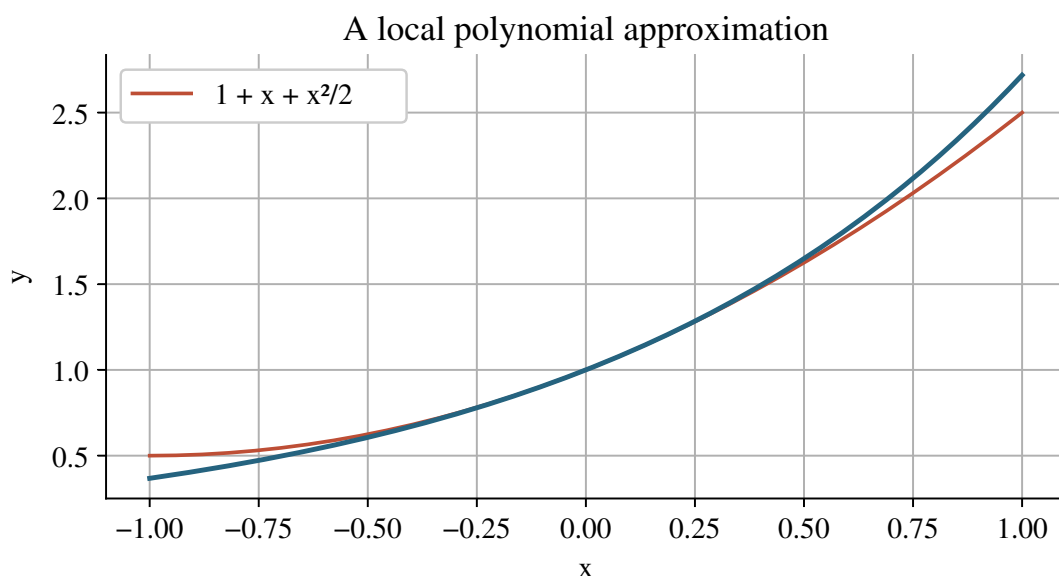
- About $x=a$, $f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\dots$ when the expansion is valid.
 - At $a=0$ it is a Maclaurin expansion.
 - Finite sums may also simplify by cancellation or standard sum formulae.
-

■ A worked case

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots$$

Example:

The first three terms of e^x are $1+x+\frac{x^2}{2}$. At $x=0.1$ this gives 1.105, close to $e^{0.1} \approx 1.105170$. For the sum of k^2 from $k=1$ to 5, $\frac{n(n+1)(2n+1)}{6}$ gives $5 \times 6 \times 11 / 6 = 55$.



$1 + x + x^2/2$ matches e^x closely near $x = 0$ and less well further away

■ Where it goes wrong

- The factorial belongs in the denominator of every Taylor coefficient.
- The expansion centre is not always zero.
- An approximation close to its centre can be poor far away.

Not true: A Taylor polynomial is always exact for the original function at every input. A Taylor polynomial is an approximation; it is usually accurate only near its centre.

■ Applying it

- For a telescoping series, write several terms and identify both surviving ends.
- For a Taylor approximation, state its order and compare with a known value or a remainder estimate where required.

■ In your course

- Analysis and Approaches HL, the course first assessed in 2021.
- Paper 1 is non-calculator: keep fractions, surds and π exact and show each step. On calculator papers, still write the expression before its value.

Key:

- **Taylor polynomial:** A polynomial formed from a function's derivatives at a chosen centre.
- About $x=a$, $f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\dots$ when the expansion is valid.
- Watch for: the factorial belongs in the denominator of every Taylor coefficient.

■ **Write from a square sum of 30**

- Find the sum of k squared from $k = 1$ to 4.
- Use $1 + x + \frac{x^2}{2}$ to estimate e to the power 0.2. A Taylor polynomial is not exact at every input.

■ **Write from a sum of 5050**

- Find the sum of k from $k = 1$ to 100.
- Gauss's pairing: $1 + 100, 2 + 99, \dots$ gives 50 pairs of 101, so 5 050.

■ **Write from a second-order estimate of 1.22**

- Use $1 + x + \frac{x^2}{2}$ at $x = 0.2$.
- The estimate is 1.22 against a true value near 1.2214 – the missing x cubed over 6 (about 0.0013) is the truncation error's size.