5.6 Implicit differentiation, related rates and limits

Name: _____ Class: _____ Date: _____ **Total: 24 marks**

KEY WORDS first principles • implicit differentiation • related rates • limit • L'Hôpital's rule • inverse trigonometric

Objective

Build the skills to answer exam questions on **further differentiation** : first principles, implicit differentiation, related rates, more derivatives and L'Hôpital's rule.

You must be able to:

- differentiate from first principles, $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$
- differentiate implicitly and find gradients of curves such as $x^2 + xy + y^2 = 7$
- solve related-rates problems with the chain rule
- differentiate $\tan x$, $\sec x$, $\arcsin x$, $\arctan x$, a^x, and use L'Hôpital's rule

1 Key knowledge and worked examples

■ Further differentiation

$f(x)$	$\tan x$	$\sec x$	$\arcsin x$	$\arctan x$	a^x	$\log_a x$
$f'(x)$	$\sec^2 x$	$\sec x \tan x$	$\dfrac{1}{\sqrt{1-x^2}}$	$\dfrac{1}{1+x^2}$	$a^x \ln a$	$\dfrac{1}{x \ln a}$

Implicit: differentiate every term with respect to x; $\dfrac{d}{dx}(y^2) = 2y\dfrac{dy}{dx}$ and $\dfrac{d}{dx}(xy) = y + x\dfrac{dy}{dx}$.

Related rates: link the rates with the chain rule, e.g. $\dfrac{dV}{dt} = \dfrac{dV}{dr} \cdot \dfrac{dr}{dt}$.

L'Hôpital's rule: if $\dfrac{f(x)}{g(x)}$ gives $\frac{0}{0}$ (or $\frac{\infty}{\infty}$) at a, then $\lim \dfrac{f(x)}{g(x)} = \lim \dfrac{f'(x)}{g'(x)}$; use it again if needed.

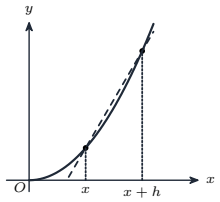
exercises.lol exercises.com.cn

Worked example. $\lim\limits_{x\to 0} \dfrac{\sin 3x}{x} = \lim\limits_{x\to 0} \dfrac{3\cos 3x}{1} = \mathbf{3}$.

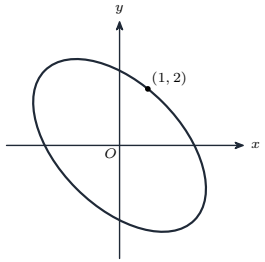
Answer shapes: in a rates question, list the rate you know, the rate you want and the formula that links them.

2 Practice

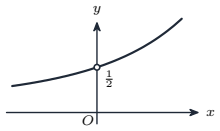
2.1 Differentiate $f(x) = x^2$ from first principles. [2]



2.2 Find the gradient of the curve $x^2 + xy + y^2 = 7$ at the point $(1, 2)$. [3]



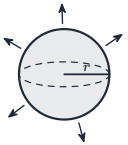
2.3 Find $\lim\limits_{x\to 0} \dfrac{e^x - 1 - x}{x^2}$. [3]



3 Exam-style questions

3.1 A spherical balloon is blown up so that its volume increases at 50 $\text{cm}^3\ \text{s}^{-1}$. ($V = \frac{4}{3}\pi r^3$, $S = 4\pi r^2$.)

(a) Find the rate at which the radius increases when $r = 10$ cm. [4]



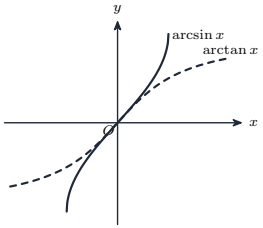
(b) Find the rate at which the surface area increases at that moment. [4]

3.2 More derivatives.

(a) Differentiate $\arctan(2x)$. [2]

(b) Differentiate 3^x. [1]

(c) $y = \arcsin x$ for $-1 < x < 1$. By writing $\sin y = x$, show that $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$. [3]



(d) Differentiate $\sec x$, using $\sec x = (\cos x)^{-1}$. [2]

exercises.lol exercises.com.cn