# 1.8 Proof by induction, contradiction and counterexample

Name: _____ Class: _____ Date: _____ **Total: 16 marks**

**KEY WORDS** mathematical induction • base case • inductive step • contradiction • counterexample • irrational

## Objective

Build the skills to answer exam questions on **proof** : mathematical induction, proof by contradiction, and disproof by counterexample.

**You must be able to:**

- write a complete proof by mathematical induction for sums, divisibility and similar results
- prove a statement by contradiction, such as the irrationality of $\sqrt{2}$
- disprove a statement with a single counterexample

## 1 Key knowledge and worked examples

### ■ Proof

**Mathematical induction** (four parts, all needed for full marks):

1. **Base case:** show the statement is true for $n = 1$.
2. **Assumption:** assume it is true for $n = k$.
3. **Inductive step:** use the assumption to show it is true for $n = k + 1$.
4. **Conclusion:** "Since it is true for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, it is true for all $n \in \mathbb{Z}^+$."

**Contradiction:** assume the statement is false and reach something impossible. Example: if $\sqrt{2} = \frac{p}{q}$ in lowest terms, then $p^2 = 2q^2$, so $p$ is even, $p = 2m$, so $q^2 = 2m^2$ and $q$ is even, contradicting "lowest terms".

**Counterexample:** one example where the statement fails is enough to disprove it.

**Worked example.** Prove $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$. Base: $1 = \frac{1\times 2}{2}$. Assume $\sum_{r=1}^{k} r = \frac{k(k+1)}{2}$. Then



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$$\sum_{r=1}^{k+1} r = \frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+2)}{2},$$

which is the formula with $n = k+1$. Conclusion as above.

**Answer shapes:** show where you use the assumption; write the target expression before you start the step.

## 2 Practice

**2.1** Find a counterexample to the statement "$n^2 + n + 41$ is prime for every positive integer $n$". [2]

| $n$ | 1 | 2 | 3 |
|---|---|---|---|
| $n^2 + n + 41$ | 43 | 47 | 53 |

**2.2** Prove by contradiction that if $n^2$ is even, then $n$ is even. [3]

$n^2$ even $\Rightarrow$ $n$ even

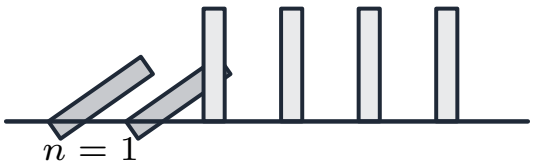
assume $n$ odd: $n = 2k + 1$



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## 3 Exam-style questions

**3.1** Prove by mathematical induction that $\sum_{r=1}^{n} r(r+1) = \frac{n(n+1)(n+2)}{3}$ for all $n \in \mathbb{Z}^+$. [6]




**3.2** Prove by mathematical induction that $7^n - 1$ is divisible by 6 for all $n \in \mathbb{Z}^+$. [5]

| $n = 1$ | $n = 2$ | $n = 3$ |
|---|---|---|
| 6 | 48 | 342 |