5.7 Maclaurin series

Name: _____ Class: _____ Date: _____ **Total: 24 marks**

KEY WORDS Maclaurin series • approximation • term • percentage error • converge • factorial

Objective

Build the skills to answer exam questions on **Maclaurin series** : building series from derivatives, combining known series, and using them for limits and approximations.

You must be able to:

- find a Maclaurin series from $f(0)$, $f'(0)$, $f''(0)$, …
- use the standard series for e^x, $\sin x$, $\cos x$, $\ln(1+x)$ and $(1+x)^p$
- multiply, substitute into and differentiate known series
- use series to estimate values, integrals and limits

1 Key knowledge and worked examples

Maclaurin series

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots$$

function	series	valid for
e^x	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$	all x
$\sin x$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$	all x
$\cos x$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$	all x
$\ln(1+x)$	$x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$	$-1 < x \leq 1$
$(1+x)^p$	$1 + px + \frac{p(p-1)}{2!}x^2 + \cdots$	$\lvert x\rvert < 1$

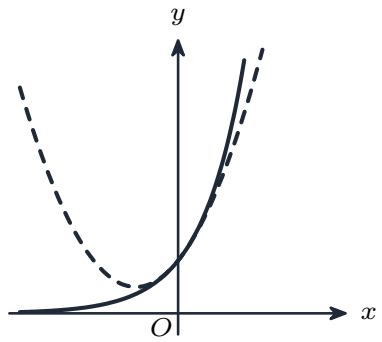
Worked example. e^{x^2}: put x^2 in place of x: $\mathbf{1 + x^2 + \frac{x^4}{2} + \cdots}$.

Answer shapes: when you multiply two series, keep only the powers you were asked for.

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2 Practice

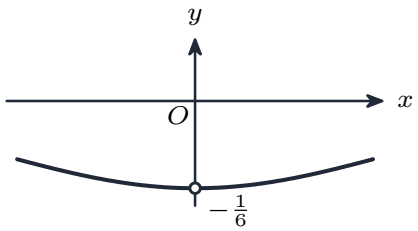
2.1 Write down the first three terms of the Maclaurin series for e^{2x}. [2]



2.2 Use derivatives to find the Maclaurin series for $\ln(1+x)$ up to the term in x^3. [3]

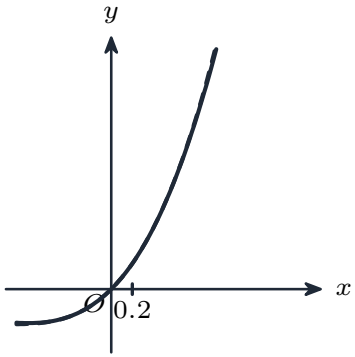
f	f'	f''	$f^{(3)}$
0	?	?	?

2.3 Use series to find $\lim_{x \to 0} \dfrac{\sin x - x}{x^3}$. [3]



3 Exam-style questions

3.1 $f(x) = e^x \sin x$.

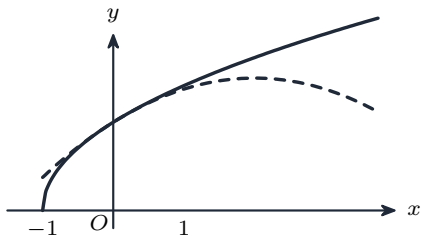


(a) Show that the Maclaurin series for $f(x)$ up to the term in x^3 is $x + x^2 + \frac{1}{3}x^3$. [4]

(b) Use your series to estimate $\displaystyle\int_0^{0.2} e^x \sin x \, dx$. [2]

(c) Explain why the estimate in (b) is good. [2]

3.2 $f(x) = (1 + x)^{1/2}$.



(a) Find the Maclaurin series for $f(x)$ up to the term in x^2. [2]

(b) Hence estimate $\sqrt{1.1}$. [2]

(c) Find the percentage error of your estimate. [2]

(d) Explain why the series cannot be used to estimate $\sqrt{3}$. [2]

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Solutions

Marking: **M1** a correct method, **A1** the correct answer, **B1** an independent mark.

Each answer shows the steps, then the **result**.

2.1

replace x by $2x$; $\mathbf{1 + 2x + 2x^2}$

2.2

$f(0) = 0$, $f'(0) = 1$, $f''(0) = -1$, $f^{(3)}(0) = 2$; $\mathbf{x - \frac{x^2}{2} + \frac{x^3}{3}}$

2.3

$\sin x - x = -\frac{x^3}{6} + \frac{x^5}{120} - \cdots$; $\dfrac{\sin x - x}{x^3} = -\frac{1}{6} + \frac{x^2}{120} - \cdots$; limit $\mathbf{-\frac{1}{6}}$

3.1

(a) $\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6}\right)\left(x - \frac{x^3}{6}\right)$; $x + x^2 + \frac{x^3}{2} - \frac{x^3}{6} = x + x^2 + \frac{1}{3}x^3$

(b) $\left[\frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{12}\right]_0^{0.2} = \mathbf{0.0228}$

(c) x is small (at most 0.2) (R1), so the missing terms in x^4 and above are very small (R1)

3.2

(a) $1 + \frac{1}{2}x + \dfrac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2}x^2 = \mathbf{1 + \frac{x}{2} - \frac{x^2}{8}}$

(b) $x = 0.1$; $1 + 0.05 - 0.00125 = \mathbf{1.04875}$

(c) $\dfrac{|1.04875 - 1.048809|}{1.048809} \times 100 = \mathbf{0.0056}\%$

(d) $\sqrt{3}$ needs $x = 2$; the series is valid only for $|x| < 1$ (R1)

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