5.6 Implicit differentiation, related rates and limits

Name: _____ Class: _____ Date: _____ **Total: 24 marks**

KEY WORDS first principles • implicit differentiation • related rates • limit • L'Hôpital's rule • inverse trigonometric

Objective

Build the skills to answer exam questions on **further differentiation** : first principles, implicit differentiation, related rates, more derivatives and L'Hôpital's rule.

You must be able to:

- differentiate from first principles, $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$
- differentiate implicitly and find gradients of curves such as $x^2 + xy + y^2 = 7$
- solve related-rates problems with the chain rule
- differentiate $\tan x$, $\sec x$, $\arcsin x$, $\arctan x$, a^x, and use L'Hôpital's rule

1 Key knowledge and worked examples

■ Further differentiation

$f(x)$	$\tan x$	$\sec x$	$\arcsin x$	$\arctan x$	a^x	$\log_a x$
$f'(x)$	$\sec^2 x$	$\sec x \tan x$	$\dfrac{1}{\sqrt{1-x^2}}$	$\dfrac{1}{1+x^2}$	$a^x \ln a$	$\dfrac{1}{x \ln a}$

Implicit: differentiate every term with respect to x; $\dfrac{d}{dx}(y^2) = 2y\dfrac{dy}{dx}$ and $\dfrac{d}{dx}(xy) = y + x\dfrac{dy}{dx}$.

Related rates: link the rates with the chain rule, e.g. $\dfrac{dV}{dt} = \dfrac{dV}{dr} \cdot \dfrac{dr}{dt}$.

L'Hôpital's rule: if $\dfrac{f(x)}{g(x)}$ gives $\frac{0}{0}$ (or $\frac{\infty}{\infty}$) at a, then $\lim \dfrac{f(x)}{g(x)} = \lim \dfrac{f'(x)}{g'(x)}$; use it again if needed.

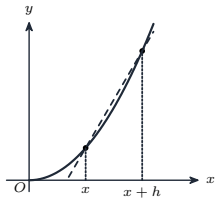
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Worked example. $\lim\limits_{x\to 0} \dfrac{\sin 3x}{x} = \lim\limits_{x\to 0} \dfrac{3\cos 3x}{1} = \mathbf{3}$.

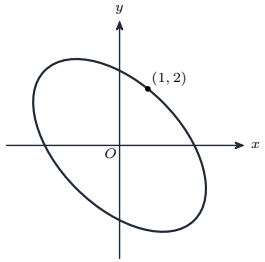
Answer shapes: in a rates question, list the rate you know, the rate you want and the formula that links them.

2 Practice

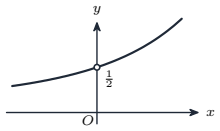
2.1 Differentiate $f(x) = x^2$ from first principles. [2]



2.2 Find the gradient of the curve $x^2 + xy + y^2 = 7$ at the point $(1, 2)$. [3]



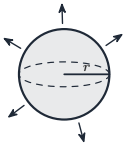
2.3 Find $\lim\limits_{x\to 0} \dfrac{e^x - 1 - x}{x^2}$. [3]



3 Exam-style questions

3.1 A spherical balloon is blown up so that its volume increases at 50 $\text{cm}^3\ \text{s}^{-1}$. ($V = \frac{4}{3}\pi r^3$, $S = 4\pi r^2$.)

(a) Find the rate at which the radius increases when $r = 10$ cm. [4]



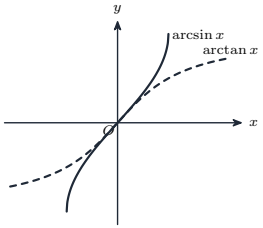
(b) Find the rate at which the surface area increases at that moment. [4]

3.2 More derivatives.

(a) Differentiate $\arctan(2x)$. [2]

(b) Differentiate 3^x. [1]

(c) $y = \arcsin x$ for $-1 < x < 1$. By writing $\sin y = x$, show that $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$. [3]



(d) Differentiate $\sec x$, using $\sec x = (\cos x)^{-1}$. [2]

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Solutions

Marking: **M1** a correct method, **A1** the correct answer, **B1** an independent mark.

Each answer shows the steps, then the **result**.

2.1

$\dfrac{(x+h)^2 - x^2}{h} = \dfrac{2xh + h^2}{h} = 2x + h$; as $h \to 0$, $f'(x) = \mathbf{2x}$

2.2

$2x + y + x\dfrac{dy}{dx} + 2y\dfrac{dy}{dx} = 0$; $4 + 5\dfrac{dy}{dx} = 0$, $\dfrac{dy}{dx} = -\frac{4}{5}$

2.3

$\frac{0}{0}$, so $\lim \dfrac{e^x - 1}{2x}$; still $\frac{0}{0}$: $\lim \dfrac{e^x}{2} = \frac{1}{2}$

3.1

(a) $\dfrac{dV}{dr} = 4\pi r^2$; $50 = 4\pi(10)^2\dfrac{dr}{dt}$; $\dfrac{dr}{dt} = \frac{1}{8\pi} = \mathbf{0.0398}$ cm s^{-1}

(b) $\dfrac{dS}{dr} = 8\pi r$; $\dfrac{dS}{dt} = 8\pi(10) \times \frac{1}{8\pi} = \mathbf{10}$ cm^2 s^{-1}

3.2

(a) $\dfrac{1}{1 + (2x)^2} \times 2 = \dfrac{\mathbf{2}}{\mathbf{1 + 4x^2}}$

(b) $\mathbf{3^x \ln 3}$

(c) $\cos y\dfrac{dy}{dx} = 1$; $\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$, positive because $-\frac{\pi}{2} < y < \frac{\pi}{2}$ (R1)

(d) $-(\cos x)^{-2}(-\sin x) = \sec \mathbf{x} \tan \mathbf{x}$

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