5.4 Substitution, parts and partial fractions

Name: _____ Class: _____ Date: _____ **Total: 13 marks**

KEY WORDS integration by parts

Objective

Substitution, parts and partial fractions. Original classroom practice for IB HL; not an official IB paper.

1 Key knowledge and worked examples

Can we integrate the two factors separately?

- A simple-looking product such as xe^x cannot be integrated by separately integrating each factor.
- This lesson studies **integration by parts** : An integration method based on the derivative of a product.

The method

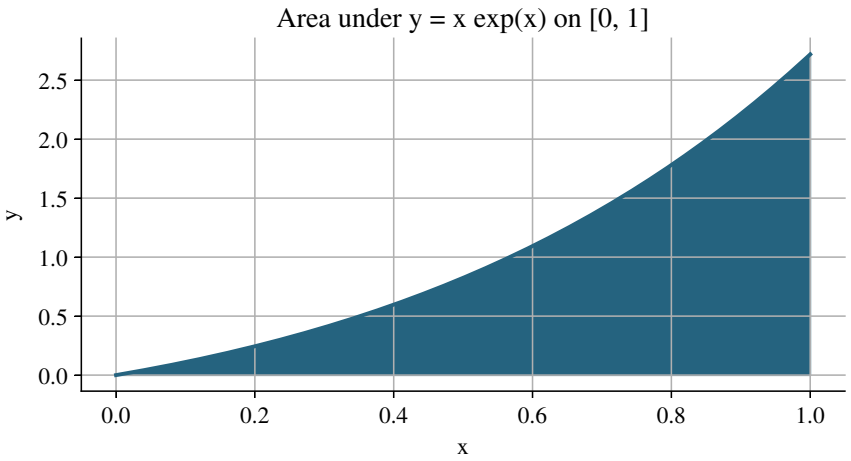
- Use substitution when an inner derivative appears as a factor.
- By parts, integral u v prime =uv-integral u prime v.
- For a rational function, divide first if needed, then decompose into partial fractions.

A worked case

$$\int uv'\,dx = uv - \int u'v\,dx$$

Example:

For integral xe^x dx, take u=x and v prime=e^x. Then the integral is xe^x-e^x+C. For integral $2x/(x^2+1)$ dx, substitute $w=x^2+1$, dw=2x dx; the answer is $\ln(x^2+1)+C$.



Where it goes wrong

- Choosing u and v prime well matters: the remaining integral should become simpler.
- In a definite substitution, either change the limits or return to x before applying the original limits.

Not true: The integral of a product equals the product of its separate integrals. A product needs a method such as integration by parts or substitution: $\int x \cdot x\,dx = x^3/3$, not $(x^2/2)(x^2/2)$.

Applying it

- Check by differentiating.
- For volumes of revolution around the x-axis use $V=\pi$ integral y^2 dx; do not confuse the square of a function with the integral of the function.

In your course

- Analysis and Approaches HL, the course first assessed in 2021.
- Paper 1 is non-calculator: keep fractions, surds and π exact and show each step. On calculator papers, still write the expression before its value.

Key:

- **Integration by parts**: An integration method based on the derivative of a product.
- Use substitution when an inner derivative appears as a factor.
- Watch for: choosing u and v prime well matters: the remaining integral should become simpler.

■ **Write from a volume factor of 18**

- The line y = 3 is revolved about the x-axis from 0 to 2. Find the volume divided by pi.
- The integral of a product is not the product of the integrals.

2 Practice

2.1 Evaluate the integral of xe^x from 0 to 1. Show the relation, substitution and units. [2]

2.2 Find the coefficient of $\ln(x^2+1)$ in the integral of $2x/(x^2+1)$. Show the relation, substitution and units. [2]

2.3 Find the volume divided by π when y=x is revolved about the x-axis from 0 to 3. Show the relation, substitution and units. [2]

2.4 Decide whether this claim is true and justify: The integral of a product equals the product of its separate integrals. [2]

2.5 Why does the substitution $w=x^2+1$ fit the integral of $2x/(x^2+1)$? Choose one answer.

- **A** Because x^2+1 has no antiderivative, a substitution is the only option
- **B** Substitution works for every quotient, whatever the numerator is
- **C** The numerator 2x is exactly the derivative of the inner expression, so it cancels [1]

3 Exam-style questions

3.1 Evaluate integral of x ln x from 1 to e by parts and differentiate the antiderivative as a check. [4]

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Solutions

2.1 An antiderivative is (x-1)e^x. At 1 and 0 it gives 0 and -1, so the integral is 1.

2.2 Substitution $w=x^2+1$ gives $dw=2x\,dx$, so the coefficient of $\ln(x^2+1)$ is 1.

2.3 $V/\pi=\int_0^3 x^2\,dx=[x^3/3]=9$.

2.4 False. A product needs a method such as integration by parts or substitution: $\int x \cdot x\,dx=x^3/3$, not $(x^2/2)(x^2/2)$.

2.5 C With $w=x^2+1$, $dw=2x\,dx$, and the integral becomes $\int dw/w=\ln|w|+C$. Substitution is powerful precisely when the inner derivative appears as a factor.

3.1 Take $u=\ln x$, $dv=x\,dx$:antiderivative$(x^2/2)\ln x-x^2/4$. Bounds give$(e^2+1)/4$. Differentiation returns $x\ln x$; domain $x>0$ matters.

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