5.3 Chain, product, quotient and implicit differentiation

Name: _____ Class: _____ Date: _____ **Total: 13 marks**

KEY WORDS chain rule

Objective

Chain, product, quotient and implicit differentiation. Original classroom practice for IB HL; not an official IB paper.

1 Key knowledge and worked examples

Does the inside expression change too?

- A cost curve is a power of a changing expression. Differentiating the outer power alone misses the rate of its input.
- This lesson studies **chain rule** : The rule that multiplies the outer derivative by the inner derivative for a composite function.

The method

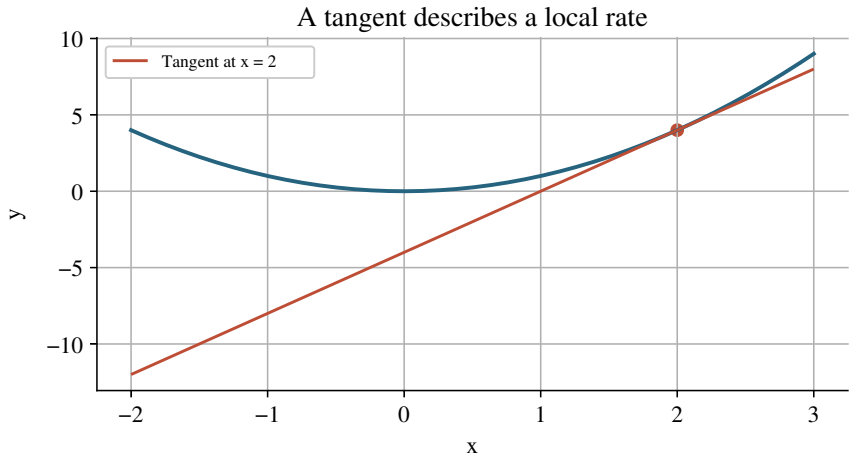
- For y=f(g(x)), y prime=f prime(g(x))g prime(x).
- For uv, differentiate to u prime v+uv prime.
- For u/v, use (u prime v-uv prime)/v^2, where v≠0.

A worked case

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$$

Example:

For y=(3x+1)^4, y prime=4(3x+1)^3×3=12(3x+1)^3. At x=0, the gradient is 12. For $x^2+y^2=25$, differentiate implicitly: 2x+2y y prime=0, so y prime=-x/y when y≠0.



Where it goes wrong

- A derivative of a product is not the product of the derivatives.
- In implicit differentiation, every differentiated function of y brings a dy/dx factor.
- A quotient's denominator is squared.

Not true: The derivative of u(x)v(x) is always u prime times v prime. The product rule is (uv)′=u′v+uv′: for x · x the derivative is 2x, not 1×1=1.

Applying it

- Choose a useful form before differentiating: expanding a short polynomial may be simpler.
- For related rates, write the relation in symbols, differentiate with respect to time, then substitute measured values.

In your course

- Analysis and Approaches HL, the course first assessed in 2021.
- Paper 1 is non-calculator: keep fractions, surds and π exact and show each step. On calculator papers, still write the expression before its value.

Key:

- **Chain rule**: The rule that multiplies the outer derivative by the inner derivative for a composite function.
- For y=f(g(x)), y prime=f prime(g(x))g prime(x).
- Watch for: A derivative of a product is not the product of the derivatives.

■ **Write from a product derivative of 11**

- For y = (2x plus 1)(x plus 3), find the derivative at x = 1.
- The derivative of a product is not the product of the derivatives.

2 Practice

2.1 Find the derivative of (3x+1)^4 at x=0. Show the relation, substitution and units. [2]

2.2 For $x^2+y^2=25$, find dy/dx at (3,4). Show the relation, substitution and units. [2]

2.3 For y=xe^x, find dy/dx at x=0. Show the relation, substitution and units. [2]

2.4 Decide whether this claim is true and justify: The derivative of u(x)v(x) is always u prime times v prime. [2]

2.5 Why is the derivative of $x \cdot e^x$ not simply $1 \cdot e^x$? Choose one answer.

- **A** The product rule adds a second term: $u'v+uv'$ gives e^x+xe^x
- **B** Because e^x cannot be differentiated
- **C** The derivative of a product is the product of the derivatives [1]

3 Exam-style questions

3.1 For $x^2+y^2=25$, find dy/dx and discuss its values at(3, 4) and(5, 0), using geometry to check. [4]

Solutions

2.1 Chain rule gives $4(3x+1)^3\times 3$; at zero this is 12.

2.2 Implicit differentiation gives 2x+2y y prime=0, so y prime=-3/4.

2.3 Product rule gives e^x+xe^x; at zero this is 1.

2.4 False. The product rule is $(uv)'=u'v+uv'$: for $x \cdot x$ the derivative is 2x, not $1\times 1=1$.

2.5 A Each factor varies, so both contribute: with u=x and $v=e^x$, the product rule gives $u'v+uv'=e^x+xe^x$. Multiplying the separate derivatives drops the uv' term.

3.1 $y'=-x/y$ where $y\neq 0$. At(3, 4), $-3/4$. At(5, 0) the tangent is vertical, so finite dy/dx is undefined; implicit differentiation does not license division by zero.