4.5 Bayes' theorem and continuous random variables

Name: _____ Class: _____ Date: _____ **Total: 22 marks**

KEY WORDS Bayes' theorem • probability density function • continuous • median • mode • false positive

Objective

Build the skills to answer exam questions on **Bayes' theorem** and **continuous random variables** .

You must be able to:

- use Bayes' theorem for two or three events
- use a probability density function $f(x)$, with $\int f(x)\,dx = 1$
- find the mode, median, mean and variance of a continuous random variable
- use $E(aX+b) = aE(X)+b$ and $\mathrm{Var}(aX+b) = a^2\,\mathrm{Var}(X)$

1 Key knowledge and worked examples

■ Bayes' theorem and continuous random variables

Bayes' theorem: $P(B \mid A) = \dfrac{P(B)\,P(A \mid B)}{P(B)\,P(A \mid B) + P(B')\,P(A \mid B')}$. On a tree diagram: the branch you want, divided by all branches that give A.

result	**continuous (f on $a \le x \le b$)**
total probability	$\int_a^b f(x)\,dx = 1$
$P(c < X < d)$	$\int_c^d f(x)\,dx$
mean	$E(X) = \int_a^b x\,f(x)\,dx$
variance	$\mathrm{Var}(X) = E(X^2) - \big(E(X)\big)^2$
median m	$\int_a^m f(x)\,dx = \frac{1}{2}$
mode	where $f(x)$ is greatest

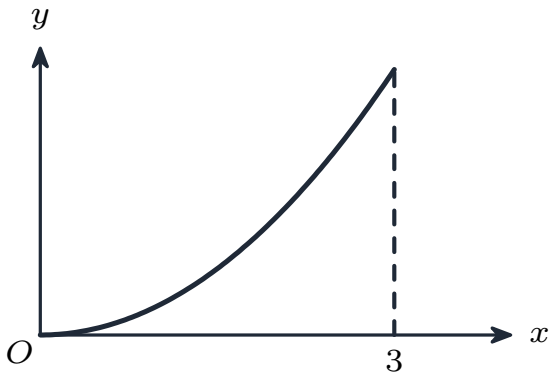
Worked example. $f(x) = \frac{1}{2}x$ for $0 \le x \le 2$: $E(X) = \int_0^2 \frac{1}{2}x^2\,dx = \frac{4}{3}$.

Answer shapes: write each integral with its limits before you use the GDC.

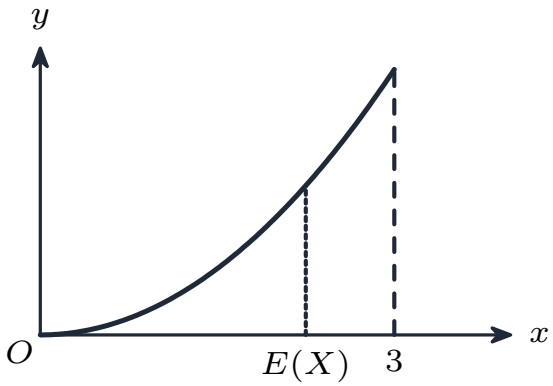
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2 Practice

2.1 $f(x) = kx^2$ for $0 \leq x \leq 3$ is a probability density function. Find k. [2]

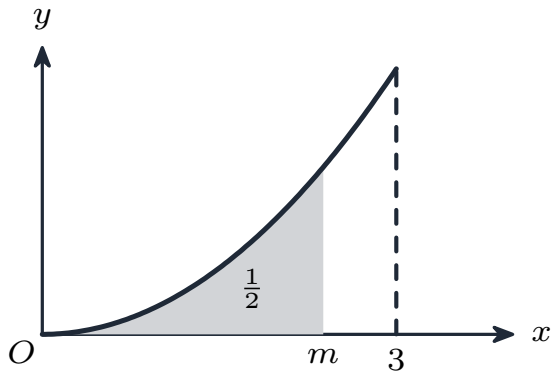


2.2 For $f(x) = \frac{1}{9}x^2$, $0 \leq x \leq 3$, find $E(X)$ and $\text{Var}(X)$. [3]



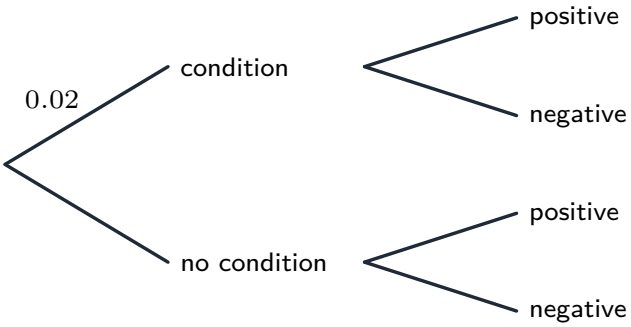
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2.3 For $f(x) = \frac{1}{9}x^2$, $0 \leq x \leq 3$, find the median of X. [3]



3 Exam-style questions

3.1 2% of people have a condition. A test is positive for 95% of people with the condition and for 8% of people without it.



(a) Draw a tree diagram. [2]

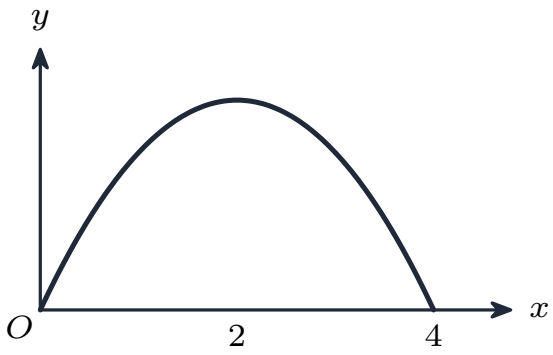
(b) Find the probability that a person tests positive. [2]

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(c) Given that a person tests positive, find the probability that they have the condition. [2]

(d) Comment on your answer to (c). [1]

3.2 The continuous random variable X has density $f(x) = k(4x - x^2)$ for $0 \leq x \leq 4$, and $f(x) = 0$ otherwise.



(a) Show that $k = \frac{3}{32}$. [2]

(b) Write down the mode and the mean of X. [1]

(c) Find $\text{Var}(X)$. [3]

(d) Find $\text{Var}(3X + 1)$. [1]

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Solutions

Marking: **M1** a correct method, **A1** the correct answer, **B1** an independent mark.

Each answer shows the steps, then the **result**.

2.1

$\int_0^3 kx^2\,dx = 9k = 1;\ k = \mathbf{\frac{1}{9}}$

2.2

$E(X) = \int_0^3 \frac{1}{9}x^3\,dx = \mathbf{\frac{9}{4}};\ E(X^2) = \frac{27}{5};\ \mathrm{Var}(X) = \frac{27}{5} - \frac{81}{16} = \mathbf{0.3375}$

2.3

$\int_0^m \frac{1}{9}x^2\,dx = \frac{1}{2};\ \frac{m^3}{27} = \frac{1}{2};\ m = \mathbf{2.38}$

3.1

(a) 0.02, 0.98; 0.95, 0.05 and 0.08, 0.92

(b) $0.02 \times 0.95 + 0.98 \times 0.08 = \mathbf{0.0974}$

(c) $\dfrac{0.019}{0.0974} = \mathbf{0.195}$

(d) most positive results are false positives, so a positive result needs a second test (R1)

3.2

(a) $\int_0^4 (4x - x^2)\,dx = 32 - \frac{64}{3} = \frac{32}{3};\ k \times \frac{32}{3} = 1,\ k = \frac{3}{32}$

(b) the graph is symmetric about $x = 2$: mode = mean = **2**

(c) $E(X^2) = \frac{3}{32}\int_0^4 (4x^3 - x^4)\,dx = \frac{24}{5};\ \mathrm{Var}(X) = \frac{24}{5} - 4 = \mathbf{0.8}$

(d) $9 \times 0.8 = \mathbf{7.2}$