3.6 Vectors: the vector product and planes

Name: _____ Class: _____ Date: _____ **Total: 28 marks**

KEY WORDS vector product • normal vector • plane • Cartesian equation • line of intersection • parallelogram

Objective

Build the skills to answer exam questions on **the vector product and planes** : areas, equations of planes, and intersections of lines and planes.

You must be able to:

- find the vector product $\mathbf{a} \times \mathbf{b}$ and use $|\mathbf{a} \times \mathbf{b}|$ for areas
- write the equation of a plane in vector and Cartesian form
- find the angle between a line and a plane, and between two planes
- find the intersection of a line and a plane, and of two or three planes

1 Key knowledge and worked examples

■ Vector product and planes

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}, \qquad |\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$$

$\mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. Area of a parallelogram $= |\mathbf{a} \times \mathbf{b}|$; of a triangle $= \frac{1}{2}|\mathbf{a} \times \mathbf{b}|$.

object	equation
plane with normal $\mathbf{n}$ through A	$\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$, or $ax + by + cz = d$
plane through A, B, C	$\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC}$
angle between line (direction $\mathbf{b}$) and plane	$\sin\theta = \dfrac{\lvert\mathbf{b} \cdot \mathbf{n}\rvert}{\lvert\mathbf{b}\rvert\lvert\mathbf{n}\rvert}$
line of two planes	direction $\mathbf{n}_1 \times \mathbf{n}_2$; find one point by setting $z = 0$

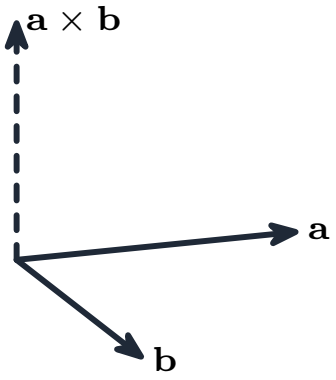
Line meets plane: substitute the line's x, y, z (in terms of λ) into the plane's equation.

Worked example. $\begin{pmatrix}1\\2\\3\end{pmatrix} \times \begin{pmatrix}0\\1\\4\end{pmatrix} = \begin{pmatrix}8-3\\0-4\\1-0\end{pmatrix} = \begin{pmatrix}5\\-4\\1\end{pmatrix}$.

Answer shapes: check a normal by dotting it with both direction vectors: each gives 0.

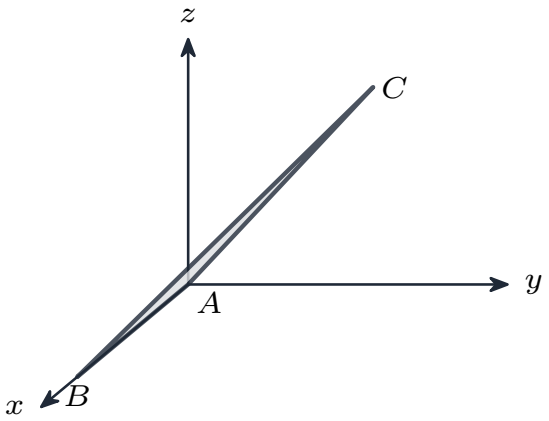
2 Practice

2.1 Find $\begin{pmatrix}2\\0\\1\end{pmatrix} \times \begin{pmatrix}1\\3\\-1\end{pmatrix}$. [2]

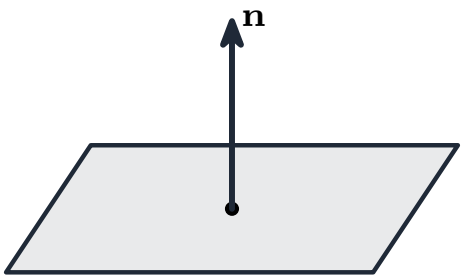


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2.2 Find the area of the triangle with vertices $A(0,0,0)$, $B(2,0,0)$ and $C(0,3,4)$. [3]



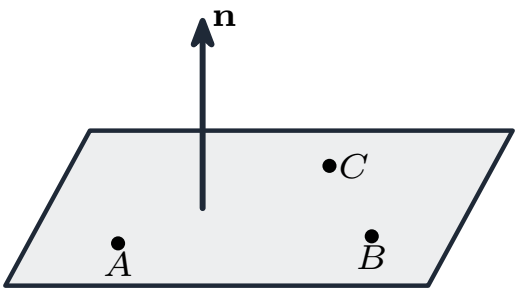
2.3 Find the Cartesian equation of the plane through $(1,2,-1)$ with normal $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$. [3]



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3 Exam-style questions

3.1 The points $A(1, 0, 2)$, $B(3, 1, 0)$ and $C(2, -1, 1)$ lie in a plane Π.



(a) Find $\overrightarrow{AB}$ and $\overrightarrow{AC}$. [2]

(b) Find $\overrightarrow{AB} \times \overrightarrow{AC}$. [2]

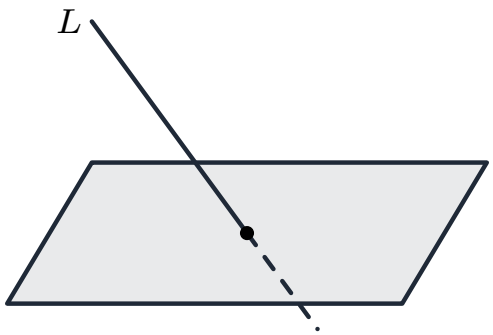
(c) Find the Cartesian equation of Π. [2]

(d) Find the area of triangle ABC. [2]

(e) Find the point where the line $\mathbf{r} = \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ meets Π. [2]

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3.2 The line L is $\mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ and the plane Π_1 is $x + 2y - z = 7$.



(a) Find the acute angle between L and Π_1. [4]

(b) Find the point where L meets Π_1. [3]

(c) The plane Π_2 is $x - y + z = 2$. Find a vector equation of the line of intersection of Π_1 and Π_2. [3]

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Solutions

Marking: **M1** a correct method, **A1** the correct answer, **B1** an independent mark.

Each answer shows the steps, then the **result**.

2.1

$\begin{pmatrix} 0-3 \\ 1+2 \\ 6-0 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \\ 6 \end{pmatrix}$

2.2

$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 0 \\ -8 \\ 6 \end{pmatrix}$; area $= \frac{1}{2}(10) = \mathbf{5}$

2.3

$2x - y + 3z = d$; $d = 2 - 2 - 3$; $\mathbf{2x - y + 3z = -3}$

3.1

(a) $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$; $\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$

(b) $\begin{pmatrix} -1-2 \\ -2+2 \\ -2-1 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \\ -3 \end{pmatrix}$

(c) normal $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $x + z = 1 + 2$; $\mathbf{x + z = 3}$

(d) $\frac{1}{2}\sqrt{9+0+9} = \mathbf{2.12}$

(e) $\lambda + \lambda = 3$, $\lambda = 1.5$; $(\mathbf{1.5}, \mathbf{1.5}, \mathbf{1.5})$

3.2

(a) $\mathbf{b} \cdot \mathbf{n} = 2 + 2 - 1 = 3$; $\sin\theta = \dfrac{3}{\sqrt{6}\sqrt{6}}$; $\theta = \mathbf{30}^\circ$

(b) $(1 + 2t) + 2(-1 + t) - (2 + t) = 7$; $t = \frac{10}{3}$; $\left(\frac{\mathbf{23}}{\mathbf{3}}, \frac{\mathbf{7}}{\mathbf{3}}, \frac{\mathbf{16}}{\mathbf{3}}\right)$

(c) direction $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}$; with $z = 0$: $x + 2y = 7$, $x - y = 2$, point $\left(\frac{11}{3}, \frac{5}{3}, 0\right)$; $\mathbf{r} = \begin{pmatrix} 11/3 \\ 5/3 \\ 0 \end{pmatrix} + s\begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}$