3.5 Vectors: lines and the scalar product

Name: _____ Class: _____ Date: _____ **Total: 28 marks**

KEY WORDS vector • magnitude • scalar product • position vector • skew lines • velocity

Objective

Build the skills to answer exam questions on **vectors** : magnitude, the scalar product, angles, vector equations of lines, intersecting and skew lines, and motion.

You must be able to:

- find magnitudes, unit vectors and position vectors
- use the scalar product to find angles and test for perpendicular vectors
- write vector equations of lines and decide whether two lines intersect, are parallel or are skew
- model motion with constant velocity, $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$

1 Key knowledge and worked examples

■ Vectors and lines

$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$; $|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$; a unit vector is $\dfrac{\mathbf{v}}{|\mathbf{v}|}$.

Scalar product: $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3 = |\mathbf{a}||\mathbf{b}| \cos\theta$. Perpendicular $\Leftrightarrow \mathbf{a} \cdot \mathbf{b} = 0$.

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Line: $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ (a point plus multiples of a direction). For two lines:

test	result
directions are multiples	parallel (or the same line)
equations for x, y, z all agree for one λ, μ	intersect
not parallel and the equations do not all agree	**skew**

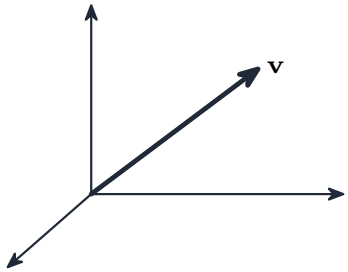
Motion: position $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$; speed $= |\mathbf{v}|$.

Worked example. $\mathbf{a} = \begin{pmatrix}1\\2\\2\end{pmatrix}$, $\mathbf{b} = \begin{pmatrix}3\\0\\4\end{pmatrix}$: $\mathbf{a} \cdot \mathbf{b} = 11$, $|\mathbf{a}| = 3$, $|\mathbf{b}| = 5$, $\cos\theta = \frac{11}{15}$, $\theta = \mathbf{42.8}°$.

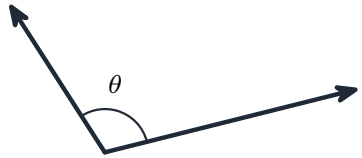
Answer shapes: for intersection, solve two components and **check the third**.

2 Practice

2.1 Find the magnitude of $\mathbf{v} = \begin{pmatrix}2\\-1\\2\end{pmatrix}$ and a unit vector in its direction. [2]



2.2 Find the angle between $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 4 \\ 8 \end{pmatrix}$. [3]



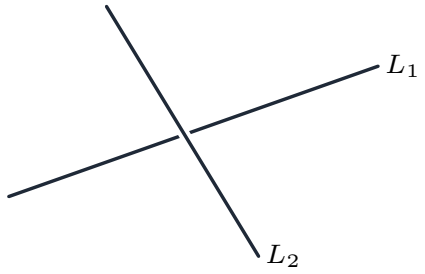
2.3 Find p such that $\begin{pmatrix} p \\ 2 \\ -1 \end{pmatrix}$ is perpendicular to $\begin{pmatrix} 3 \\ p \\ 4 \end{pmatrix}$. [3]



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3 Exam-style questions

3.1 $L_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ and $L_2 : \mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$.



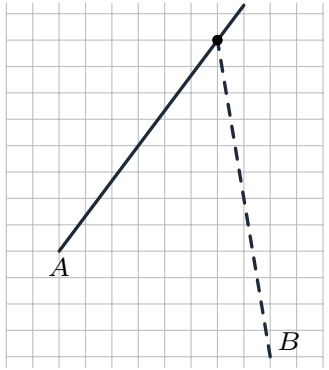
(a) Show that L_1 and L_2 are not parallel. [1]

(b) Show that L_1 and L_2 are skew. [4]

(c) Find the acute angle between the directions of L_1 and L_2. [3]

(d) Find the point on L_1 where $x = 7$. [2]

3.2 Boat A has position $\begin{pmatrix}2\\1\end{pmatrix}+t\begin{pmatrix}3\\4\end{pmatrix}$ and boat B has position $\begin{pmatrix}10\\-3\end{pmatrix}+t\begin{pmatrix}-1\\6\end{pmatrix}$, in km, t hours after noon.



(a) Find the speed of boat A. [2]

(b) Show that the boats collide, and find where. [4]

(c) Find the angle between their paths. [4]

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Solutions

Marking: **M1** a correct method, **A1** the correct answer, **B1** an independent mark.

Each answer shows the steps, then the **result**.

2.1

$|\mathbf{v}| = 3$; $\frac{1}{3}\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$

2.2

scalar product $2 + 4 - 16 = -10$; $\cos\theta = \dfrac{-10}{3 \times 9}$; $\theta = \mathbf{111.7}°$

2.3

$3p + 2p - 4 = 0$; $p = \mathbf{0.8}$

3.1

(a) $\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ is not a multiple of $\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ (R1)

(b) $1 + 2\lambda = 3 + \mu$ and $2 + \lambda = -1 - 2\mu$; $\lambda = 0.2$, $\mu = -1.6$; z: -0.2 and $4 - 4.8 = -0.8$; not equal, and not parallel, so skew (R1)

(c) $\dfrac{|2 - 2 - 3|}{\sqrt{6}\sqrt{14}}$; $\mathbf{70.9}°$

(d) $\lambda = 3$; $(\mathbf{7}, \mathbf{5}, \mathbf{-3})$

3.2

(a) $\sqrt{3^2 + 4^2} = \mathbf{5}$ km h^{-1}

(b) $2 + 3t = 10 - t$, $t = 2$; y: $1 + 8 = 9$ and $-3 + 12 = 9$, so they are at the same place at the same time (R1); at $(\mathbf{8}, \mathbf{9})$

(c) $\begin{pmatrix} 3 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 6 \end{pmatrix} = 21$; $\cos\theta = \dfrac{21}{5\sqrt{37}}$; $\theta = \mathbf{46.3}°$

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