1.8 Proof by induction, contradiction and counterexample

Name: _____ Class: _____ Date: _____ **Total: 16 marks**

KEY WORDS mathematical induction • base case • inductive step • contradiction • counterexample • irrational

Objective

Build the skills to answer exam questions on **proof** : mathematical induction, proof by contradiction, and disproof by counterexample.

You must be able to:

- write a complete proof by mathematical induction for sums, divisibility and similar results
- prove a statement by contradiction, such as the irrationality of $\sqrt{2}$
- disprove a statement with a single counterexample

1 Key knowledge and worked examples

Proof

Mathematical induction (four parts, all needed for full marks):

1. **Base case:** show the statement is true for $n = 1$.
2. **Assumption:** assume it is true for $n = k$.
3. **Inductive step:** use the assumption to show it is true for $n = k + 1$.
4. **Conclusion:** "Since it is true for $n = 1$, and true for $n = k$ implies true for $n = k + 1$, it is true for all $n \in \mathbb{Z}^+$."

Contradiction: assume the statement is false and reach something impossible. Example: if $\sqrt{2} = \frac{p}{q}$ in lowest terms, then $p^2 = 2q^2$, so p is even, $p = 2m$, so $q^2 = 2m^2$ and q is even, contradicting "lowest terms".

Counterexample: one example where the statement fails is enough to disprove it.

Worked example. Prove $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$. Base: $1 = \frac{1\times 2}{2}$. Assume $\sum_{r=1}^{k} r = \frac{k(k+1)}{2}$. Then

$$\sum_{r=1}^{k+1} r = \frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+2)}{2},$$

which is the formula with $n = k + 1$. Conclusion as above.

Answer shapes: show where you use the assumption; write the target expression before you start the step.

2 Practice

2.1 Find a counterexample to the statement "$n^2 + n + 41$ is prime for every positive integer n". [2]

n	1	2	3
$n^2 + n + 41$	43	47	53

2.2 Prove by contradiction that if n^2 is even, then n is even. [3]

n^2 even $\Rightarrow$ n even

assume n odd: $n = 2k + 1$

exercises.lol exercises.com.cn

3 Exam-style questions

3.1 Prove by mathematical induction that $\sum_{r=1}^{n} r(r+1) = \frac{n(n+1)(n+2)}{3}$ for all $n \in \mathbb{Z}^+$. [6]



3.2 Prove by mathematical induction that $7^n - 1$ is divisible by 6 for all $n \in \mathbb{Z}^+$. [5]

$n = 1$	$n = 2$	$n = 3$
6	48	342

exercises.lol exercises.com.cn

Solutions

Marking: **M1** a correct method, **A1** the correct answer, **B1** an independent mark.

Each answer shows the steps, then the **result**.

2.1

e.g. $n = 40$: $40^2 + 40 + 41 = 1681 = 41^2$, not prime

2.2

assume n is odd, $n = 2k + 1$; $n^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$, which is odd; this contradicts n^2 even, so n is even (R1)

3.1

$n = 1$: $1 \times 2 = 2$ and $\frac{1 \times 2 \times 3}{3} = 2$; assume true for $n = k$; $\sum_{r=1}^{k+1} r(r+1) = \frac{k(k+1)(k+2)}{3} + (k+1)(k+2)$; $= \frac{(k+1)(k+2)(k+3)}{3}$; true for $n = 1$ and true for k implies true for $k + 1$, so true for all $n \in \mathbb{Z}^+$ (R1)

3.2

$n = 1$: $7 - 1 = 6$; assume $7^k - 1 = 6m$, $m \in \mathbb{Z}$; $7^{k+1} - 1 = 7(6m + 1) - 1$; $= 42m + 6 = 6(7m + 1)$; so true for all $n \in \mathbb{Z}^+$ by induction (R1)

exercises.lol exercises.com.cn