1.7 Series, finite differences and Taylor expansions

Name: _____ Class: _____ Date: _____ **Total: 13 marks**

KEY WORDS Taylor polynomial

Objective

Series, finite differences and Taylor expansions. Original classroom practice for IB HL; not an official IB paper.

1 Key knowledge and worked examples

Can a polynomial replace an exponential nearby?

- Near a known point, a complicated function can be replaced by a polynomial approximation. The approximation needs a centre and a valid scale.
- This lesson studies **Taylor polynomial** : A polynomial formed from a function's derivatives at a chosen centre.

The method

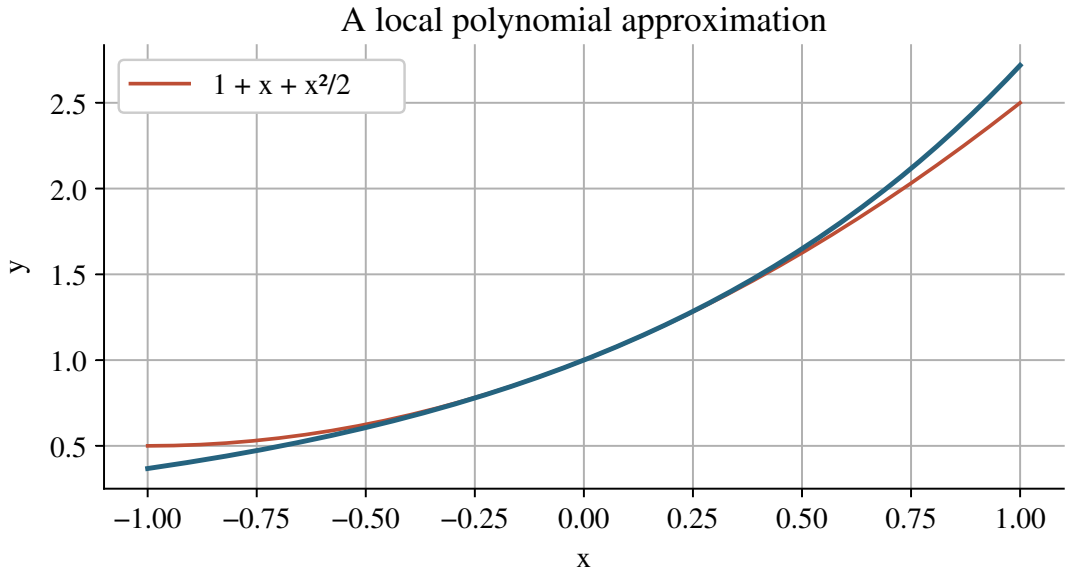
- About x=a, f(x)=f(a)+f prime(a)(x-a)+f double prime(a)(x-a)^2/2!+... when the expansion is valid.
- At a=0 it is a Maclaurin expansion.
- Finite sums may also simplify by cancellation or standard sum formulae.

A worked case

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots$$

Example:

The first three terms of e^x are 1+x+x^2/2. At x=0.1 this gives 1.105, close to e^0.1≈1.105170. For the sum of k^2 from k=1 to 5, n(n+1)(2n+1)/6 gives 5×6×11/6=55.



Where it goes wrong

- The factorial belongs in the denominator of every Taylor coefficient.
- The expansion centre is not always zero.
- An approximation close to its centre can be poor far away.

Not true: A Taylor polynomial is always exact for the original function at every input. A Taylor polynomial is an approximation; it is usually accurate only near its centre.

Applying it

- For a telescoping series, write several terms and identify both surviving ends.
- For a Taylor approximation, state its order and compare with a known value or a remainder estimate where required.

In your course

- Analysis and Approaches HL, the course first assessed in 2021.
- Paper 1 is non-calculator: keep fractions, surds and π exact and show each step. On calculator papers, still write the expression before its value.

Key:

- **Taylor polynomial**: A polynomial formed from a function's derivatives at a chosen centre.
- About x=a, f(x)=f(a)+f prime(a)(x-a)+f double prime(a)(x-a)^2/2!+... when the expansion is valid.
- Watch for: the factorial belongs in the denominator of every Taylor coefficient.

■ **Write from a square sum of 30**

- Find the sum of k squared from k = 1 to 4.
- Use 1 plus x plus x squared over 2 to estimate e to the power 0.2. A Taylor polynomial is not exact at every input.

2 Practice

2.1 Use 1+x+$x^2/2$ to estimate e^0.1. Show the relation, substitution and units. [2]

2.2 Find the sum of k^2 for k=1 to 5. Show the relation, substitution and units. [2]

2.3 Find the coefficient of x^2 in the Maclaurin expansion of e^x. Show the relation, substitution and units. [2]

2.4 Decide whether this claim is true and justify: A Taylor polynomial is always exact for the original function at every input. [2]

2.5 Why does 1+x+$x^2/2$ only approximate e^x? Choose one answer.

- **A** The factorials belong in the numerators, so the true series grows faster
- **B** The Maclaurin series continues with $x^3/6$ and beyond; truncating drops those terms
- **C** e^x is not a function, so no polynomial can represent it exactly [1]

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3 Exam-style questions

3.1 Approximate e^0.2 with $1+x+x^2/2$. Give a remainder bound using e^ξ $x^3/6$ with $0<\xi<0.2$ and explain what the bound certifies. [4]

Solutions

2.1 The truncated polynomial is $1+0.1+0.1^2/2=1.105$.

2.2 Add the squared terms: $1+4+9+16+25=55$.

2.3 The second derivative of e^x at zero is 1; divide by $2!=2$ to get $1/2$.

2.4 False. A Taylor polynomial is an approximation; it is usually accurate only near its centre.

2.5 B The full series is infinite: $e^x=1+x+x^2/2!+x^3/3!+\ldots$. Stopping after three terms discards the tail, so the result is an approximation good only near the centre $x=0$.

3.1 Approximation 1.22; $0<R<e^{0.2}(0.2)^3/6\approx0.001629$. The quadratic underestimates and the exact value lies between 1.22 and 1.221629. The bound depends on the stated Taylor theorem conditions.

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