1.6 Binomial expansion and valid approximations

Name: _____ Class: _____ Date: _____ **Total: 13 marks**

KEY WORDS binomial coefficient

Objective

Binomial expansion and valid approximations. Original classroom practice for IB HL; not an official IB paper.

1 Key knowledge and worked examples

How does a small change affect a power?

- A small measurement change affects a power of a quantity. An expansion can show the size of first and second effects.
- This lesson studies **binomial coefficient** : The number of ways to choose a specified number of objects from a set.

The method

- For positive integer n, (a+b)^n is a finite binomial expansion.
- For noninteger n, expand (1+x)^n as 1+nx+n(n-1)x^2/2+... with |x|<1.
- Factor out constants before using this form.

A worked case

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \cdots$$

Example:

(1+2x)^5=1+10x+40x^2+80x^3+80x^4+32x^5. For (1+x)^(-1), the first three terms are 1-x+x^2; at x=0.1 this gives 0.91 versus the exact 1/1.1≈0.909091.

$(1 + 2x)^5$

Coefficients: 1, 10, 40, 80, 80, 32

The x^2 coefficient includes $(2x)^2$

Where it goes wrong

- An expansion in 2x requires |2x|<1 for the infinite series, not merely |x|<1.
- Positive-integer expansions are finite and do not have that convergence restriction.

Not true: Every binomial expansion is a finite polynomial, including powers that are not positive integers. For a negative or fractional power the expansion is an infinite series, valid only for |x|<1.

Applying it

- State the range of validity alongside an approximation.
- Retain enough terms to support the requested accuracy, and distinguish a coefficient from the whole term containing x.

In your course

- Analysis and Approaches HL, the course first assessed in 2021.
- Paper 1 is non-calculator: keep fractions, surds and π exact and show each step. On calculator papers, still write the expression before its value.

Key:

- **Binomial coefficient**: The number of ways to choose a specified number of objects from a set.
- For positive integer n, (a+b)^n is a finite binomial expansion.
- Watch for: an expansion in 2x requires |2x|<1 for the infinite series, not merely |x|<1.

■ **Write from a coefficient of 54**

- Find the coefficient of x squared in (1 plus 3x) to the power 4.
- A binomial expansion is finite only for a non-negative integer power.

2 Practice

2.1 Find the coefficient of x^2 in (1+2x)^5. Show the relation, substitution and units. [2]

2.2 Use $1-x+x^2$ to estimate 1/(1+x) at x=0.1. Show the relation, substitution and units. [2]

2.3 How many terms are in the full expansion of (a+b)^5? Show the relation, substitution and units. [2]

2.4 Decide whether this claim is true and justify: Binomial coefficient means the number of ways to choose a specified number of objects from a set. [2]

2.5 For the infinite expansion of (1+2x)^(-1), which condition is required? Choose one answer.

- **A** No condition: it always converges
- **B** Only |x|<1
- **C** |2x|<1, so |x|<1/2 [1]

3 Exam-style questions

3.1 Use $1-x+x^2$ to approximate 1/(1+x) at x=0.2. Find the exact error algebraically and explain its sign. [4]

Solutions

2.1 Choose two of five factors and include 2^2: C(5,2)×4=40.

2.2 Substitute 0.1: 1-0.1+0.01=0.91.

2.3 The powers run from 0 through 5, giving six terms.

2.4 True. True: The number of ways to choose a specified number of objects from a set.

2.5 C The validity condition $|u|<1$ applies to the whole bracket term $u=2x$, so $|x|<1/2$. Using $|x|<1$ would wrongly accept $x=0.9$, where the series diverges.

3.1 Approximation 0.84; exact 5/6. Identity$(1-x+x^2)-1/(1+x)=x^3/(1+x)$, so error $1/150\approx0.006667$, positive. The infinite geometric expansion requires$|x|<1$.